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Seasonal Unit Root Tests and the Role of Initial Conditions*

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Abstract

In the context of regression-based (quarterly) seasonal unit root tests, we examine the impact of initial conditions (one for each quarter) of the process on test power. We investigate the behaviour of the OLS detrended HEGY seasonal unit root tests of Hylleberg *et al.* (1990) and the corresponding quasi-differenced (QD) detrended tests of Rodrigues and Taylor (2007), when the initial conditions are not asymptotically negligible. We show that the asymptotic local power of a test at a given frequency depends on the value of particular linear (frequency-specific) combinations of the initial conditions. Consistent with previous findings in the non-seasonal case (see, *inter alia*, Harvey *et al.*, 2008, Elliott and Müller, 2006), the QD detrended test at a given spectral frequency dominates on power for relatively small values of this combination, while the OLS detrended test dominates for larger values. Since, in practice, the seasonal initial conditions are not observed, in order to maintain good power across both small and large initial conditions, we extend the idea of Harvey *et al.* (2008) to the seasonal case, forming tests based on a union of rejections decision rule; rejecting the unit root null at a given frequency (or group of frequencies) if either of the relevant QD and OLS detrended HEGY tests rejects. This procedure is shown to perform well in practice, simultaneously exploiting the superior power of the QD (OLS) detrended HEGY test for small (large) combinations of the initial conditions. Moreover, our procedure is particularly adept in the seasonal context since, by design, it exploits the power advantage of the QD (OLS) detrended HEGY tests at a particular frequency when the relevant initial condition is small (large) without imposing that same method of detrending on tests at other frequencies.

Keywords: HEGY seasonal unit root tests; initial conditions; asymptotic local power; union of rejections decision rule.

JEL Classifications: C22.

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1 Introduction

The role of the initial condition (defined as the the deviation of the first observation from its deterministic component) on standard (zero frequency) unit root tests has attracted considerable attention in recent years. While unit root tests which include a constant in their detrending procedure are exact similar with respect to the initial condition, their local power functions depend crucially on the magnitude of the initial condition, even asymptotically; see, *inter alia*, Elliott *et al.* (1996), Elliott (1999), Müller and Elliott (2003), Elliott and Müller (2006), Harvey and Leybourne (2005, 2006), and Harvey *et al.* (2008).

As discussed in Elliott and Müller (2006,pp.286-90), while there may be situations in which one would not necessarily expect the initial condition to be unusually large or, indeed, unusually small, relative to the other data points, equally the initial condition might be relatively large in other situations. The former case occurs, for example, where the first observation in the sample is dated quite some time after the inception of a mean-reverting process, while the latter can happen if the sample data happen to be chosen to start after a break (perceived or otherwise) in the series or where the beginning of the sample coincides with the start of the process. This latter example can also allow for the case where an unusually small (even zero) initial condition occurs. In practice it is therefore hard to rule out small or large initial conditions, *a priori*. This is problematic, given the substantial impact of the magnitude of the initial condition on the power properties of standard unit root tests.

In the seasonal unit root testing context we have not one initial condition but S (seasonal) initial conditions, one for each of the S seasons. It therefore seems worthwhile and of practical relevance to investigate the role played by the magnitude of the initial conditions in determining the power properties of seasonal unit root tests. Working with a rather general formulation for the seasonal initial conditions (which includes seasonal extensions of the non-seasonal set-ups of Elliott, 1999, Müller and Elliott, 2003, and Elliott and Müller, 2006, as special cases), we find that for a test at a given frequency it is the magnitude of a specific linear combination of the seasonal initial conditions that matters, rather than the initial conditions themselves. For example, in the case of the zero frequency it is the sum of the initial conditions for each season that turns out to be the important quantity. We term these quantities *spectral initial conditions*. Where the spectral initial condition for a test at a given frequency is not asymptotically negligible, the quasi-differenced (QD) detrended Hylleberg *et al.* (1992) [HEGY]-type tests of Rodrigues and Taylor (2007) can perform very badly indeed, with their power against a given alternative rapidly decreasing towards zero as the magnitude of the spectral initial condition is increased. In sharp contrast, the OLS detrended HEGY tests show an increase in power, other things equal, as the magnitude of the spectral initial condition increases, albeit their powers are considerably lower than those of corresponding QD detrended tests when the initial condition is small. Powers of joint frequency unit root tests of the type proposed in Ghysels *et al.* (1994) [GLN] are also shown to depend on the method of detrending and relevant set of spectral initial conditions.

Our findings are made relevant because in practice the seasonal initial conditions are neither known nor are they amenable to estimation. Consequently, uncertainty surrounds the appropriate choice of detrending method. In the non-seasonal setting, such considerations led Harvey *et al.* (2008) to investigate whether it is possible in practice to construct unit root test strategies that maintain good power properties across both large and small initial conditions. They showed that a *union of rejections* decision rule between the QD- and OLS-based ADF tests (whereby the unit root null is rejected if either of the QD detrended ADF and OLS detrended ADF tests rejects) works well. This approach exploits the superior power properties of the QD (OLS) detrended tests when the initial condition is small (large) and is capable of outperforming the more sophisticated testing procedures proposed in Elliott and Müller (2006) and Harvey and Leybourne (2005, 2006). Our findings on the relative power behaviour of the QD and OLS detrended HEGY tests indicates that a union of rejections decision rule between the QD and OLS detrended HEGY tests, either at a given frequency or set of frequencies, can also be fruitfully employed in a seasonal context. We provide asymptotic and finite sample evidence to suggest that this procedure is again highly effective, despite its relative simplicity.

The plan of the remainder of the paper is as follows. In section 2 we outline our reference seasonal unit root testing model and detail the unit root tests on which we focus our attention. These are the OLS detrended seasonal unit root tests of HEGY and the corresponding QD detrended HEGY-type tests of Rodrigues and Taylor (2007). Although we restrict our analysis to the case of quarterly ($S = 4$) data, generalisations to an arbitrary seasonal aspect follow quite straightforwardly. The limiting distributions of these statistics are derived under near-seasonal integration in section 3. This enables us to show, and to illustrate numerically, the precise nature of the dependence of the asymptotic local power functions of these tests on the initial conditions of the process. In section 4 we detail our union of rejections testing strategy and compare its large sample performance with that of the corresponding OLS and QD detrended HEGY tests. Section 5 reports corresponding finite sample results. We offer some conclusions in section 6. Proofs of the main technical results in this paper are given in an Appendix.

Throughout the paper we use the following notation: '$x := y$' to indicate that x is defined by y; $\lfloor \cdot \rfloor$ to denote the integer part of the argument; '$\xrightarrow{p}$' and '$\xrightarrow{d}$' denote convergence in probability and weak convergence, respectively, as the sample size diverges, and $\mathbb{I}(\cdot)$ to denote the indicator function.

2 The Seasonal Unit Root Framework

2.1 The Seasonal Model

Consider the case where we have $T := 4N$ observations on the quarterly time series process $\{x_{4t+s}\}$, where N denotes the span in years of the sample data, generated according to the model

$$\begin{aligned} x_{4t+s} &= \mu_{4t+s} + v_{4t+s},\ s = -3, ..., 0,\ t = 1, 2, \ldots, N, & (1) \\ a(L)v_{4t+s} &= u_{4n+s},\ s = -3, ..., 0,\ t = 2, ..., N, & (2) \\ v_i &= \xi_i,\ i = 1, ..., 4, & (3) \end{aligned}$$

where $a(L) := 1 - \sum_{j=1}^{4} a_j L^j$ is a fourth order AR polynomial in the lag operator L, $L^{4j+k}x_{4t+s} := x_{4(t-j)+s-k}$, and the deterministic component $\mu_{4t+s} = \gamma_s + \delta(4t + s)$; that is, seasonal intercepts and a (non-seasonal) time trend.[1] The shocks, $\{u_{4t+s}\}$, are assumed to follow a stationary AR(p), $0 \leq p < \infty$, process, *viz.*,

$$\phi(L)u_{4t+s} = \varepsilon_{4t+s}, \quad (4)$$

where $\phi(z) := 1 - \sum_{i=1}^{p} \phi_i z^i$, the roots of $\phi(z) = 0$ all lie outside the unit circle, $|z| = 1$, and the error process, $\{\varepsilon_{4t+s}\}$, is a martingale difference sequence with constant conditional variance, σ^2; see Fuller (1996, Theorem 5.3.5,pp.236-37) for precise assumptions on $\{\varepsilon_{4t+s}\}$. We denote the long run variance of u_t by $\omega_u^2 := \sigma^2\psi(1)^2$, where $\psi(z)$ denotes the (unique) inverse of $\phi(z)$. The initial conditions of the process are given by $\xi_1, ..., \xi_4$ in (3), so that ξ_1 is the initial condition associated with the first quarter, ξ_2 the second quarter, and so on. Precise assumptions on the initial conditions will be detailed and discussed in section 2.3 below.

2.2 The Seasonal Unit Root Hypotheses

In this paper we are concerned with the behaviour of tests for seasonal unit roots in the $AR(S)$ polynomial, $\alpha(L)$, against near seasonally integrated alternatives; that is, the null hypothesis of interest is

$$H_0 : a(L) = 1 - L^4 =: \Delta_4, \quad (5)$$

while, following Tanaka (1996,pp.355-356), Rodrigues (2001), Taylor (2002) and Rodrigues and Taylor (2004b), *inter alia*, the near seasonally integrated alternative takes

[1]For expositional purposes we have chosen to focus our attention on the case of most practical relevance where the deterministic component consists of seasonal intercepts and a (non-seasonal) trend. Other choices of the deterministic component are possible; see, in particular, the typology of cases in Smith and Taylor, 1998. However, Smith and Taylor (1998) show that allowing for seasonal intercepts ensures that the resulting seasonal unit root tests will be exact similar with respect to the initial conditions, which is especially important given our focus in this paper. If the drift should appear seasonal, then μ_{4t+s} could be augmented with seasonal time trends, as in Smith and Taylor (1998), while if no drift was apparent the linear trend could be omitted from μ_{4t+s}.

the form,

$$H_c : a(L) = \left[1 - \left(1 + \frac{c}{N}\right) L^4\right], \ c \leq 0. \tag{6}$$

Notice that H_c of (6) reduces to H_0 of (5) for $c = 0$.

Under H_0 of (5) the DGP (1)-(2) of $\{x_{4t+s}\}$ is that of a quarterly random walk process with (non-seasonal) drift δ, admitting unit roots at each of the zero frequency, $\omega_0 = 0$, the Nyquist (or biannual) frequency, $\omega_2 = \pi$ and the annual frequency $\omega_1 = \pi/2$. Under H_c of (6) the process $\{x_{4t+s}\}$ is locally stationary. Rodrigues and Taylor (2004b) demonstrate that H_c of (6) can be partitioned into $H_c \equiv \cap_{k=0}^{2} H_{c,k}$, where the hypotheses $H_{c,0}$ and $H_{c,2}$ correspond to a local to unit root at the zero and biannual frequencies respectively, while $H_{c,1}$ yields a pair of complex conjugate local to unit roots at the annual frequency. The null hypothesis of unit roots at the zero, biannual and annual frequencies are therefore individually denoted as $H_{0,0}$, $H_{0,2}$ and $H_{0,1}$, respectively.

2.3 The Initial Conditions

As discussed in the introduction, a number of recent papers have highlighted the strong dependence of the power functions of non-seasonal unit root tests on the deviation of the initial observation of the series from its underlying deterministic component (see, *inter alia,* Elliott, 1999, Müller and Elliott, 2003, Elliott and Müller, 2006, and Harvey and Leybourne, 2005, 2006). The following assumption provides a generalisation of the conditions discussed by these authors to the seasonal case, and contains as special cases the assumptions made by previous authors in the seasonal case.

Assumption 1 *Under H_c of (6) with $c < 0$, the initial conditions in (3) are generated according to*

$$\xi_i = \alpha_i \sqrt{\omega_u^2/(1-\rho_N^2)}, \quad i = 1, ..., 4, \tag{7}$$

where $\rho_N := 1 + \frac{c}{N}$, and where $\alpha_i \sim IN(\mu_{\alpha,i}\mathbb{I}(\sigma_\alpha^2 = 0), \sigma_\alpha^2)$, $i = 1, ..., 4$, independently of u_{4t+s}, $s = -3, ..., 0$, $t = 2, ..., N$. For $c = 0$, that is under H_0 of (5), we may set $\xi_i = 0$, $i = 1, .., 4$, without loss of generality, due to the exact similarity of the seasonal unit root tests considered in this paper to the initial conditions; see Smith and Taylor (1998) and Rodrigues and Taylor (2007).

In Assumption 1, α_i controls the magnitude of the initial condition in season i, ξ_i, relative to the magnitude of the standard deviation of a stationary seasonal $AR(1)$ process with parameter ρ_N and innovation long-run variance ω_u^2. The form given for the ξ_i allow the initial conditions to be either random and of $O_p(N^{1/2})$, or fixed and of $O(N^{1/2})$. If $\sigma_\alpha^2 > 0$, then the initial conditions are random; $\sigma_\alpha^2 = 1$ yields the so-called unconditional case considered in the non-seasonal case by Elliott (1999) and in the seasonal case by Rodrigues and Taylor (2004b), *inter alia.* If, on the other hand, $\sigma_\alpha^2 = 0$ then the ξ_i are non-random and of the form given in Müller and Elliott (2003), Elliott and Müller (2006). By considering both the random and fixed scenarios in this way, we try to allow for some flexibility in how the initial conditions may be generated. Notice finally that Rodrigues and Taylor (2007) assume that the initial conditions are asymptotically vanishing, such that $N^{-1/2}\xi_i \xrightarrow{p} 0$, $i = 1, ..., 4$, which is equivalent to setting $\alpha_i = 0$, $i = 1, ..., 4$, in (7).

2.4 Regression-Based Seasonal Unit Root Tests

Following HEGY, Smith and Taylor (1998) and Rodrigues and Taylor (2007), *inter alia*, the regression-based approach to testing for seasonal unit roots in $\alpha(L)$ consists of two stages. In the first stage one detrends the data in order to achieve (exact) invariance to the seasonal intercept and linear trend parameters, γ_s, $s = -3, ..., 0$ and δ of (1). In the case of the OLS detrending approach of HEGY and Smith and Taylor (1998), the detrended series is given by $\hat{x}_{4t+s} := x_{4t+s} - \widehat{\boldsymbol{\beta}}'\mathbf{z}_{4t+s}$, where $\mathbf{z}_{4t+s} := (D_{1,4t+s}, ..., D_{4,4t+s}, (4t+s))'$ where $D_{j,4t+s} := \mathbb{I}(j = s)$, $j = -3, ..., 0$, and $\widehat{\boldsymbol{\beta}}$ is the OLS estimator of $\boldsymbol{\beta} := (\gamma_{-3}, ..., \gamma_0, \delta)'$, obtained from regressing x_{4t+s} onto $\mathbf{z}_{4t+s}$ along $4t + s = 1, ..., T$. Under the QD detrending approach of Rodrigues and Taylor (2007), $\hat{x}_{4t+s} := x_{4t+s} - \widetilde{\boldsymbol{\beta}}'\mathbf{z}_{4t+s}$, where $\widetilde{\boldsymbol{\beta}}$ is the QD estimator of $\boldsymbol{\beta}$ obtained from the OLS regression of $\mathbf{x_c}$ on $\mathbf{Z_c}$, where

$$\begin{aligned}
\mathbf{x_c} &:= (x_1, x_2 - \alpha_1^{\mathbf{c}} x_1, x_3 - \alpha_1^{\mathbf{c}} x_2 - \alpha_2^{\mathbf{c}} x_1, x_4 - \alpha_1^{\mathbf{c}} x_3 - \cdots - \alpha_4^{\mathbf{c}} x_1, \Delta_{\mathbf{c}} x_5, ...\Delta_{\mathbf{c}} x_T)' \\
\mathbf{Z_c} &:= (\mathbf{z}_1, \mathbf{z}_2 - \alpha_1^{\mathbf{c}} \mathbf{z}_1, \mathbf{z}_3 - \alpha_1^{\mathbf{c}} \mathbf{z}_2 - \alpha_2^{\mathbf{c}} \mathbf{z}_1, \mathbf{z}_4 - \alpha_1^{\mathbf{c}} \mathbf{z}_3 - \cdots - \alpha_4^{\mathbf{c}} \mathbf{z}_1, \Delta_{\mathbf{c}} \mathbf{z}_5, ..., \Delta_{\mathbf{c}} \mathbf{z}_T)'
\end{aligned}$$

and

$$\Delta_{\mathbf{c}} := \left(1 - \left(1 + \frac{\bar{c}_1}{T}\right) L\right)\left(1 + \left(1 + \frac{\bar{c}_2}{T}\right) L\right)\left(1 + \left(1 + \frac{\bar{c}_3}{T}\right)^2 L^2\right) =: 1 - \sum_{j=1}^{4} \alpha_j^{\mathbf{c}} L^j$$

where for tests run at the 5% level,[2] $\bar{c}_1 = -13.5$, $\bar{c}_2 = -7$ and $\bar{c}_3 = -3.75$.

