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Local asymptotic power of the Im-Pesaran-Shin panel unit root test and the impact of initial observations

by

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Local Asymptotic Power of the Im-Pesaran-Shin Panel Unit Root Test and the Impact of Initial Observations[*]

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Abstract

In this note we derive the local asymptotic power function of the standardized averaged Dickey-Fuller panel unit root statistic of Im, Pesaran and Shin (2003, *Journal of Econometrics*, 115, 53-74), allowing for heterogeneous deterministic intercept terms. We consider the situation where the deviation of the initial observation from the underlying intercept term in each individual time series may not be asymptotically negligible. We find that power decreases monotonically as the absolute values of the initial conditions increase in magnitude, in direct contrast to the univariate case. Finite sample simulations confirm the relevance of this result for practical applications, demonstrating that the power of the test can be very low for values of T and N typically encountered in practice.

1 Introduction

In this note we consider the large sample behaviour of the standardized averaged Dickey-Fuller (DF) unit root test of Im, Pesaran and Shin (2003) (IPS) for panels allowing heterogeneous deterministic intercept terms. We derive the local asymptotic power function for this statistic where the time series dimension $T \to \infty$ followed by the cross-sectional dimension $N \to \infty$. Allowance is made for the fact that the deviation of the initial observation from the underlying intercept term (referred to as the initial condition) in each individual time series may not be asymptotically negligible, thereby generalizing the univariate model of Müller and Elliott (2003) to the panel environment. We find that the local asymptotic power function of the IPS statistic is a monotonically decreasing function of the magnitude of the absolute values of the initial conditions. Moreover, local asymptotic power falls below the nominal size of the IPS test for plausible values of the absolute initial conditions. This behaviour is in direct contrast to the univariate case, where Müller and Elliott (2003) demonstrate that the local asymptotic power of the DF statistic is an increasing function of the absolute initial condition. To show that our large sample results are of more than just theoretical interest, we supplement our asymptotic study with a finite sample analysis. This clearly demonstrates that the IPS test can also have very low power in situations where T and N assume the sort of values typically encountered in practice. Since applied researchers are unable to stipulate the nature of the initial conditions they face, they should be fully aware of the potential for poor power performance of the IPS test in such circumstances.

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As a by-product of our analysis, we also show that when the initial conditions are asymptotically negligible, the representation of the limit distribution of the IPS test as stated in Breitung and Pesaran (2008) is actually incorrect, since it omits important contributions from a number of non-negligible expectation terms.

2 The Panel Model and IPS Statistic

Consider the following data generating process for N cross-sectional series y_{it}, $i = 1, \ldots, N$, observed over $t = 1, \ldots, T$ time periods

$$
\begin{aligned}
y_{it} &= \mu_i + w_{it} \\
w_{it} &= \rho_i w_{i,t-1} + v_{it}, \quad t = 2, \ldots, T \\
w_{i1} &= \xi_i
\end{aligned}
\tag{1}
$$

where the innovations v_{it} are assumed to satisfy the following assumption

Assumption 1. *The v_{it} are i.i.d.($0, \sigma_{v,i}^2$) across $t = 1, ..., T$ and are independently distributed across $i = 1, ..., N$.*

We consider the case where the initial conditions ξ_i are governed by

Assumption 2. *Let the initial conditions be $\xi_i = \alpha \sigma_{w,i}$, where $\sigma_{w,i}^2$ denotes the short run variance of w_{it} for $\rho_i < 1$, i.e. $\xi_i = \alpha \sqrt{\frac{\sigma_{v,i}^2}{1-\rho_i^2}}$.*

This initial value specification implies that each initial condition ξ_i is proportional to the standard deviation of the corresponding process w_{it}. For tractability in the analysis, we assume that the constant of proportionality α is common across $i = 1, ..., N$, and we treat α as a fixed parameter.

The null and alternative hypotheses for the panel unit root testing problem are

$$
\begin{aligned}
H_0 &: \quad \rho_i = 1 \quad \text{for all } i \\
H_1 &: \quad \rho_i < 1 \quad \text{for at least one } i.
\end{aligned}
$$

Denoting the standard DF statistic that allows for an intercept by t_i^μ for series i, the IPS test statistic is given by

$$
Z^\mu = \frac{N^{1/2}\{\bar{t}^\mu - E(t_0^\mu)\}}{\sqrt{V(t_0^\mu)}}
$$

where $\bar{t}^\mu = N^{-1} \sum_{i=1}^{N} t_i^\mu$ and where $E(t_0^\mu)$ and $V(t_0^\mu)$ denote the mean and variance, respectively, of t_i^μ under the null hypothesis. Under Assumption 1 these moments do not depend on i and hence the subscript is omitted. We use this convention for all expectation terms thoughout the paper.

3 Asymptotic Local Power of the IPS Statistic

We specify the local alternative hypothesis by letting ρ_i be governed by the following assumption

Assumption 3. *Let $\rho_i = 1 + \frac{c_i}{T\sqrt{N}}$ for $c_i < 0$, $i = 1, ..., N$*

noting that the null hypothesis holds for $c_i = 0 \; \forall i$.

Remark 1. Under Assumptions 2–3, the order of the initial conditions is given by $\xi_i = O(N^{1/4}T^{1/2})$.