In the second stage, using the Proposition in HEGY (pp.221-222), we expand $a(L)$ of (1) around the seasonal unit roots $\pm 1, \pm i$, $i := \sqrt{(-1)}$, to obtain the auxiliary regression equation

$$\Delta_4 \hat{x}_{4t+s} = \sum_{j=1}^{4} \pi_j \, \hat{x}_{j,4t+s-1} + \sum_{j=1}^{p} \phi_j^* \, \Delta_4 \hat{x}_{4t+s-j} + \hat{u}_{4t+s} \tag{8}$$

where $\Delta_4 \hat{x}_{4t+s} := \hat{x}_{4t+s} - \hat{x}_{4(t-1)+s}$ and, corresponding to the zero and biannual frequencies

$$\hat{x}_{1,4t+s} := a_1(L)\hat{x}_{4t+s}, \quad a_1(L) := \left(1 + L + L^2 + L^3\right) \tag{9}$$

and

$$\hat{x}_{2,4t+s} := -a_2(L)\hat{x}_{4t+s}, \quad a_2(L) := \left(1 - L + L^2 - L^3\right) \tag{10}$$

[2]If, as discussed in footnote 1, the linear trend variable is omitted from z_{4t+s}, then $\bar{c}_1$ should be changed to -7, while if seasonal trends are also included in z_{4t+s}, $\bar{c}_2$ and $\bar{c}_3$ should be changed to -13.5 and -8.65, respectively.

respectively, and corresponding to the annual frequency,

$$\begin{aligned} \hat{x}_{3,4t+s} &:= -a_3(L)\hat{x}_{4t+s}, \;\; a_3(L): \; = L(1-L^2) \\ \hat{x}_{4,4t+s} &:= -a_4(L)\hat{x}_{4t+s}, \;\; a_4(L): \; = (1-L^2) \end{aligned} \tag{11}$$

cf. HEGY and Smith and Taylor (1998).

The parameters π_j, $j = 1, ..., 4$, of (8) are of focal interest. As demonstrated in HEGY, a unit root occurs at the zero and biannual frequencies when $\pi_1 = 0$ and $\pi_2 = 0$ respectively, while a pair of complex conjugate unit roots occur at the annual frequency when $\pi_3 = \pi_4 = 0$. In order to test H_0 of (5) against the alternative of stationarity at at least one of the zero, biannual and harmonic seasonal frequencies, HEGY therefore propose using the following regression statistics in (8): t_1 (left-sided) for the exclusion of $\hat{x}_{1,4t+s-1}$; t_2 (left-sided) for the exclusion of $\hat{x}_{2,4t+s-1}$, and F_{34} for the exclusion of $\hat{x}_{3,4t+s-1}$ *and* $\hat{x}_{3,4t+s-1}$.[3] GLN also propose the joint frequency F-statistics, F_{234}, for the exclusion of $\hat{x}_{2,4t+s-1}$, $\hat{x}_{3,4t+s-1}$ and $\hat{x}_{4,4t+s-1}$, and F_{1234}, for the exclusion of all of $\hat{x}_{1,4t+s-1}$, $\hat{x}_{2,4t+s-1}$, $\hat{x}_{3,4t+s-1}$ and $\hat{x}_{4,4t+s-1}$. The former tests the null hypothesis of unit roots at all of the seasonal frequencies, while the latter tests the null hypothesis, H_0 of (5).

In what follows, we use a superscript OLS (QD) on these tests to denote that OLS (QD) detrending has been performed in the first stage, so that for example t_2^{QD} denotes the QD detrended biannual frequency test of Rodrigues and Taylor (2007), while t_2^{OLS} denotes the corresponding OLS detrended test of HEGY. Where no superscript is present, reference to the test is understood to be made in a generic sense. Finite sample and asymptotic null critical values for these tests are provided in Table 1, Panels A and B. The finite sample critical values were obtained via Monte Carlo simulation, setting $p = 0$ in the fitted regression (8) with $\phi(z) = 1$ and $\gamma_{-3} = \cdots \gamma_0 = \delta = 0$ in (1) and generating $\{\varepsilon_{4t+s}\}$ as an $NIID(0,1)$ sequence. Here and throughout the paper, simulations were programmed in Gauss 7.0 using 50,000 replications. See the discussion following Remark 4 below, regarding computation of the asymptotic critical values.

3 Asymptotic Representations

For the set of OLS and QD detrended seasonal unit root tests considered in section 2, the following lemma details their asymptotic behaviour.

Lemma 1 *Let* $\{x_{4t+s}\}$ *be generated according to (1)-(3) and let Assumption 1 hold.*

[3] In their original article HEGY also suggest a testing procedure for the annual frequency pair of unit roots based on the pair of regression statistics t_3 for the exclusion of $\hat{x}_{3,4t+s-1}$ and t_4 for the exclusion of $\hat{x}_{4,4t+s-1}$. However, these statistics have subsequently been shown to have non-pivotal asymptotic limiting null distributions when $p > 0$ in (4), rendering them unusable in practice; see, for example, Smith *et al.* (2007) and Burridge and Taylor (2001).

For $i = 1, 2, 3, 4$, let

$$K_{ic}(r) := \begin{cases} W_i(r) & c = 0 \\ \bar{\alpha}_i(e^{rc} - 1)(-2c)^{-1/2} + W_{ic}(r) & c < 0 \end{cases}$$

where $W_i(r)$, $i = 1, ..., 4$, are independent standard Brownian motion processes, $W_{ic}(r)$, $i = 1, ..., 4$, are independent standard Ornstein-Uhlenbeck processes, and the spectral magnitudes are defined as,

$$\begin{aligned} \bar{\alpha}_1 &:= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/2 \\ \bar{\alpha}_2 &:= (-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4)/2 \\ \bar{\alpha}_3 &:= (\alpha_4 - \alpha_2)/\sqrt{2} \\ \bar{\alpha}_4 &:= (\alpha_3 - \alpha_1)/\sqrt{2}. \end{aligned}$$

Also define

$$K^{\mu}_{ic}(r) := K_{ic}(r) - \int_0^1 K_{ic}(s)ds \qquad i = 1, 2, 3, 4.$$

Then under H_c of (6), the asymptotic distributions of the OLS and QD detrended t_1 and t_2 statistics from (8) are given by

$$t^j_i \xrightarrow{d} \frac{K^j_{ic}(1)^2 - K^j_{ic}(0)^2 - 1}{2\sqrt{\int_0^1 K^j_{ic}(r)^2 dr}} =: \tau^j_i \qquad i = 1, 2; \quad j = OLS,\ QD$$

where

$$\begin{aligned} K^{OLS}_{1c}(r) &:= K^{\mu}_{1c}(r) - 12\left(r - \frac{1}{2}\right)\int_0^1 \left(s - \frac{1}{2}\right) K_{1c}(s)ds \\ K^{OLS}_{2c}(r) &:= K^{\mu}_{2c}(r) \\ K^{QD}_{1c}(r) &:= K_{1c}(r) - \bar{c}^*_1 r K_{1c}(1) - 3(1 - \bar{c}^*_1) r \int_0^1 s K_{1c}(s)ds \\ K^{QD}_{2c}(r) &:= K_{2c}(r). \end{aligned}$$

with $\bar{c}^*_1 := (1 - \bar{c}_1)(1 - \bar{c}_1 + \bar{c}^2_1/3)$. Moreover, the asymptotic distributions of the F_{34}, F_{234} and F_{1234} statistics from (8) under H_c are given by

$$\begin{aligned} F^j_{34} &\xrightarrow{d} \frac{1}{2}\left[(A^j)^2 + (B^j)^2\right] =: \tau^j_{34}, \qquad j = OLS,\ QD \\ F^j_{234} &\xrightarrow{d} \frac{1}{3}\left[(\tau^j_2)^2 + (A^j)^2 + (B^j)^2\right] =: \tau^j_{234}, \qquad j = OLS,\ QD \\ F^j_{1234} &\xrightarrow{d} \frac{1}{4}\left[(\tau^j_1)^2 + (\tau^j_2)^2 + (A^j)^2 + (B^j)^2\right] =: \tau^j_{1234}, \qquad j = OLS,\ QD \end{aligned}$$

where

$$A^{OLS} := \quad c\sqrt{\int_0^1 K^{\mu}_{3c}(r)^2dr + \int_0^1 K^{\mu}_{4c}(r)^2dr} + \frac{\int_0^1 K^{\mu}_{3c}(r)dW_3(r) + \int_0^1 K^{\mu}_{4c}(r)dW_4(r)}{\sqrt{\int_0^1 K^{\mu}_{3c}(r)^2dr + \int_0^1 K^{\mu}_{4c}(r)^2dr}}$$

$$A^{QD} := \quad c\sqrt{\int_0^1 K_{3c}(r)^2dr + \int_0^1 K_{4c}(r)^2dr}$$
$$+\frac{\bar{\alpha}_3 c(-2c)^{-1/2}\int_0^1 K_{3c}(r)dr + \bar{\alpha}_4 c(-2c)^{-1/2}\int_0^1 K_{4c}(r)dr}{\sqrt{\int_0^1 K_{3c}(r)^2dr + \int_0^1 K_{4c}(r)^2dr}}$$
$$+\frac{\int_0^1 K_{3c}(r)dW_3(r) + \int_0^1 K_{4c}(r)dW_4(r)}{\sqrt{\int_0^1 K_{3c}(r)^2dr + \int_0^1 K_{4c}(r)^2dr}}$$

$$B^{OLS} := \quad \frac{\int_0^1 K^{\mu}_{3c}(r)dW_4(r) - \int_0^1 K^{\mu}_{4c}(r)dW_3(r)}{\sqrt{\int_0^1 K^{\mu}_{3c}(r)^2dr + \int_0^1 K^{\mu}_{4c}(r)^2dr}}$$

$$B^{QD} := \quad \frac{\bar{\alpha}_4 c(-2c)^{-1/2}\int_0^1 K_{3c}(r)dr - \bar{\alpha}_3 c(-2c)^{-1/2}\int_0^1 K_{4c}(r)dr}{\sqrt{\int_0^1 K_{3c}(r)^2dr + \int_0^1 K_{4c}(r)^2dr}}$$
$$+\frac{\int_0^1 K_{3c}(r)dW_4(r) - \int_0^1 K_{4c}(r)dW_3(r)}{\sqrt{\int_0^1 K_{3c}(r)^2dr + \int_0^1 K_{4c}(r)^2dr}}.$$

Remark 1. Under the null hypothesis H_0 of (5) the test statistics do not depend on the initial conditions $\{\xi_j\}_{j=1}^4$ (see footnote 1), so they play no role in their asymptotic null distributions. It is under the alternative hypothesis, H_c of (6) with $c < 0$, that the initial conditions have an effect. For a given statistic, setting the relevant value(s) of the $\{\bar{\alpha}_i\}_{i=1}^4$ to zero, the limiting representation given in Lemma 1 reduces to the corresponding representation for the statistic when the initial conditions are asymptotically negligible, as given in, *inter alia*, Rodrigues and Taylor (2004b, 2007).

Remark 2. Observe that the limiting distributions of the OLS and QD detrended HEGY tests from (8) do not depend on the magnitudes, α_i, of the initial conditions, ξ_i, $i = 1, ..., 4$, of (7) directly. Rather, they depend on the magnitude of frequency specific linear combinations of these initial conditions, what we will term *spectral initial conditions*. The zero frequency initial condition is given by $\bar{\xi}_1 := \xi_1 + \xi_2 + \xi_3 + \xi_4$, that for the biannual frequency by $\bar{\xi}_2 := -\xi_1 + \xi_2 - \xi_3 + \xi_4$, and those for the annual frequency by $\bar{\xi}_3 := \xi_4 - \xi_2$ and $\bar{\xi}_4 := \xi_3 - \xi_1$. Notice from (7) that the spectral magnitudes therefore satisfy $\bar{\alpha}_i \sim IN(\bar{\mu}_i \mathbb{I}(\sigma^2_\alpha = 0), \sigma^2_\alpha)$, $i = 1, ..., 4$, with $\bar{\mu}_1 := (\mu_{\alpha,1} + \mu_{\alpha,2} + \mu_{\alpha,3} + \mu_{\alpha,4})/2$, $\bar{\mu}_2 := (-\mu_{\alpha,1} + \mu_{\alpha,2} - \mu_{\alpha,3} + \mu_{\alpha,4})/2$, $\bar{\mu}_3 := (\mu_{\alpha,4} - \mu_{\alpha,2})/\sqrt{2}$ and $\bar{\mu}_4 := (\mu_{\alpha,3} - \mu_{\alpha,1})/\sqrt{2}$. Consequently if, for example, the magnitude of the initial conditions from each of the seasons happened to sum to zero (which would imply that $\bar{\alpha}_1 = 0$), then the asymptotic local power functions of the t_1^{OLS} and t_1^{QD} tests would be the same as if these initial conditions were asymptotically vanishing. Notice that the asymptotic local power function of the joint frequency F_{234} test depends on the spectral initial conditions relating to both the biannual and annual frequencies, while that for the F_{1234} test additionally depends on the zero frequency initial condition.

Remark 3. From Lemma 1, the limiting distributions of the t_1^{QD}, t_2^{QD} and F_{34}^{QD} statistics are mutually independent under H_c, as are the limiting distributions of the t_1^{OLS}, t_2^{OLS} and F_{34}^{OLS} statistics. Moreover, the limiting distributions of the t_1^{QD} and F_{234}^{QD} and t_1^{OLS} and F_{1234}^{OLS} statistics are also mutually independent. In each case this follows from the independence of the $K_{ic}(r)$, $i = 1, ..., 4$, limiting processes. Indeed, this implies more generally that the limiting distributions of different frequency statistics will be mutually independent regardless of whether they be based on OLS or QD detrended data, so that, for example, the t_2^{QD} and F_{34}^{OLS} statistics also have independent limiting distributions. However, it should be noted that, for example, the t_1^{OLS} and t_1^{QD} statistics will not have independent limiting distributions owing to the fact that they are both functionals of $K_{1c}(r)$.

Remark 4. If, as discussed in footnote 1, the linear trend variable is omitted from z_{4t+s}, then the representation given in Lemma 1 for t_1^{OLS} would hold on re-defining $K_{1c}^{OLS} := K_{1c}^{\mu}(r)$, and for t_1^{QD} by re-defining $K_{1c}^{OLS} := K_{1c}(r)$. In this case the stated representations given for both the OLS and QD detrended versions of the t_2, F_{34} and F_{234} statistics would be unchanged, while those for the OLS and QD detrended F_{1234} statistics would still be of the form given in Lemma 1, noting the change in τ_1^{OLS} and τ_1^{QD} from above. Should seasonal trends be included in z_{4t+s}, then while the limiting distributions of t_1^{OLS} and t_1^{QD} would remain unchanged, the limiting distributions for t_2^{OLS} and t_2^{GLS} would obtain on re-defining $K_{2c}^{OLS} := K_{2c}^{\mu}(r) - 12\left(r - \frac{1}{2}\right)\int_0^1 \left(s - \frac{1}{2}\right) K_{2c}(s)ds$ and $K_{2c}^{QD} := K_{2c}(r) - \bar{c}_2^* r K_{2c}(1) - 3(1-\bar{c}_2^*) r \int_0^1 s K_{2c}(s)ds$, where $\bar{c}_2^* := (1-\bar{c}_2)(1-\bar{c}_2+\bar{c}_2^2/3)$. Similarly, in this case the representations for the F_{34}^{QD} and F_{34}^{OLS} statistics would hold (and, hence, together with the changes for t_2^{QD} and t_2^{OLS}, for the F_{1234}^{QD}, F_{1234}^{OLS}, F_{234}^{QD} and F_{234}^{OLS} statistics) on replacing $K_{jc}(r)$ by $K_{jc}^{QD} := K_{jc}(r) - \bar{c}_3^* r K_{jc}(1) - 3(1-\bar{c}_3^*) r \int_0^1 s K_{jc}(s)ds$, where $\bar{c}_3^* := (1-\bar{c}_3)(1-\bar{c}_3+\bar{c}_3^2/3)$, for $j = 3, 4$ in the expressions for A^{QD} and B^{QD}, and replacing $K_{jc}^{\mu}(r)$ with $K_{jc}^{OLS} := K_{jc}^{\mu}(r) - 12\left(r - \frac{1}{2}\right)\int_0^1 \left(s - \frac{1}{2}\right) K_{jc}(s)ds$, for $j = 3, 4$ in the expressions for A^{OLS} and B^{OLS}. □

In Figures 1-6 we graph the asymptotic local powers of the OLS and corresponding QD detrended HEGY tests from (8) run at the nominal 0.05 level, of each of the tests from Lemma 1 for $c = -5, -7.5, -10$. The results reported in Figures 1 ($c = -5$), 2 ($c = -7.5$) and 3 ($c = -10$) pertain to the fixed initial conditions case, while Figures 4 ($c = -5$), 5 ($c = -7.5$) and 6 ($c = -10$) are for the case of random initial conditions. In the case of the t_1^{QD} and F_{1234}^{QD} statistics, whose limiting distribution depends on the QD parameter $\bar{c}_1$, the reported results pertain to $\bar{c}_1 = -13.5$. The null critical values and local powers were obtained by direct simulation of the limiting functionals in Lemma 1 approximating the Wiener processes using $NIID(0, 1)$ random variates, and with the integrals approximated by normalized sums of 1000 steps.

For the results in Figures 1-3 we report in parts (a), (b) and (c) for the pairs of tests, (t_1^{OLS}, t_1^{QD}), (t_2^{OLS}, t_2^{QD}) and $(F_{34}^{OLS}, F_{34}^{QD})$, respectively, the local powers as functions of the absolute values[4] of the relevant magnitude parameters $\bar{\mu}_1$, $\bar{\mu}_2$ and $\bar{\mu}_3 = \bar{\mu}_4$, respectively, for $|\bar{\mu}_i| = |\bar{\mu}| = \{0.0, 0.1, 0.2, ..., 6.0\}$, $i = 1, ..., 4$. For the joint frequency $(F_{234}^{OLS}, F_{234}^{QD})$ pair of tests we report in parts (d), (e) and (f) of Figures 1-3 the local powers as functions of: $|\bar{\mu}_2| = |\bar{\mu}| = \{0.0, 0.1, 0.2, ..., 6.0\}$ with $\bar{\mu}_3 = \bar{\mu}_4 = 0$; $|\bar{\mu}_3| = |\bar{\mu}_4| = |\bar{\mu}| = \{0.0, 0.1, 0.2, ..., 6.0\}$ with $\bar{\mu}_2 = 0$, and $|\bar{\mu}_2| = |\bar{\mu}_3| = |\bar{\mu}_4| = |\bar{\mu}| = \{0.0, 0.1, 0.2, ..., 6.0\}$, respectively. Parts (g), (h) and (i) of Figures 1-3 report local powers of the $(F_{1234}^{OLS}, F_{1234}^{QD})$ pair of tests as functions of $|\bar{\mu}_1| = |\bar{\mu}| = \{0.0, 0.1, 0.2, ..., 6.0\}$ with $\bar{\mu}_2 = \bar{\mu}_3 = \bar{\mu}_4 = 0$; $|\bar{\mu}_3| = |\bar{\mu}_4| = |\bar{\mu}| = \{0.0, 0.1, 0.2, ..., 6.0\}$ with $\bar{\mu}_1 = \bar{\mu}_2 = 0$, and $|\bar{\mu}_1| = |\bar{\mu}_2| = |\bar{\mu}_3| = |\bar{\mu}_4| = |\bar{\mu}| = \{0.0, 0.1, 0.2, ..., 6.0\}$, respectively. Corresponding results for the case of random starting values are reported in Figures 4-6 as functions of $\sigma_\alpha = \{0.0, 0.1, 0.2, ..., 6.0\}$.

Consider first the results for the t_1, t_2 and F_{34} tests.[5] We immediately see from these results that with either random or fixed initial conditions, the power curves of the QD detrended HEGY tests in each case exhibit monotonic decrease in σ_α or $|\bar{\mu}_i|$, whilst the power of the OLS detrended HEGY tests increase monotonically. In the fixed case, the t_1^{OLS} test is seen to have higher power than the t_1^{QD} test when (approximately) $|\bar{\mu}_1| = 1.5, 1.4, 1.3$ for $c = -5, -7.5$ and -10, respectively. For the t_2^{OLS} and t_2^{QD} tests these crossing points all occur at about $|\bar{\mu}_2| = 1.0$, while for the F_{34}^{OLS} and F_{34}^{QD} tests these occur at about $|\bar{\mu}_3| = |\bar{\mu}_4| = 0.9$. A key feature here is the drastic speed with which the power of the QD detrended version of the tests approaches zero with $|\bar{\mu}|$: the t_1^{QD} test has power which is effectively zero for $|\bar{\mu}_1| \geq 4$, while for t_2^{QD} and F_{34}^{QD} power is effectively zero even by $|\bar{\mu}_2| \geq 2$ and $|\bar{\mu}_3| = |\bar{\mu}_4| \geq 2$, respectively. For the random case the crossing points for t_1^{OLS} and t_1^{QD} occur at about $\sigma_\alpha = 1.8, 1.7, 1.6$ for $c = -5, -7.5, -10$, respectively. For t_2^{OLS} and t_2^{QD} they occur at about 1.6, 1.5 and 1.4, respectively, while for the F_{34}^{OLS} and F_{34}^{QD} tests these occur at about 1.1, 1.0 and 0.9, respectively. For each of these pairs of tests, the extent of the power dominance of the QD detrended variant over the OLS detrended variant increases as σ_α shrinks towards zero.

Now consider the results for the joint frequency F_{234} and F_{1234} tests. As with the results for the single frequency tests discussed above, we see that in both the fixed and random cases the QD detrended HEGY tests dominate the corresponding OLS detrended tests on power for small initial conditions with the pattern reversing for large initial conditions. In the case of the fixed initial conditions, the limiting distributions of the joint tests now depend on more than one spectral initial condition (precisely,

[4]It should be clear from the representations given in Lemma 1 that the asymptotic local power functions of the tests do not depend on the signs of the $\bar{\alpha}_i$, $i = 1, ..., 4$.

[5]Notice that the powers of the t_1^{QD} and t_1^{OLS} tests are in general rather lower than the power functions of the corresponding tests at other frequencies. This is because of the presence of a non-seasonal linear trend in the detrending routine. It is well known that this causes a significant reduction in available power; cf. Elliott *et al.* (1996), Harvey *et al.* (2008) and Rodrigues and Taylor (2004, 2007).

the F_{234} tests depends on $\bar{\xi}_2, \bar{\xi}_3$ and $\bar{\xi}_4$, while the F_{1234} test additionally depends on $\bar{\xi}_1$) so that the relationship between the power properties of the QD and OLS detrended variants of the test and the underlying initial conditions is more complex than for the single frequency tests.

In the fixed case, the crossing points of the power functions of a given joint frequency tests are consequently related to the magnitude of all of the spectral intercepts which feature in the statistic's limiting distribution. As can clearly be seen by comparing, for example, parts (g) and (i) of Figure 1, the crossing point for the F_{1234} tests when $c = -5$ is at about $|\bar{\mu}_1| = 4.4$ when $\bar{\mu}_2 = \bar{\mu}_3 = \bar{\mu}_4 = 0$, but is at about $|\bar{\mu}| = 0.8$ when $|\bar{\mu}_1| = |\bar{\mu}_2| = |\bar{\mu}_3| = |\bar{\mu}_4| = |\bar{\mu}|$, indicating that, as might be expected, the point at which the QD version of the joint frequency tests becomes inferior on power to the OLS version occurs for smaller magnitudes of the spectral initial conditions the more of these there are that are non-zero. Moreover, as can be seen by comparing, for example, (d) and (e) with (f), and (g) and (h) with (i) in Figure 1 it is only when *all* of the spectral initial values relevant to a particular test are non-zero that the power of the QD test collapses to zero as the magnitude of the initial conditions increases. To explain this phenomenon, consider, for example, the F_{1234}^{QD} statistic. Now, asymptotically, this is equal to the average of the squared t_j^{QD}, $j = 1, ..., 4$, statistics. Consider then part (g) of Figures 1-3. Here while the power of the t_1^{QD} statistic (and, hence, also the power of the $(t_1^{QD})^2$ statistic), collapses to zero as $|\bar{\mu}_1|$ increases, the spectral intercepts relating to the t_2^{QD} and F_{34}^{QD} statistics are all zero and so these tests maintain power, such that the power of the F_{1234}^{QD} test will not drop to zero. This also explains why the crossing point for the joint tests moves to the left, other things equal, as the number of non-zero spectral intercepts which affect the statistic increases. In the random case, similar patterns are seen in the joint frequency F_{1234} and F_{234} tests as for the t_1, t_2 and F_{34} tests, with the crossing points occurring at about 1.0 for each of the tests for $c = -5, -7.5$, and at about 0.9 for $c = -10$.

An interesting implication of the findings above is that, depending on the magnitudes of the individual spectral initial conditions it is possible that at one frequency, due to a large spectral initial condition at that frequency, the OLS detrended variant of the HEGY test could dominate the corresponding QD detrended test on power, but that if the initial conditions at the other spectral frequencies were small, here the QD variants of the tests would dominate on power. Consequently, while constructing the HEGY regression from QD detrended data would be appropriate for tests at those frequencies where small spectral initial conditions pertained, it would be a very inefficient thing to do for any frequencies where the initial condition was large, and *vice versa*.

4 A Union of Rejections Testing Strategy

Given the clear results of Figures 1-6, it seems sensible to consider whether it is possible to devise a testing strategy which, for small values of σ_α in the random case or the relevant $|\mu_i|$, $i = 1, ..., 4$, magnitudes in the fixed case, captures the power advantages of the QD detrended HEGY tests over the corresponding OLS detrended tests and, at the same time, exploits the reverse relationship that exists between the tests' power when σ_α or $|\mu_i|$, $i = 1, ..., 4$, is large.

As noted in the introduction, in the context of initial condition uncertainty in the non-seasonal case, Harvey *et al.* (2008) suggest a simple *union of rejections* decision rule between the QD- and OLS-based ADF tests; the unit root null being rejected if either of the QD detrended ADF and OLS detrended ADF tests rejects. This approach effectively combines the superior power properties of the QD detrended ADF test when the initial condition is small with those of the OLS detrended ADF when the initial condition is large and, as such, represents a near admissible procedure; see Müller (2008). Compared with other more involved procedures, such as those of Elliott and Müller (2006) and Harvey and Leybourne (2005, 2006), it is extremely competitive in terms of power. Whilst is is not immediately clear how these competing procedures might be extended to the current seasonal case, extension of the union of rejections approach is quite straightforward. We simply take the union of rejections of the QD and OLS detrended versions of each of the t_1, t_2, F_{34}, F_{234} and F_{1234} statistics.

Let cv_ζ^{QD} and cv_ζ^{OLS} be used generically to denote the asymptotic ζ significance level critical values of the QD and OLS detrended HEGY tests for some sample size T. Then:

(i) For the zero frequency, the relevant union of rejections procedure is given by

$$t_1^{UR}(\zeta) := t_1^{QD}\mathbb{I}(t_1^{QD} < cv_\zeta^{QD}) + t_1^{OLS}\mathbb{I}(t_1^{QD} \geq cv_\zeta^{QD})$$

where if $t_1^{UR}(\zeta) = t_1^{QD}$, a rejection of $H_{0,0}$ is recorded if $t_1^{UR}(\gamma) < cv_\zeta^{QD}$; otherwise if $t_1^{UR}(\zeta) = t_1^{OLS}$, a rejection is recorded if $t_1^{UR}(\zeta) < cv_\zeta^{OLS}$. In the limit, using the relevant expressions from Lemma 1

$$t_1^{UR}(\zeta) \xrightarrow{d} \tau_1^{QD}I(\tau_1^{QD} < cv_\zeta^{QD}) + \tau_1^{OLS}I(\tau_1^{QD} \geq cv_\zeta^{QD})$$

where, for example, for tests run at the asymptotic 0.05 level, $cv_\zeta^{QD} = -2.85$ and $cv_\zeta^{OLS} = -3.42$; cf. Table 1, Panels A and B.

(ii) For the biannual frequency, the union of rejections is given by

$$t_2^{UR}(\zeta) := t_2^{QD}I(t_2^{QD} < cv_\zeta^{QD}) + t_2^{OLS}I(t_2^{QD} \geq cv_\zeta^{QD})$$

where if $t_2^{UR}(\zeta) = t_2^{QD}$, a rejection of $H_{0,2}$ is recorded if $t_2^{UR}(\zeta) < cv_\zeta^{QD}$; otherwise if $t_2^{UR}(\zeta) = t_2^{OLS}$, a rejection is recorded if $t_2^{UR}(\zeta) < cv_\zeta^{OLS}$. In the limit, from Lemma 1

$$t_2^{UR}(\zeta) \xrightarrow{d} \tau_2^{QD}\mathbb{I}(\tau_2^{QD} < cv_\zeta^{QD}) + \tau_2^{OLS}\mathbb{I}(\tau_2^{QD} \geq cv_\zeta^{QD})$$

and at the asymptotic 0.05 level, $cv_\zeta^{QD} = -1.94$ and $cv_\zeta^{OLS} = -2.86$.

(iii) For the annual frequency, the union of rejections is

$$F_{34}^{UR}(\zeta) := F_{34}^{QD}\mathbb{I}(F_{34}^{QD} > cv_\zeta^{QD}) + F_{34}^{OLS}\mathbb{I}(F_{34}^{QD} \leq cv_\zeta^{QD})$$

where if $F_{34}^{UR}(\zeta) = F_{34}^{QD}$, a rejection of $H_{0,1}$ is recorded if $F_{34}^{UR}(\zeta) > cv_\zeta^{QD}$; otherwise if $F_{34}^{UR}(\zeta) = F_{34}^{OLS}$, a rejection is recorded if $F_{34}^{UR}(\zeta) > cv_\zeta^{OLS}$. From Lemma 1, we have that

$$F_{34}^{UR}(\zeta) \xrightarrow{d} \tau_{34}^{QD}\mathbb{I}(\tau_{34}^{QD} > cv_\zeta^{QD}) + \tau_{34}^{OLS}\mathbb{I}(\tau_{34}^{QD} \leq cv_\zeta^{QD}).$$

For tests run at the asymptotic 0.05 level, $cv_\zeta^{QD} = 3.07$ and $cv_\zeta^{OLS} = 6.62$.

(iv) For testing the joint null hypothesis of unit roots at the biannual and annual frequencies, the union of rejections is

$$F_{234}^{UR}(\zeta) := F_{234}^{QD}I(F_{234}^{QD} > cv_\zeta^{QD}) + F_{234}^{OLS}I(F_{234}^{QD} \leq cv_\zeta^{QD})$$

where if $F_{234}^{UR}(\zeta) = F_{234}^{QD}$, a rejection of $H_{0,1} \cap H_{0,2}$ is recorded if $F_{234}^{UR}(\zeta) > cv_\zeta^{QD}$; otherwise if $F_{234}^{UR}(\zeta) = F_{234}^{OLS}$, a rejection is recorded if $F_{234}^{UR}(\zeta) > cv_\zeta^{OLS}$. Again using Lemma 1, we have that

$$F_{234}^{UR}(\zeta) \xrightarrow{d} \tau_{234}^{QD}I(\tau_{234}^{QD} > cv_\zeta^{QD}) + \tau_{234}^{OLS}I(\tau_{234}^{QD} \leq cv_\zeta^{QD}).$$

For tests run at the asymptotic 0.05 level, $cv_\zeta^{QD} = 2.74$ and $cv_\zeta^{OLS} = 5.87$.

(iv) For testing the joint null hypothesis of unit roots at the zero, biannual and annual frequencies, the union of rejections is

$$F_{1234}^{UR}(\zeta) := F_{1234}^{QD}I(F_{1234}^{QD} > cv_\zeta^{QD}) + F_{1234}^{OLS}I(F_{1234}^{QD} \leq cv_\zeta^{QD})$$

where if $F_{1234}^{UR}(\zeta) = F_{1234}^{QD}$, a rejection of H_0 is recorded if $F_{1234}^{UR}(\zeta) > cv_\zeta^{QD}$; otherwise if $F_{1234}^{UR}(\zeta) = F_{1234}^{OLS}$, a rejection is recorded if $F_{1234}^{UR}(\zeta) > cv_\zeta^{OLS}$. Using Lemma 1 we have that

$$F_{1234}^{UR}(\zeta) \xrightarrow{d} \tau_{1234}^{QD}I(\tau_{1234}^{QD} > cv_\zeta^{QD}) + \tau_{1234}^{OLS}I(\tau_{1234}^{QD} \leq cv_\zeta^{QD}).$$

For tests run at the asymptotic 0.05 level, $cv_\zeta^{QD} = 3.32$ and $cv_\zeta^{OLS} = 6.19$.

Observe that while these procedures share the same asymptotic independence properties as were detailed in Remark 2 (so that, for example, $t_1^{UR}(\zeta)$ is asymptotically independent of $t_2^{UR}(\zeta)$), it should also be clear (using Bonferroni's inequality) that none of these individual strategies are size controlled for $c = 0$, being oversized even asymptotically. For significance levels $\gamma = 0.10$, 0.05 and 0.01 the asymptotic sizes of $t_1^{UR}(\zeta)$, $t_2^{UR}(\zeta)$, $F_{34}^{UR}(\zeta)$, $F_{234}^{UR}(\zeta)$ and $F_{1234}^{UR}(\zeta)$ are given in Table 1, Panel C.

However, we can correct these sizes quite easily in the limit. Taking the zero frequency $t_1^{UR}(\zeta)$ test to illustrate the principle, we simply need to determine a scaling constant, λ_ζ, say, that is applied to the critical values cv_ζ^{QD} and cv_ζ^{OLS}, such that

$$t_1^{UR*}(\zeta) := t_1^{QD}I(t_1^{QD} < \lambda_\zeta cv_\zeta^{QD}) + t_1^{OLS}I(t_1^{QD} \geq \lambda_\zeta cv_\zeta^{QD})$$

which records a rejection of $H_{0,0}$ if $t_1^{UR*}(\zeta) = t_1^{QD}$ and $t_1^{UR*}(\zeta) < \lambda_\zeta cv_\zeta^{QD}$ or if $t_1^{UR*}(\zeta) = t_1^{OLS}$ and $t_1^{UR*}(\zeta) < \lambda_\zeta cv_\zeta^{OLS}$, has an asymptotic size of ζ. In order to determine λ_ζ in a computationally efficient way, we recognise that the decision rule associated with $t_1^{UR}(\zeta)$ can also be written as

$$\text{Reject } H_{0,0} \text{ if } \min\left\{t_1^{QD}, \left(\frac{cv_\zeta^{QD}}{cv_\zeta^{OLS}}\right) t_1^{OLS}\right\} < cv_\zeta^{QD}. \tag{12}$$

The representation in (12) makes it very straightforward to calculate λ_ζ. Specifically, setting $c = 0$ we find the limit distribution of the min function in (12) using the (joint) limit distributions of t_1^{QD} and t_1^{OLS}, then obtain an asymptotic ζ-level critical value from this empirical cdf, say cv_ζ^{UR}. Then λ_ζ is given by $\lambda_\zeta := cv_\zeta^{UR}/cv_\zeta^{QD}$.

The asymptotic size-corrected variants of the remaining union of rejections tests can be obtained in the same way, and we label these t_2^{UR*}, F_{34}^{UR*}, F_{234}^{UR*} and F_{1234}^{UR*}. Notice, however, for the latter three tests, that the decision rule analogous to (12) involves the maximum rather than minimum function, so that, for example, the decision rule of $F_{34}^{UR}(\zeta)$ can also be written as

$$\text{Reject } H_{0,1} \text{ if } \max\left\{F_{34}^{QD}, \left(\frac{cv_\zeta^{QD}}{cv_\zeta^{OLS}}\right) F_{34}^{OLS}\right\} > cv_\zeta^{QD}.$$

The scaling constants required for all the tests are reported in Table 1, Panel D for $\zeta = 0.10$, 0.05 and 0.01. Notice that this yields testing strategies which are correctly sized in the limit, regardless of the value of σ_α or the $\mu_{\alpha,i}$, $i = 1, ..., 4$, since the (exact) null distributions of the tests involved do not depend on these parameters.

The asymptotic power curves for the union of rejections tests are shown in Figures 1-6. Both the raw and size-corrected variants of each of the tests are given. As we would conjecture, the power curves of the basic t_1^{UR}, t_2^{UR}, F_{34}^{UR}, F_{234}^{UR} and F_{1234}^{UR} tests tend to mimic (lie slightly above due to the oversizing phenomenon noted above) those of t_1^{QD}, t_2^{QD}, F_{34}^{QD}, F_{234}^{QD} and F_{1234}^{QD}, respectively, for small magnitudes of the relevant spectral initial conditions, then mimic those of t_1^{OLS}, t_2^{OLS}, F_{34}^{OLS}, F_{234}^{OLS} and F_{1234}^{OLS}, respectively, for large initial conditions. Thus, the union of rejections tests capture the power advantage of the QD detrended tests relative to the OLS detrended tests when the magnitude of the relevant spectral initial conditions is small, whilst avoiding the severe power losses that the QD detrended tests frequently exhibit relative to the OLS detrended tests when the converse is true. The size-corrected tests t_1^{UR*}, t_2^{UR*}, F_{34}^{UR*}, F_{234}^{UR*} and F_{1234}^{UR*} obviously have lower power across the board than their uncorrected counterparts, but the qualitative picture here is similar; for the smaller initial conditions the size corrected tests pick up a good deal of the extra power available to the QD detrended tests over OLS detrended variants, while for the larger initial conditions they avoid the dramatic power losses often associated with QD detrended tests and follow, with only a modest loss in power, the (typically rising) power profile of the OLS detrended tests. It is interesting to observe, also, that when the initial conditions are fixed, there is an almost exact common point of intersection for the QD detrended, OLS detrended and size-corrected union of rejections tests.

5 Finite Sample Comparisons

In this section we investigate the finite sample size and power properties of the QD and OLS detrended HEGY-type tests of section 2.4 together with the corresponding raw and size-corrected union of rejections tests from section 4. Our simulations are based on the DGP (1)-(3) under H_c of (6). Results are reported for samples of size $T = 152$ and $T = 300$. Without loss of generality, we set $\gamma_{-3} = \cdots = \gamma_0 = \delta = 0$ in (1).

Throughout these simulations, we select p in the fitted regression (8) using downward testing at the 0.10 level from a maximum lag length set at $p_{\max} = \lfloor 12(T/100)^{1/4} \rfloor$. Finite sample critical values for the tests are taken from Table 1, Panels A and B. The scaling constants applied to size-corrected union of rejections tests are the asymptotically valid ones given in Table 1, Panel D.

In Table 2 we first report the empirical sizes of the various tests for the case where $c = 0$ and where the innovation process u_{4t+s} is assumed to follow an ARMA(1,4) process of the form $(1-\phi L)u_{4t+s} = (1-\theta L^4)\varepsilon_{4t+s}$ for $\phi \in \{0.0, 0.3, 0.6\}$, $\theta \in \{\pm 0.4, 0.0\}$ and $\varepsilon_{4t+s} \sim NIID(0,1)$, with $u_1 = \varepsilon_1$ and $\varepsilon_s = 0$, $s = -3, ..., 0$. For the size results all initial conditions ξ_i, $i = 1, ..., 4$, are set to zero with no loss of generality. The sizes of the QD and OLS detrended tests are fairly similar throughout Table 2 and are generally free from significant size distortion (outside of a negative moving averge component), particularly for the larger sample size. If anything, the QD detrended tests display slightly less upward size distortion than the corresponding OLS detrended tests. For any given combination of ϕ and θ, the (asymptotically) size-corrected union of rejections tests tend to have sizes similar to the larger of the QD and OLS detrended tests individual sizes. This occurs since the union essentially selects whichever of the QD and OLS detrended tests is least favourable to the null. Obviously, compared to the other tests, the raw union of rejections always have the highest sizes. Overall, then, the results of Table 2 indicate that the (asymptotically) size-corrected union of rejections tests display pretty decent size control.

Finite sample powers are given in Figures 7-12. Here we set $\phi = \theta = 0$ in the generating process to abstract from any counfonding effects that may arise from size distortions. For brevity we report results for the random initial conditions case only, using the same sets of values for σ_α and c that underlie Figures 4-6. An issue of note here is that all the tests tend to display lower powers than in the asymptotic case, most noticeably when $T = 152$. We expect that this is partly explained by the lag selection process which is still in place. Otherwise, the finite sample relationships between the QD detrended, OLS detrended, and corresponding raw and size-corrected union of rejections tests across σ_α and c qualitatively resemble those of their asymptotic counterparts when $T = 152$. For $T = 300$, the resemblance is much closer in general.

On the basis of this finite sample evidence, it appears then that a size-corrected union of rejections approach can provide a very decent practical strategy for seasonal unit root testing in the context of uncertainty about the initial conditions and, consequently, equal uncertainty over whether it is best to employ QD or OLS detrending. This is a particularly pertinent issue in the seasonal case considered here, because in employing QD detrending, while this might constitute the best approach at one frequency, it may also be totally unsuitable for a different frequency, depending on the values of the spectral initial conditions. Taking unions of rejections at each frequency essentially ensures that we employ the most appropriate method of detrending at any particular frequency.

6 Conclusions

In this paper we have investigated the impact that the magnitude of the spectral initial condition has on the power of commonly used seasonal unit root tests. For a given frequency we have shown that when the relevant spectral initial condition of the process is not asymptotically negligible, QD detrended implementation of a HEGY-type seasonal unit root test, as developed by Rodrigues and Taylor (2007), can lead to tests that have very low power against a given alternative, typically decreasing towards zero as the magnitude of the relevant spectral initial condition(s) increase. In contrast, we showed that corresponding OLS detrended HEGY tests display increasing power, other things equal, as the magnitude of the spectral initial condition(s) increase. At the same time, the power of such tests can lie well below that of their QD detrended counterparts for small (or asymptotically negligible) values of the initial condition. The relevance of these results lies in the fact that the magnitude of the initial condition is unknown in practice and therefore uncertainty surrounds the best choice of detrending method, which can therefore also differ across frequencies. Given these considerations, we followed a strategy shown to work well in the non-seasonal case by Harvey *et al.* (2008) and proposed a union of rejections decision rule, whereby the relevant null hypothesis was rejected if either of the QD and OLS detrended variants rejected. Asymptotic and finite sample evidence suggested that, despite its simplicity, this procedure is highly effective.

Appendix

Proof of Lemma 1

Consider first the t_2^{QD} statistic. When the initial conditions are asymptotically negligible, it follows from Rodrigues and Taylor (2004a, 2004b, 2007) that under the stated conditions the limit distribution of the statistic can be written as

$$t_2^{QD} \xrightarrow{d} c\sqrt{\int_0^1 W_{2c}(r)^2 dr} + \frac{\int_0^1 W_{2c}(r) dW_{20}(r)}{\sqrt{\int_0^1 W_{2c}(r)^2 dr}}$$

where

$$W_{2c}(r) := \left\{-W^*_{-3,c}(r) + W^*_{-2,c}(r) - W^*_{-1,c}(r) + W^*_{0,c}(r)\right\}/2$$

is a standard Ornstein-Uhlenbeck [OU] process, formed from the independent standard OU processes, $W^*_{s,c}(r)$, $s = -3, ..., 0$. Noting that $dW_{2c}(r) = cW_{2c}(r)dr + dW_{20}(r)$, we can equivalently write

$$\begin{aligned} t_2^{QD} &\xrightarrow{d} \frac{\int_0^1 W_{2c}(r)dW_{2c}(r)}{\sqrt{\int_0^1 W_{2c}(r)^2 dr}} \\ &= \frac{W_{2c}(1)^2 - W_{2c}(0)^2 - 1}{2\sqrt{\int_0^1 W_{2c}(r)^2 dr}} \end{aligned} \tag{13}$$

using the Itô integral. When the initial conditions are as defined in Assumption 1, the analyis of Müller and Elliott (2003) implies that we need to replace $W^*_{s,c}(r)$ with $K^*_{s,c}(r) := \alpha_{s+4}(e^{rc} - 1)(-2c)^{-1/2} + W^*_{s,c}(r)$ for $s = -3, ..., 0$. Consequently, $W_{2c}(r)$ in (13) is replaced with

$$\begin{aligned} K_{2c}(r) &= \left\{-K^*_{-3,c}(r) + K^*_{-2,c}(r) - K^*_{-1,c}(r) + K^*_{0,c}(r)\right\}/2 \\ &= \{(-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4)/2\}(e^{rc} - 1)(-2c)^{-1/2} + W_{2c}(r) \\ &= \bar{\alpha}_2(e^{rc} - 1)(-2c)^{-1/2} + W_{2c}(r) \end{aligned}$$

which completes the proof of the stated result for t_2^{QD} in Lemma 1.

The result for the t_2^{OLS} statistic follows in exactly the same way as for t_2^{QD}, replacing $W^*_{s,c}(r)$ with $W^{*\mu}_{s,c}(r) := W^*_{s,c}(r) - \int_0^1 W^*_{s,c}(t)dt$, $s = -3, ..., 0$, and, hence, $K^*_{s,c}(r)$ with $K^{*\mu}_{s,c}(r) := K^*_{s,c}(r) - \int_0^1 K^*_{s,c}(t)dt$, $s = -3, ..., 0$. The limit of t_2^{OLS} then has the same form as that for t_2^{QD} but with $K_{2c}(r)$ now replaced by $K^{\mu}_{2c}(r) := K_{2c}(r) - \int_0^1 K_{2c}(s)ds$.

Next consider the t_1^{OLS} statistic. When the initial conditions are asymptotically negligible, the limit distribution can be written as

$$t_1^{OLS} \xrightarrow{d} \frac{W^{\tau}_{1c}(1)^2 - W^{\tau}_{1c}(0)^2 - 1}{2\sqrt{\int_0^1 W^{\tau}_{1c}(r)^2 dr}} \tag{14}$$

where

$$W^{\tau}_{1c}(r) := \left\{W^{*\tau}_{-3,c}(r) + W^{*\tau}_{-2,c}(r) + W^{*\tau}_{-1,c}(r) + W^{*\tau}_{0,c}(r)\right\}/2$$

is a demeaned and detrended standard OU process, formed from the independent demeaned and detrended standard OU processes

$$W^{*\tau}_{s,c}(r) := W^*_{s,c}(r) - \int_0^1 W^*_{s,c}(t)dt - 12\left(r - \frac{1}{2}\right)\int_0^1 \left(t - \frac{1}{2}\right) W^*_{s,c}(t)dt, \quad s = -3, ..., 0.$$

When the initial conditions are governed by Assumption 1, as before we replace $W^*_{s,c}(r)$ with $K^*_{s,c}(r) := \alpha_{s+4}(e^{rc} - 1)(-2c)^{-1/2} + W^*_{s,c}(r)$ for $s = -3, ..., 0$, thus $W^{\tau}_{1c}(r)$ in (14) is replaced with

$$\begin{aligned} K_{1c}^{OLS}(r) &= \sum_{s=-3}^{0}\left\{K_{s,c}^{*}(r)-\int_0^1 K_{s,c}^{*}(t)dt-12\left(r-\frac{1}{2}\right)\int_0^1\left(t-\frac{1}{2}\right)K_{s,c}^{*}(t)dt\right\}/2 \\ &= \bar{\alpha}_1\left\{(e^{rc}-1)(-2c)^{-1/2}-\int_0^1(e^{sc}-1)(-2c)^{-1/2}ds\right. \\ &\quad \left.-12\left(r-\frac{1}{2}\right)\int_0^1\left(s-\frac{1}{2}\right)(e^{sc}-1)(-2c)^{-1/2}ds\right\}+W_{1c}^{\tau}(r) \\ &= K_{1c}^{\mu}(r)-12\left(r-\frac{1}{2}\right)\int_0^1\left(s-\frac{1}{2}\right)K_{1c}(s)ds \end{aligned}$$

which completes the proof of the result for t_1^{OLS} in Lemma 1.

The result for the t_1^{QD} statistic follows in exactly the same way as for t_1^{OLS}, replacing $W_{s,c}^{*\tau}(r)$ with $W_{s,c}^{*\tau,\bar{c}_1}(r) := W_{s,c}^{*}(r)-\bar{c}_1^{*}rW_{s,c}^{*}(1)-3(1-\bar{c}_1^{*})r\int_0^1 tW_{s,c}^{*}(t)dt$, $s=-3,...,0$, where $\bar{c}_1^{*}$ is as defined in Lemma 1. The limit of t_1^{QD} then has the same form as that for t_1^{OLS} but with $K_{1c}^{OLS}(r)$ now replaced by $K_{1c}^{QD}(r) := K_{1c}(r)-\bar{c}_1^{*}rK_{1c}(1)-3(1-\bar{c}_1^{*})r\int_0^1 sK_{1c}(s)ds$.

Consider next the result for the F_{34}^{QD} statistic. Drawing again on results established in Rodrigues and Taylor (2004a, 2004b, 2007), we can write the limit distribution in the asymptotically negligible initial conditions case as

$$F_{34}^{QD}\xrightarrow{d}\frac{1}{2}\left[(A^{*})^2+(B^{*})^2\right]$$

where

$$\begin{aligned} A^{*} &:= c\sqrt{\int_0^1 W_{3c}(r)^2dr+\int_0^1 W_{4c}(r)^2dr}+\frac{\int_0^1 W_{3c}(r)dW_{30}(r)+\int_0^1 W_{4c}(r)dW_{40}(r)}{\sqrt{\int_0^1 W_{3c}(r)^2dr+\int_0^1 W_{4c}(r)^2dr}} \\ B^{*} &:= \frac{\int_0^1 W_{3c}(r)dW_{40}(r)-\int_0^1 W_{4c}(r)dW_{30}(r)}{\sqrt{\int_0^1 W_{3c}(r)^2dr+\int_0^1 W_{4c}(r)^2dr}} \end{aligned}$$

with

$$W_{3c}(r) := \left\{-W_{-2,c}^{*}(r)+W_{0,c}^{*}(r)\right\}/\sqrt{2}$$

and

$$W_{4c}(r) := \left\{-W_{-3,c}^{*}(r)+W_{-1,c}^{*}(r)\right\}/\sqrt{2}$$

constituting a pair of mutually independent standard OU processes defined *via* the independent standard OU processes $W_{s,c}^{*}(r)$, $s=-3,...,0$. Now since $dW_{ic}(r) =$

$cW_{ic}(r)dr + dW_{i0}(r)$, $i = 3, 4$, we can alternatively write

$$\begin{aligned} A^* &= \frac{\int_0^1 W_{3c}(r)dW_{3c}(r) + \int_0^1 W_{4c}(r)dW_{4c}(r)}{\sqrt{\int_0^1 W_{3c}(r)^2 dr + \int_0^1 W_{4c}(r)^2 dr}} \\ B^* &= \frac{\int_0^1 W_{3c}(r)dW_{4c}(r) - \int_0^1 W_{4c}(r)dW_{3c}(r)}{\sqrt{\int_0^1 W_{3c}(r)^2 dr + \int_0^1 W_{4c}(r)^2 dr}}. \end{aligned}$$

Introducing initial conditions of the form given in Assumption 1, we again need to replace $W^*_{s,c}(r)$ with $K^*_{s,c}(r)$, $s = -3, ..., 0$. The limit processes $W_{3c}(r)$ and $W_{4c}(r)$ are then correspondingly replaced with

$$\begin{aligned} K_{3c}(r) &= \left\{-K^*_{-2,c}(r) + K^*_{0,c}(r)\right\}/\sqrt{2} \\ &= \{(-\alpha_2 + \alpha_4)/\sqrt{2}\}(e^{rc} - 1)(-2c)^{-1/2} + W_{3c}(r) \\ &= \bar{\alpha}_3(e^{rc} - 1)(-2c)^{-1/2} + W_{3c}(r) \end{aligned}$$

and

$$\begin{aligned} K_{4c}(r) &= \left\{-K^*_{-3,c}(r) + K^*_{-1,c}(r)\right\}/\sqrt{2} \\ &= \{(-\alpha_1 + \alpha_3)/\sqrt{2}\}(e^{rc} - 1)(-2c)^{-1/2} + W_{4c}(r) \\ &= \bar{\alpha}_4(e^{rc} - 1)(-2c)^{-1/2} + W_{4c}(r) \end{aligned}$$

respectively. Consequently,

$$F_{34}^{QD} \xrightarrow{d} \frac{1}{2}\left[(A^{QD})^2 + (B^{QD})^2\right]$$

where

$$\begin{aligned} A^{QD} &= \frac{\int_0^1 K_{3c}(r)dK_{3c}(r) + \int_0^1 K_{4c}(r)dK_{4c}(r)}{\sqrt{\int_0^1 K_{3c}(r)^2 dr + \int_0^1 K_{4c}(r)^2 dr}} \\ B^{QD} &= \frac{\int_0^1 K_{3c}(r)dK_{4c}(r) - \int_0^1 K_{4c}(r)dK_{3c}(r)}{\sqrt{\int_0^1 K_{3c}(r)^2 dr + \int_0^1 K_{4c}(r)^2 dr}}. \end{aligned}$$

Now, it is straightforward to show that

$$dK_{ic}(r) = cK_{ic}(r)dr + \bar{\alpha}_i c(-2c)^{-1/2}dr + dW_i(r), \quad i = 3, 4$$

and so we obtain on substitution that

$$
\begin{aligned}
A^{QD} &= c\sqrt{\int_0^1 K_{3c}(r)^2 dr + \int_0^1 K_{4c}(r)^2 dr} \\
&\quad + \frac{\bar{\alpha}_3 c(-2c)^{-1/2} \int_0^1 K_{3c}(r)dr + \bar{\alpha}_4 c(-2c)^{-1/2} \int_0^1 K_{4c}(r)dr}{\sqrt{\int_0^1 K_{3c}(r)^2 dr + \int_0^1 K_{4c}(r)^2 dr}} \\
&\quad + \frac{\int_0^1 K_{3c}(r)dW_3(r) + \int_0^1 K_{4c}(r)dW_4(r)}{\sqrt{\int_0^1 K_{3c}(r)^2 dr + \int_0^1 K_{4c}(r)^2 dr}} \\
B^{QD} &= \frac{\bar{\alpha}_4 c(-2c)^{-1/2} \int_0^1 K_{3c}(r)dr - \bar{\alpha}_3 c(-2c)^{-1/2} \int_0^1 K_{4c}(r)dr}{\sqrt{\int_0^1 K_{3c}(r)^2 dr + \int_0^1 K_{4c}(r)^2 dr}} \\
&\quad + \frac{\int_0^1 K_{3c}(r)dW_4(r) - \int_0^1 K_{4c}(r)dW_3(r)}{\sqrt{\int_0^1 K_{3c}(r)^2 dr + \int_0^1 K_{4c}(r)^2 dr}}
\end{aligned}
$$

which completes the result.

The result for F_{34}^{OLS} again follows in exactly the same way as for F_{34}^{QD}, replacing $W_{s,c}^*(r)$ with $W_{s,c}^{*\mu}(r)$, $s = -3, ..., 0$, and $K_{s,c}^*(r)$ with $K_{s,c}^{*\mu}(r)$, $s = -3, ..., 0$. The limit of F_{34}^{OLS} is then obtained as that for F_{34}^{QD} above, but with $K_{ic}(r)$ replaced by $K_{ic}^{\mu}(r) := K_{ic}(r) - \int_0^1 K_{ic}(s)ds$, $i = 3, 4$. Note that $\int_0^1 K_{ic}^{\mu}(r)dr = 0$, $i = 3, 4$, so that the expression simplifies to the form given in Lemma 1.

The stated representations for the F_{234}^{OLS}, F_{234}^{QD}, F_{1234}^{OLS} and F_{1234}^{QD} statistics then follow immediately from the representations given above, noting the asymptotic orthogonality of the HEGY regressors $\hat{x}_{1,4t+s-1}$, $\hat{x}_{2,4t+s-1}$, $\hat{x}_{3,4t+s-1}$ and $\hat{x}_{4,4t+s-1}$ from (8) under H_c of (6) for both OLS and QD detrended data; see, Rodrigues and Taylor (2004b,2007).

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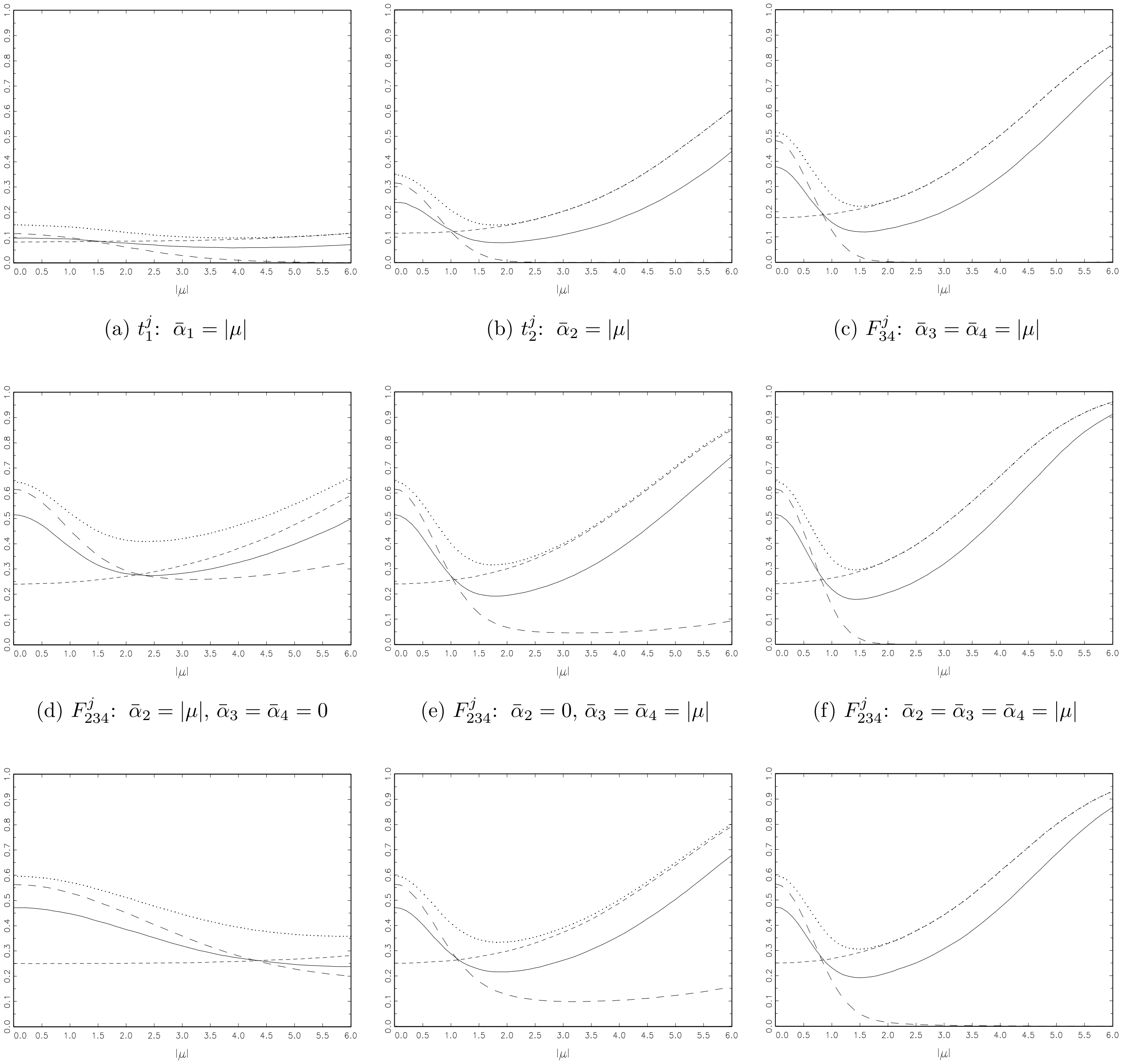
T.1

Table 1. Critical values, asymptotic UR sizes, and λ_ζ values for ζ-level seasonal unit root tests.

	t_1			t_2			F_{34}			F_{234}			F_{1234}		
T	$\zeta = 0.10$	$\zeta = 0.05$	$\zeta = 0.01$	$\zeta = 0.10$	$\zeta = 0.05$	$\zeta = 0.01$	$\zeta = 0.10$	$\zeta = 0.05$	$\zeta = 0.01$	$\zeta = 0.10$	$\zeta = 0.05$	$\zeta = 0.01$	$\zeta = 0.10$	$\zeta = 0.05$	$\zeta = 0.01$
Panel A. Critical values for OLS detrended tests															
52	−3.18	−3.49	−4.12	−2.63	−2.96	−3.62	6.01	7.23	9.97	5.75	6.79	9.12	6.41	7.40	9.61
100	−3.15	−3.45	−4.04	−2.61	−2.92	−3.53	5.82	6.92	9.33	5.42	6.33	8.30	5.95	6.77	8.54
152	−3.15	−3.44	−4.00	−2.59	−2.90	−3.52	5.71	6.73	8.97	5.30	6.14	7.90	5.80	6.53	8.18
300	−3.14	−3.43	−3.99	−2.58	−2.87	−3.48	5.68	6.71	9.00	5.22	6.04	7.84	5.68	6.41	7.90
∞	−3.13	−3.42	−3.96	−2.57	−2.86	−3.44	5.62	6.62	8.78	5.13	5.87	7.52	5.52	6.19	7.61
Panel B. Critical values for QD detrended tests															
52	−3.07	−3.37	−4.00	−2.34	−2.64	−3.27	3.69	4.53	6.50	3.84	4.57	6.23	4.79	5.51	7.21
100	−2.91	−3.19	−3.75	−2.13	−2.41	−3.02	3.14	3.92	5.72	3.14	3.77	5.24	3.96	4.56	5.92
152	−2.83	−3.11	−3.67	−2.01	−2.31	−2.90	2.91	3.66	5.38	2.82	3.42	4.75	3.62	4.18	5.38
300	−2.72	−3.01	−3.59	−1.86	−2.16	−2.75	2.66	3.38	5.07	2.50	3.08	4.34	3.25	3.77	4.91
∞	−2.56	−2.85	−3.41	−1.62	−1.94	−2.56	2.39	3.07	4.70	2.20	2.74	3.89	2.81	3.32	4.35
Panel C. Sizes of UR tests															
∞	0.155	0.080	0.017	0.173	0.089	0.018	0.178	0.092	0.019	0.177	0.091	0.019	0.170	0.088	0.019
Panel D. λ_ζ values for UR tests*															
∞	1.070	1.058	1.043	1.126	1.095	1.065	1.197	1.163	1.118	1.163	1.131	1.101	1.118	1.100	1.075

Table 2. Empirical sizes of nominal 0.05-level seasonal unit root tests.

				t_1^j				t_2^j				F_{34}^j				F_{234}^j				F_{1234}^j			
T	ϕ	θ	j:	*OLS*	*GLS*	*UR*	*UR**	*OLS*	*GLS*	*UR*	*UR**	*OLS*	*GLS*	*UR*	*UR**	*OLS*	*GLS*	*UR*	*UR**	*OLS*	*GLS*	*UR*	*UR**
152	0.0	−0.4		0.126	0.110	0.159	0.118	0.093	0.090	0.135	0.090	0.100	0.073	0.144	0.092	0.117	0.093	0.166	0.115	0.152	0.123	0.204	0.150
		0.0		0.060	0.066	0.087	0.057	0.049	0.053	0.080	0.046	0.054	0.056	0.097	0.054	0.052	0.057	0.092	0.054	0.056	0.063	0.097	0.057
		0.4		0.065	0.071	0.092	0.061	0.052	0.057	0.084	0.050	0.053	0.057	0.095	0.053	0.054	0.059	0.095	0.056	0.061	0.068	0.103	0.063
	0.3	−0.4		0.112	0.102	0.145	0.107	0.102	0.100	0.147	0.099	0.105	0.078	0.151	0.096	0.127	0.103	0.179	0.125	0.155	0.125	0.207	0.154
		0.0		0.058	0.063	0.084	0.054	0.050	0.051	0.079	0.045	0.053	0.057	0.096	0.053	0.052	0.057	0.092	0.054	0.057	0.062	0.096	0.056
		0.4		0.075	0.082	0.106	0.072	0.047	0.052	0.076	0.045	0.052	0.057	0.094	0.053	0.050	0.057	0.091	0.052	0.064	0.073	0.108	0.067
	0.6	−0.4		0.092	0.090	0.125	0.088	0.102	0.097	0.145	0.099	0.107	0.078	0.152	0.100	0.134	0.104	0.183	0.130	0.155	0.122	0.205	0.151
		0.0		0.058	0.064	0.085	0.055	0.047	0.050	0.076	0.043	0.052	0.056	0.094	0.053	0.050	0.055	0.089	0.053	0.055	0.061	0.094	0.056
		0.4		0.063	0.070	0.090	0.061	0.047	0.054	0.077	0.045	0.051	0.055	0.092	0.052	0.050	0.056	0.090	0.051	0.057	0.066	0.098	0.061
300	0.0	−0.4		0.090	0.086	0.124	0.085	0.070	0.073	0.113	0.070	0.067	0.059	0.110	0.062	0.076	0.068	0.121	0.074	0.097	0.090	0.147	0.097
		0.0		0.057	0.060	0.084	0.056	0.049	0.053	0.084	0.049	0.051	0.055	0.095	0.053	0.050	0.053	0.091	0.052	0.053	0.059	0.093	0.056
		0.4		0.056	0.060	0.083	0.054	0.049	0.053	0.084	0.048	0.049	0.055	0.093	0.051	0.050	0.052	0.090	0.051	0.054	0.060	0.094	0.056
	0.3	−0.4		0.080	0.079	0.112	0.076	0.076	0.078	0.120	0.076	0.070	0.060	0.113	0.065	0.083	0.073	0.130	0.081	0.097	0.089	0.147	0.097
		0.0		0.057	0.059	0.084	0.054	0.048	0.051	0.082	0.048	0.050	0.054	0.094	0.052	0.049	0.052	0.089	0.051	0.053	0.059	0.093	0.055
		0.4		0.058	0.061	0.084	0.056	0.049	0.052	0.083	0.048	0.050	0.054	0.093	0.051	0.049	0.052	0.089	0.051	0.054	0.060	0.095	0.058
	0.6	−0.4		0.071	0.073	0.102	0.070	0.074	0.078	0.119	0.076	0.069	0.060	0.112	0.066	0.082	0.071	0.128	0.081	0.093	0.085	0.141	0.094
		0.0		0.056	0.060	0.084	0.055	0.048	0.051	0.081	0.048	0.050	0.054	0.093	0.051	0.049	0.052	0.089	0.051	0.052	0.059	0.092	0.055
		0.4		0.052	0.056	0.078	0.051	0.049	0.054	0.084	0.049	0.049	0.055	0.093	0.051	0.049	0.053	0.089	0.051	0.051	0.057	0.091	0.053



(a) t_1^j: $\bar{\alpha}_1 = |\mu|$

(b) t_2^j: $\bar{\alpha}_2 = |\mu|$

(c) F_{34}^j: $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

(d) F_{234}^j: $\bar{\alpha}_2 = |\mu|$, $\bar{\alpha}_3 = \bar{\alpha}_4 = 0$

(e) F_{234}^j: $\bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

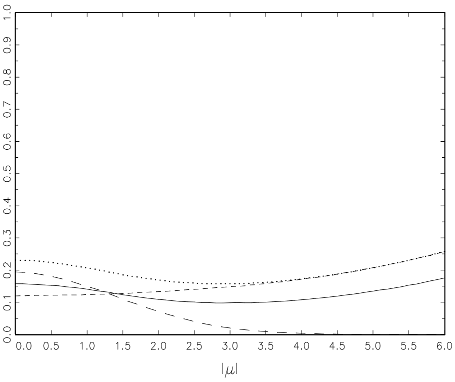
(f) F_{234}^j: $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

(g) F_{1234}^j: $\bar{\alpha}_1 = |\mu|$, $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = 0$

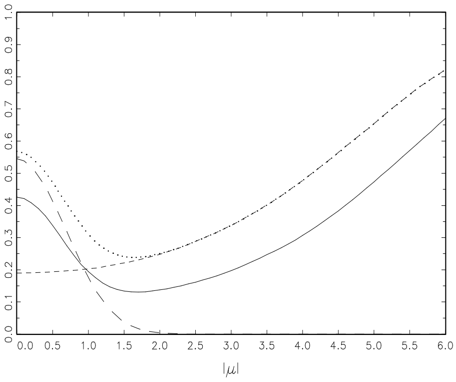
(h) F_{1234}^j: $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

(i) F_{1234}^j: $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

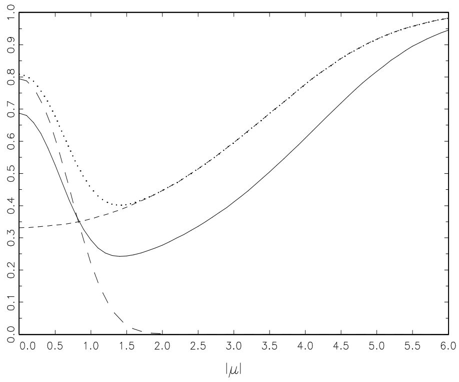
Figure 1. Asymptotic local power: $c = -5$; $j = OLS$: - - - , $j = GLS$: – – , $j = UR$: $\cdots$, $j = UR^*$: —



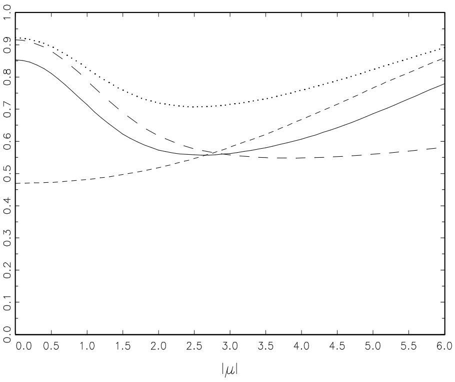
(a) t_1^j: $\bar{\alpha}_1 = |\mu|$



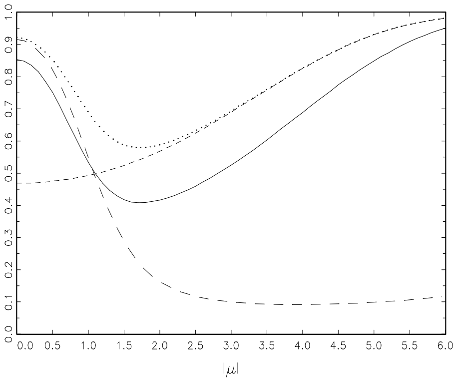
(b) t_2^j: $\bar{\alpha}_2 = |\mu|$



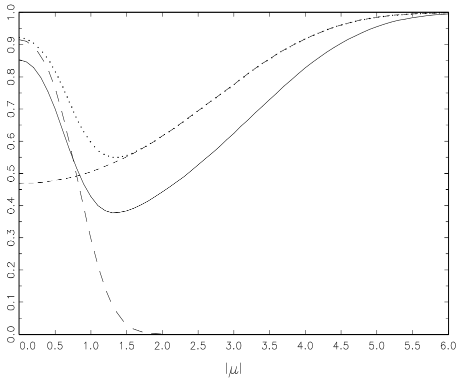
(c) F_{34}^j: $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



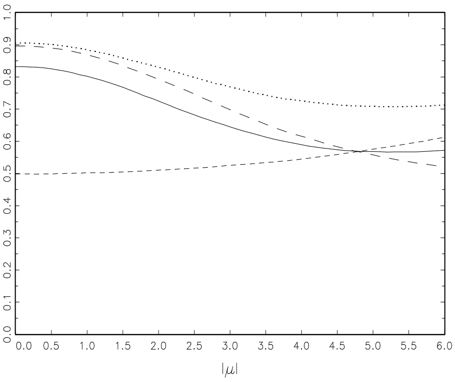
(d) F_{234}^j: $\bar{\alpha}_2 = |\mu|$, $\bar{\alpha}_3 = \bar{\alpha}_4 = 0$



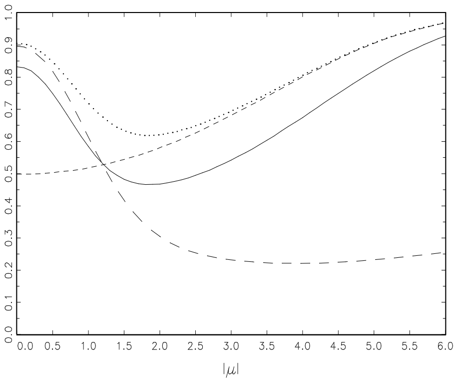
(e) F_{234}^j: $\bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



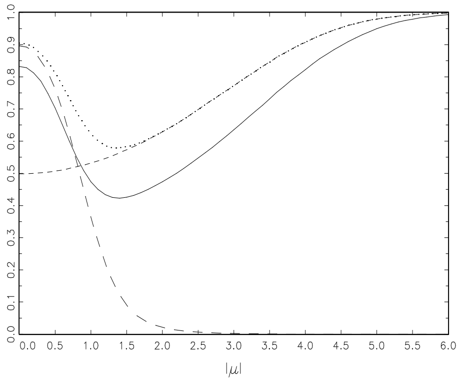
(f) F_{234}^j: $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



(g) F_{1234}^j: $\bar{\alpha}_1 = |\mu|$, $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = 0$

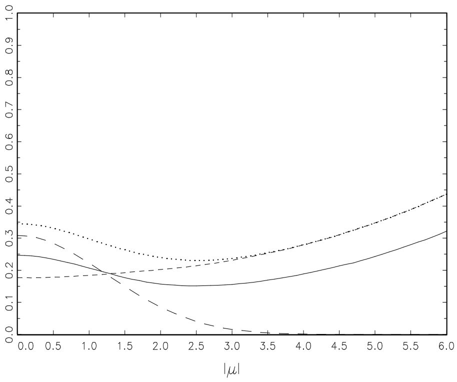


(h) F_{1234}^j: $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

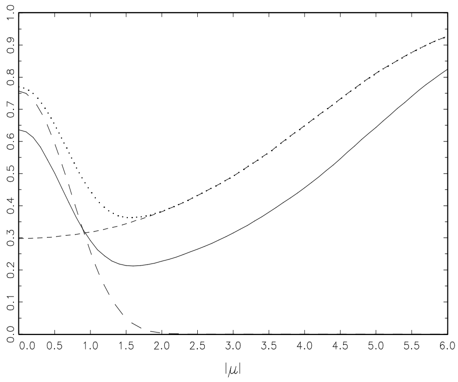


(i) F_{1234}^j: $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

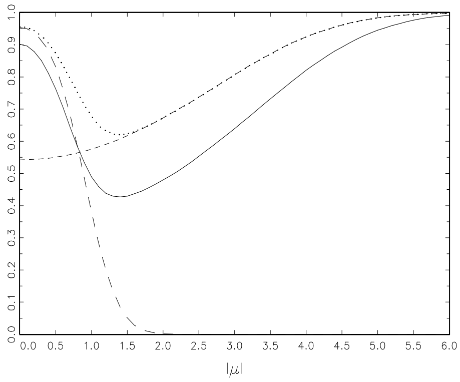
Figure 2. Asymptotic local power: $c = -7.5$; $j = OLS$: - - - , $j = GLS$: – – , $j = UR$: $\cdots$, $j = UR^*$: ——



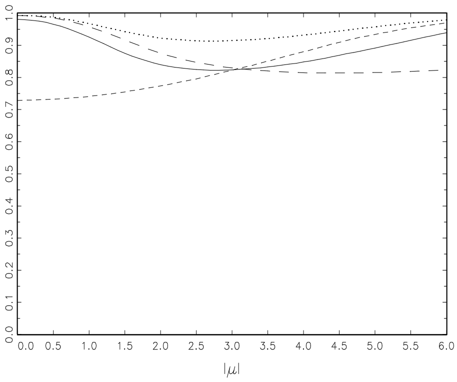
(a) t_1^j: $\bar{\alpha}_1 = |\mu|$



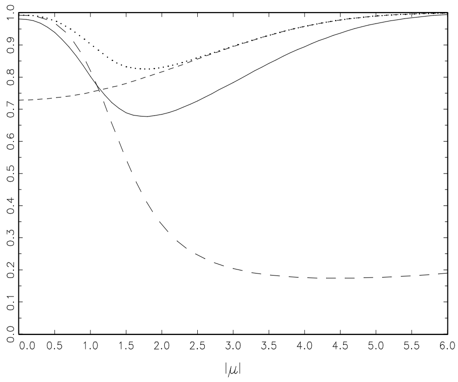
(b) t_2^j: $\bar{\alpha}_2 = |\mu|$



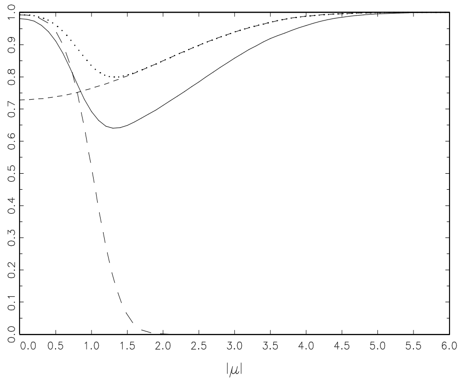
(c) F_{34}^j: $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



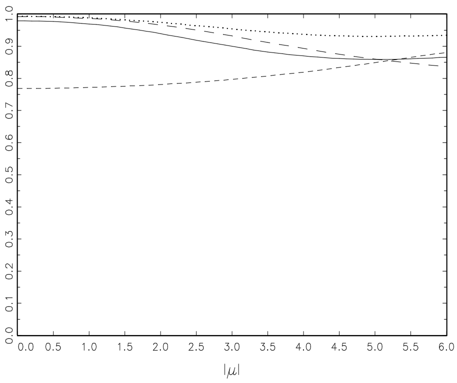
(d) F_{234}^j: $\bar{\alpha}_2 = |\mu|$, $\bar{\alpha}_3 = \bar{\alpha}_4 = 0$



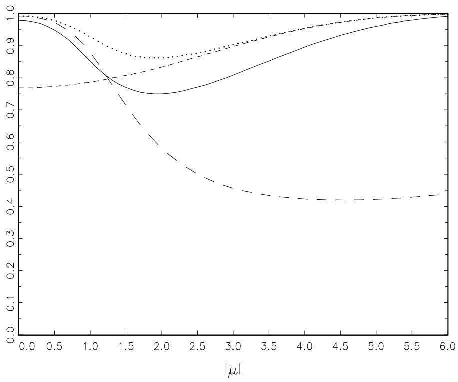
(e) F_{234}^j: $\bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



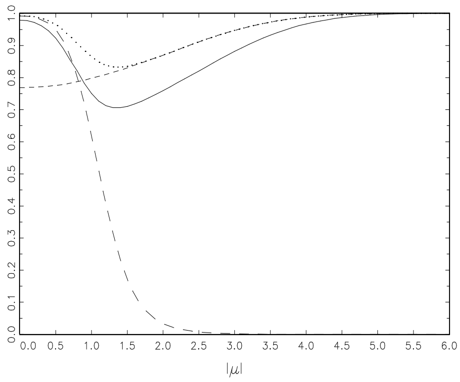
(f) F_{234}^j: $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$



(g) F_{1234}^j: $\bar{\alpha}_1 = |\mu|$, $\bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = 0$

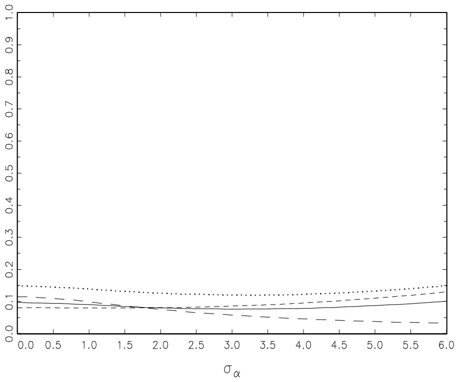


(h) F_{1234}^j: $\bar{\alpha}_1 = \bar{\alpha}_2 = 0$, $\bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

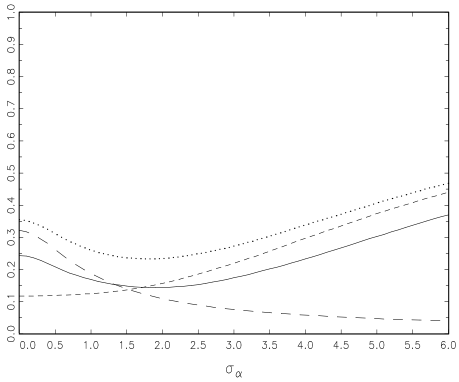


(i) F_{1234}^j: $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\alpha}_4 = |\mu|$

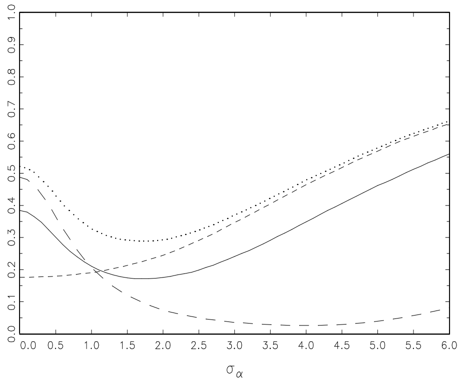
Figure 3. Asymptotic local power: $c = -10$; $j = OLS$: - - - , $j = GLS$: $--$, $j = UR$: $\cdots$, $j = UR^*$: ——



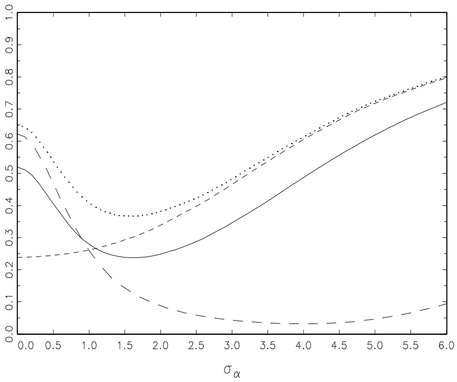
(a) t_1^j



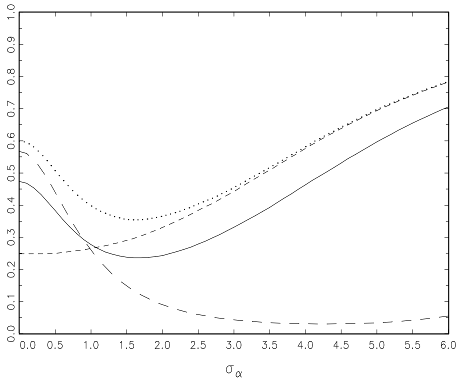
(b) t_2^j



(c) F_{34}^j

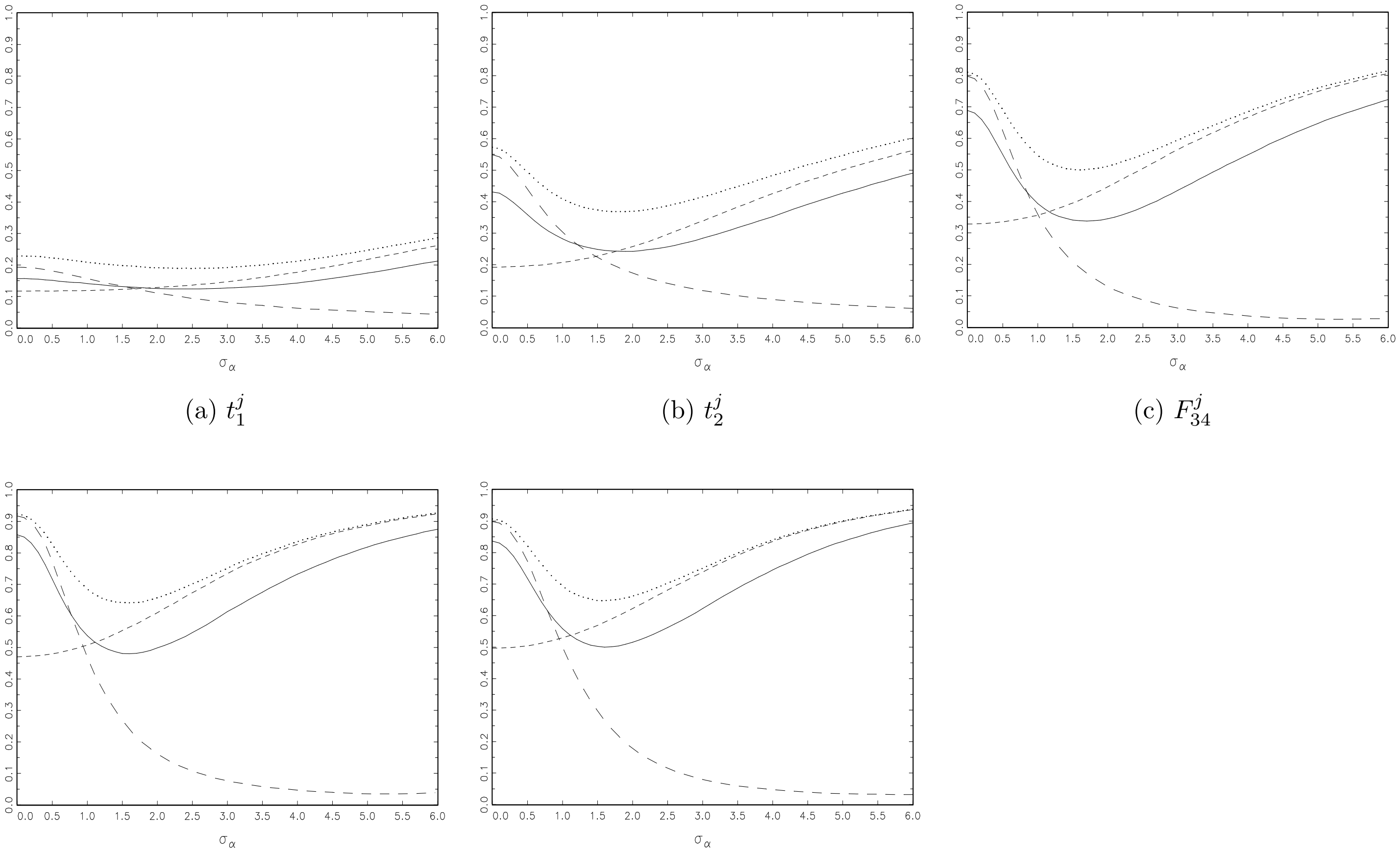


(d) F_{234}^j



(e) F_{1234}^j

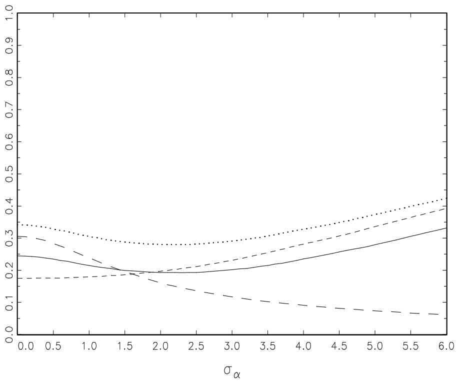
Figure 4. Asymptotic local power: $c = -5$, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$;
$j = OLS$: - - - , $j = GLS$: $--$, $j = UR$: $\cdots$, $j = UR^*$: —



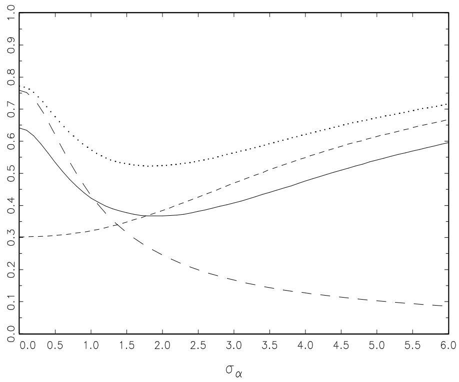
(a) t_1^j (b) t_2^j (c) F_{34}^j

(d) F_{234}^j (e) F_{1234}^j

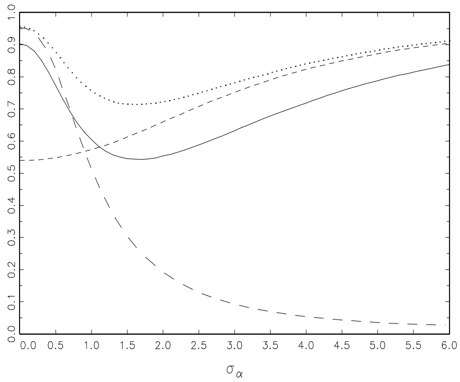
Figure 5. Asymptotic local power: $c = -7.5$, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$;
$j = OLS$: - - - , $j = GLS$: $--$, $j = UR$: $\cdots$, $j = UR^*$: ——



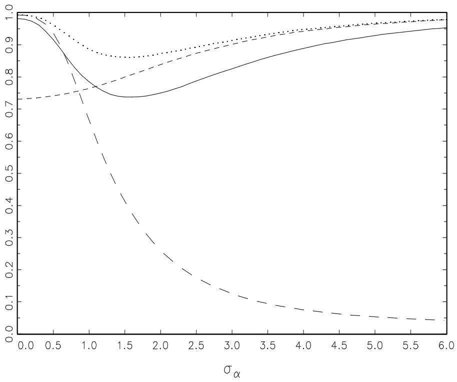
(a) t_1^j



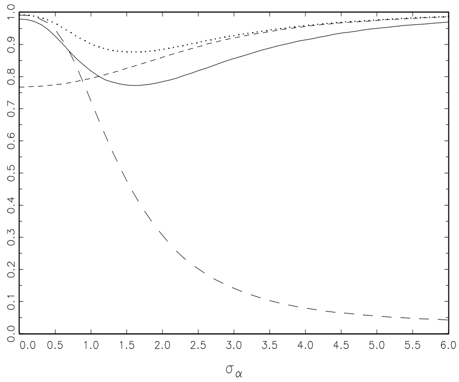
(b) t_2^j



(c) F_{34}^j

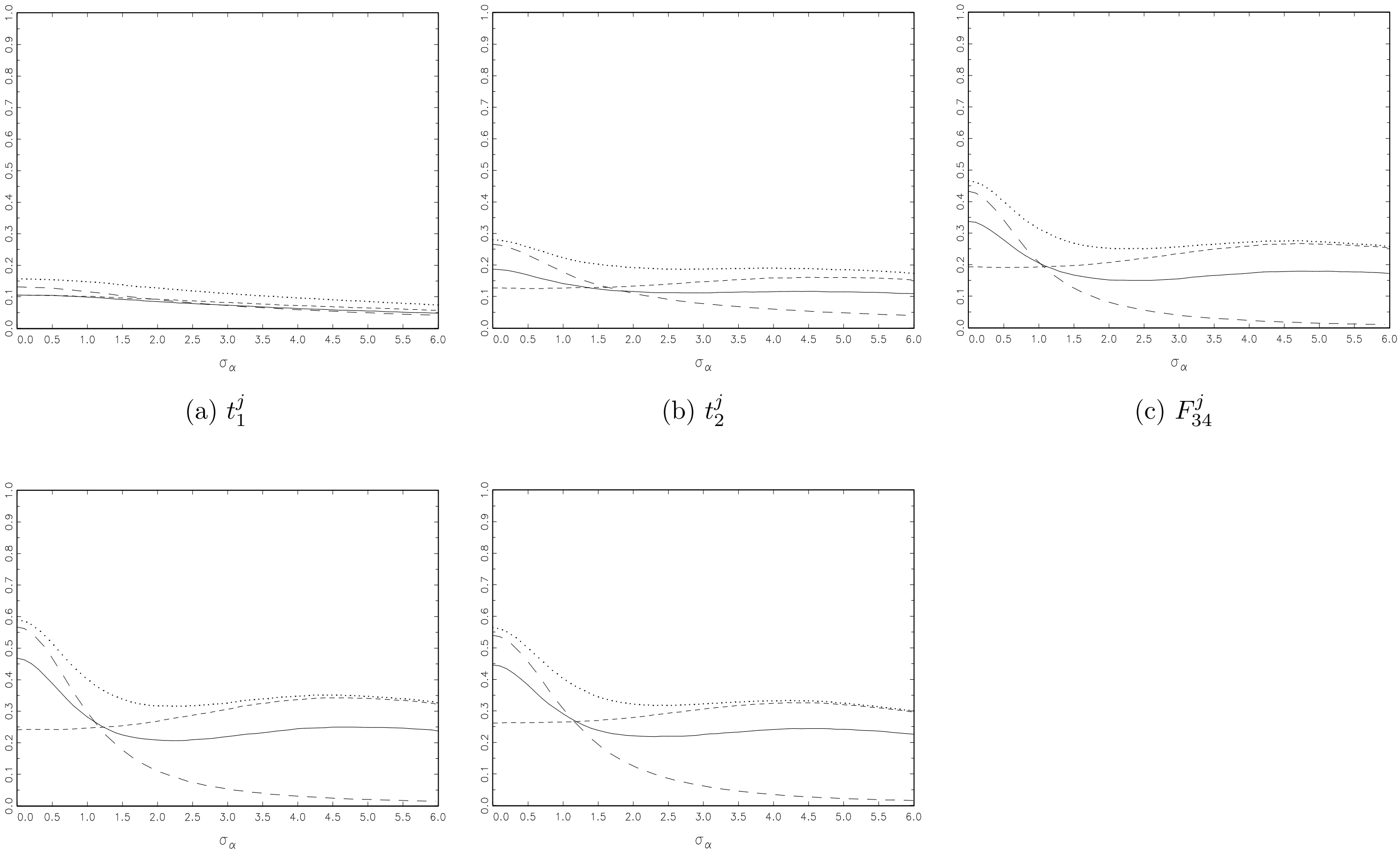


(d) F_{234}^j



(e) F_{1234}^j

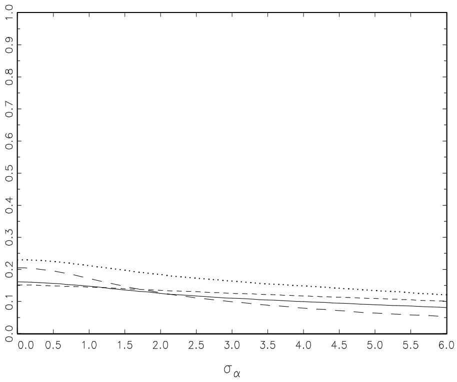
Figure 6. Asymptotic local power: $c = -10$, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$;
$j = OLS$: - - - , $j = GLS$: $--$, $j = UR$: $\cdots$, $j = UR^*$: ——



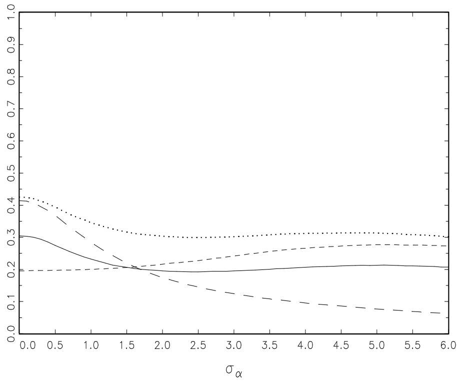
(a) t_1^j (b) t_2^j (c) F_{34}^j

(d) F_{234}^j (e) F_{1234}^j

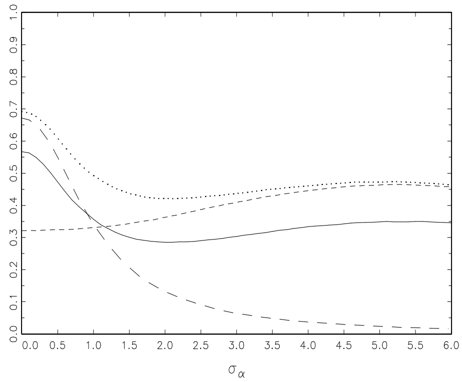
Figure 7. Finite sample power: $T = 152$, $c = -5$, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$;
$j = OLS$: - - - , $j = GLS$: $--$, $j = UR$: $\cdots$, $j = UR^*$: —



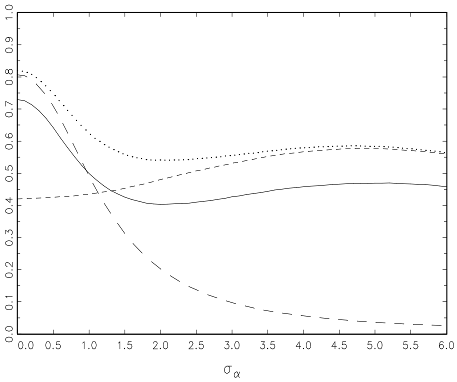
(a) t_1^j



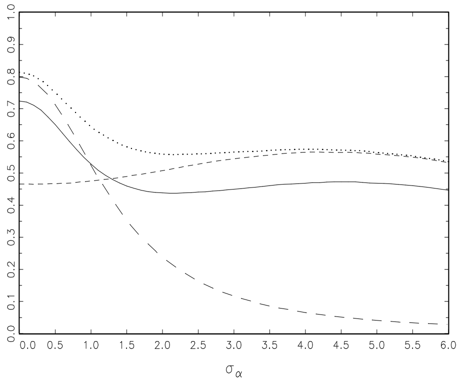
(b) t_2^j



(c) F_{34}^j

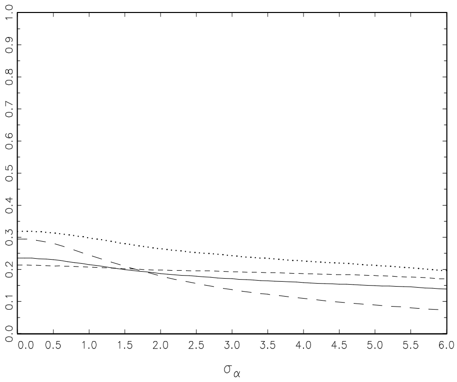


(d) F_{234}^j

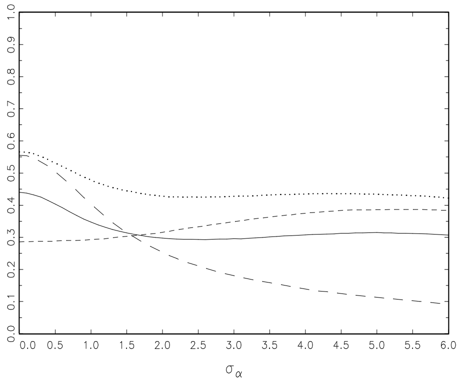


(e) F_{1234}^j

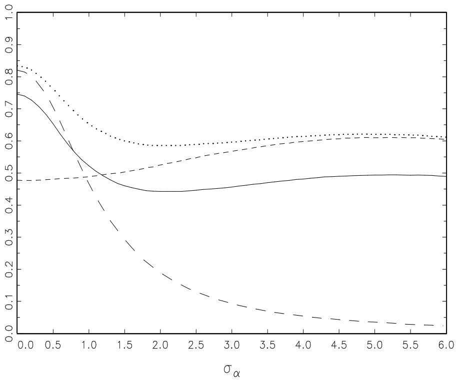
Figure 8. Finite sample power: $T = 152$, $c = -7.5$, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$;
$j = OLS$: - - - , $j = GLS$: $--$, $j = UR$: $\cdots$, $j = UR^*$: —



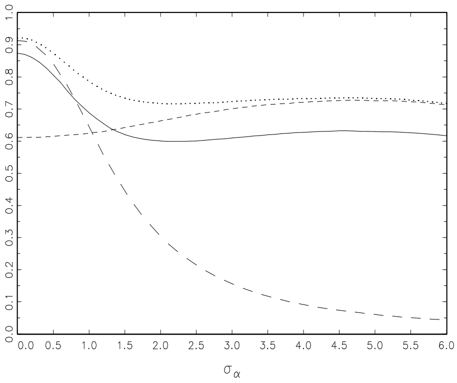
(a) t_1^j



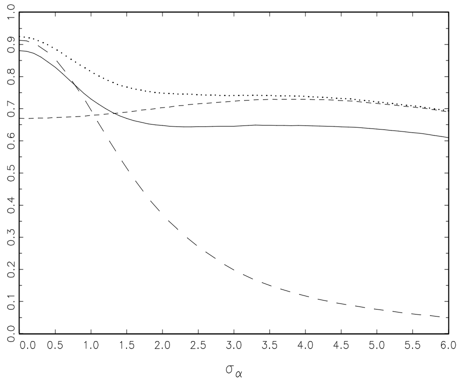
(b) t_2^j



(c) F_{34}^j

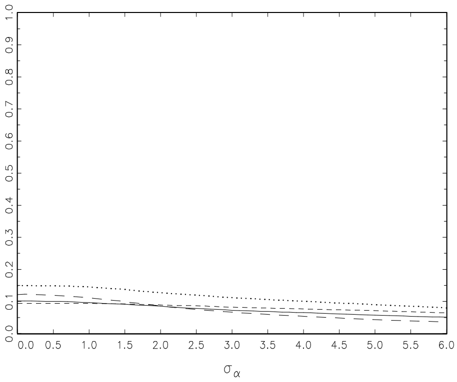


(d) F_{234}^j

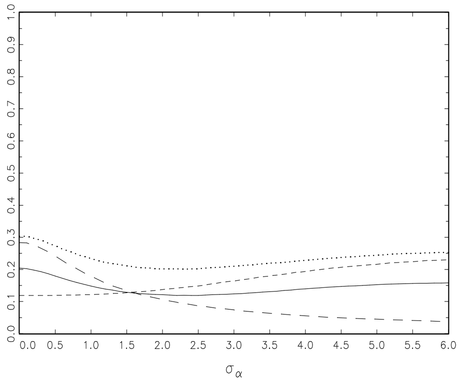


(e) F_{1234}^j

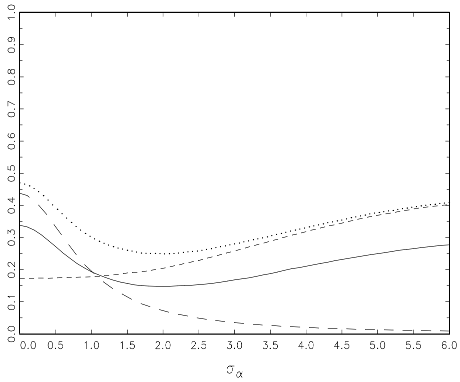
Figure 9. Finite sample power: $T = 152$, $c = -10$, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$;
$j = OLS$: - - - , $j = GLS$: $--$, $j = UR$: $\cdots$, $j = UR^*$: ——



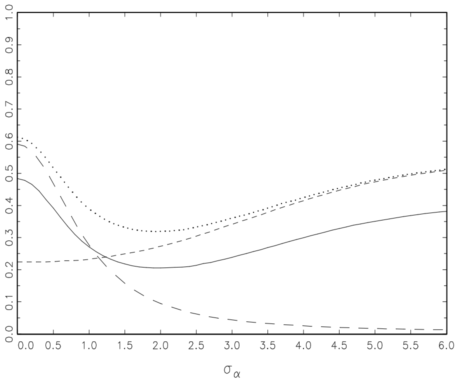
(a) t_1^j



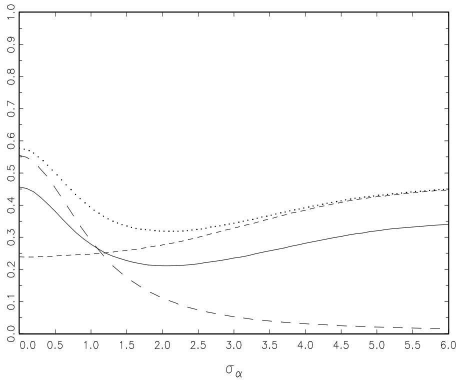
(b) t_2^j



(c) F_{34}^j

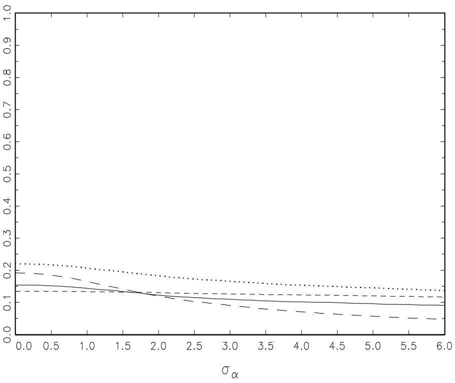


(d) F_{234}^j

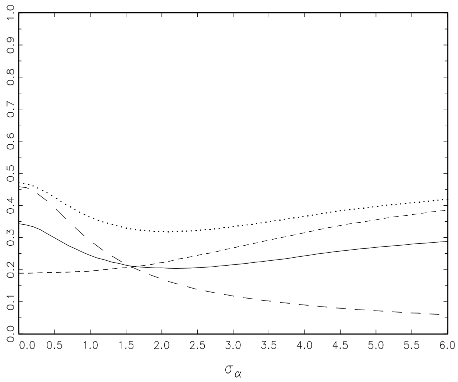


(e) F_{1234}^j

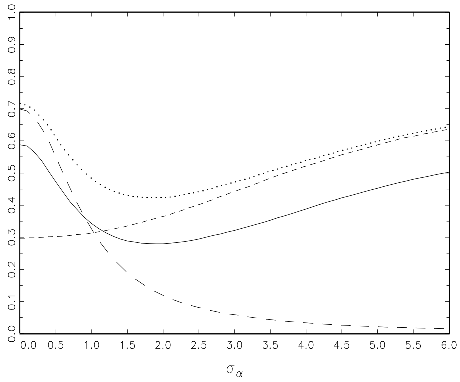
Figure 10. Finite sample power: $T = 300$, $c = -5$, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$;
$j = OLS$: - - - , $j = GLS$: $--$, $j = UR$: $\cdots$, $j = UR^*$: —



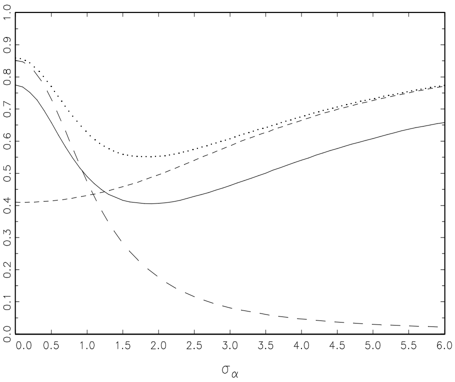
(a) t_1^j



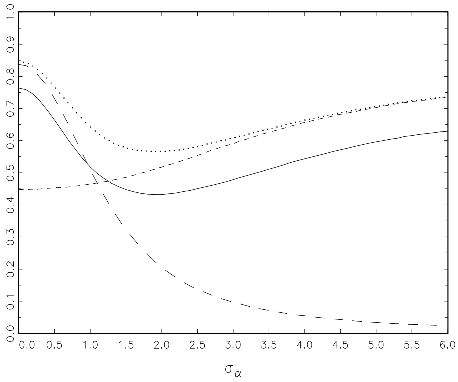
(b) t_2^j



(c) F_{34}^j

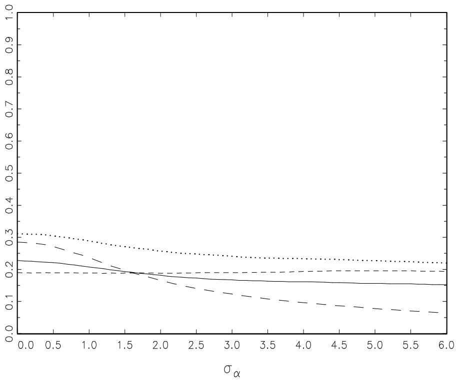


(d) F_{234}^j

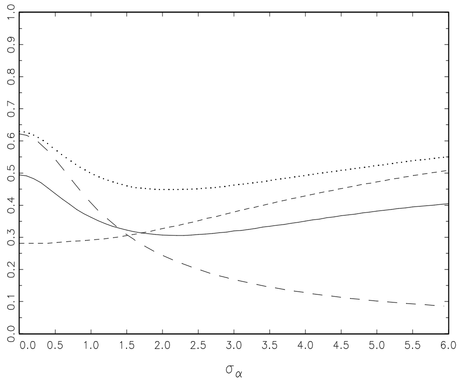


(e) F_{1234}^j

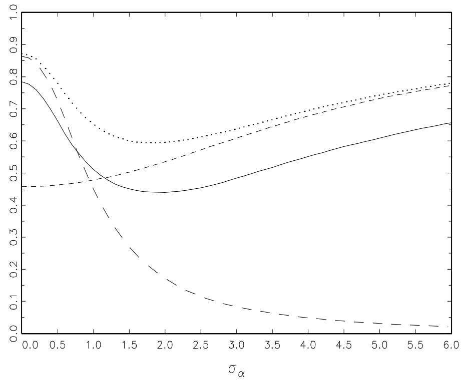
Figure 11. Finite sample power: $T = 300$, $c = -7.5$, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$;
$j = OLS$: - - - , $j = GLS$: $--$, $j = UR$: $\cdots$, $j = UR^*$: ——



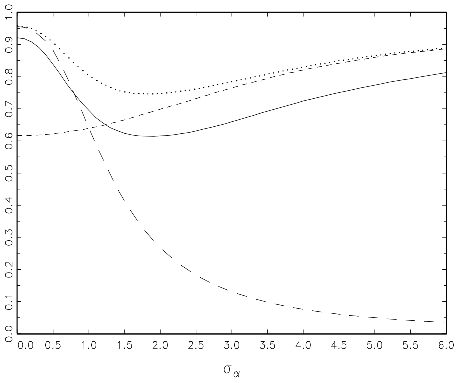
(a) t_1^j



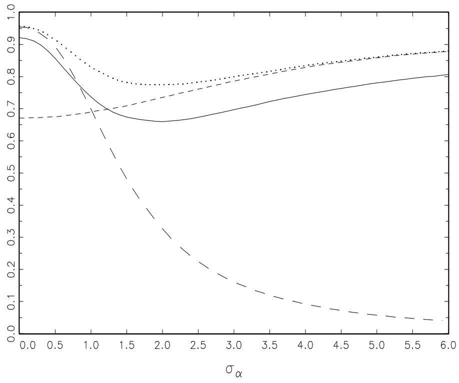
(b) t_2^j



(c) F_{34}^j



(d) F_{234}^j



(e) F_{1234}^j

Figure 12. Finite sample power: $T = 300$, $c = -10$, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $i = 1, 2, 3, 4$;
$j = OLS$: - - - , $j = GLS$: $--$, $j = UR$: $\cdots$, $j = UR^*$: —