We consider sequential asymptotic theory, where $T \to \infty$ followed by $N \to \infty$. All proofs are given in the Appendix. The following lemma gives the distribution of the IPS statistic as $T \to \infty$

Lemma 1. *Under Assumptions 1–3*

$$
\sqrt{V(t_0^\mu)}Z^\mu \stackrel{T \to \infty}{\Rightarrow} N^{1/2} \left\{ N^{-1} \sum_{i=1}^{N} \frac{A_{1i}}{\sqrt{B_i}} - E\left(\tau^\mu\right) \right\}
$$

$$
+ N^{-1} \sum_{i=1}^{N} c_i \sqrt{B_i} + N^{-1} \sum_{i=1}^{N} c_i \frac{A_{2i}}{\sqrt{B_i}} - N^{-1} \sum_{i=1}^{N} c_i \frac{A_{1i}A_{3i}}{\sqrt{B_i^3}}
$$

$$
+ \frac{1}{48}\alpha^2 N^{-1} \sum_{i=1}^{N} c_i \frac{A_{1i}}{\sqrt{B_i^3}} - \frac{3}{4}\alpha^2 N^{-1} \sum_{i=1}^{N} c_i \frac{A_{1i}A_{4i}^2}{\sqrt{B_i^5}}
$$

$$
+ \frac{1}{\sqrt{2}}\alpha N^{1/4} N^{-1} \sum_{i=1}^{N} (-c_i)^{1/2} \frac{A_{1i}A_{4i}}{\sqrt{B_i^3}} - \frac{1}{\sqrt{2}}\alpha N^{1/4} N^{-1} \sum_{i=1}^{N} (-c_i)^{1/2} \frac{A_{5i}}{\sqrt{B_i}}
$$

$$
+ \frac{1}{2}\alpha^2 N^{-1} \sum_{i=1}^{N} c_i \frac{A_{4i}A_{5i}}{\sqrt{B_i^3}} + O_p(N^{-1/4})
$$

where τ^μ represents the limit distribution of the DF statistic with intercept and

$$
A_{1i} = \int_0^1 W_i^\mu(r)dW_i(r), \quad A_{2i} = \int_0^1 \left\{ \int_0^r W_i(s)ds - \int_0^1 \int_0^t W_i(s)dsdt \right\} dW_i(r),
$$

$$
A_{3i} = \int_0^1 W_i^\mu(r) \left\{ \int_0^r W_i(s)ds - \int_0^1 \int_0^t W_i(s)dsdt \right\} dr, \quad A_{4i} = \int_0^1 (r - \tfrac{1}{2})W_i^\mu(r)dr,
$$

$$
A_{5i} = \int_0^1 (r - \tfrac{1}{2})dW_i(r), \quad B_i = \int_0^1 W_i^\mu(r)^2 dr, \quad W_i^\mu(r) = W_i(r) - \int_0^1 W_i(s)ds
$$

and $W_i(r)$ is a standard Brownian motion process.

The behaviour as $N \to \infty$ of the constituent terms in the limit in Lemma 1 is given by the following series of lemmas

Lemma 2.
$$
N^{1/2} \left\{ N^{-1} \sum_{i=1}^{N} \frac{A_{1i}}{\sqrt{B_i}} - E\left(\tau^\mu\right) \right\} \stackrel{N \to \infty}{\Rightarrow} N(0, V(\tau^\mu)).
$$

Lemma 3. *Let $c = \lim_{N \to \infty} N^{-1}\sum_{i=1}^{N} c_i$. Then*

$$
(i) \;\; N^{-1} \sum_{i=1}^{N} c_i \sqrt{B_i} \stackrel{N \to \infty}{\Rightarrow} cE\left(\sqrt{B}\right), \quad (ii) \;\; N^{-1} \sum_{i=1}^{N} c_i \frac{A_{2i}}{\sqrt{B_i}} \stackrel{N \to \infty}{\Rightarrow} cE\left(\frac{A_2}{\sqrt{B}}\right),
$$

$$
(iii) \;\; N^{-1} \sum_{i=1}^{N} c_i \frac{A_{1i}A_{3i}}{\sqrt{B_i^3}} \stackrel{N \to \infty}{\Rightarrow} cE\left(\frac{A_1 A_3}{\sqrt{B^3}}\right), \quad (iv) \;\; N^{-1} \sum_{i=1}^{N} c_i \frac{A_{1i}}{\sqrt{B_i^3}} \stackrel{N \to \infty}{\Rightarrow} cE\left(\frac{A_1}{\sqrt{B^3}}\right),
$$

$$
(v) \;\; N^{-1} \sum_{i=1}^{N} c_i \frac{A_{1i}A_{4i}^2}{\sqrt{B_i^5}} \stackrel{N \to \infty}{\Rightarrow} cE\left(\frac{A_1 A_4^2}{\sqrt{B^5}}\right).
$$

3

Lemma 4.

$$(i) \quad N^{-1} \sum_{i=1}^{N} (-c_i)^{1/2} \frac{A_{1i}A_{4i}}{\sqrt{B_i^3}} = O_p(N^{-1/2}), \quad (ii) \quad N^{-1} \sum_{i=1}^{N} (-c_i)^{1/2} \frac{A_{5i}}{\sqrt{B_i}} = O_p(N^{-1/2}),$$

$$(iii) \quad N^{-1} \sum_{i=1}^{N} (-c_i) \frac{A_{4i}A_{5i}}{\sqrt{B_i^3}} = O_p(N^{-1/2}).$$

The local asymptotic limit of the IPS test as $T \to \infty$ followed by $N \to \infty$ is given by

Theorem 1. *Under Assumptions 1–3, with $c = \lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} c_i$*

$$Z^\mu \overset{T \to \infty, N \to \infty}{\Rightarrow} N(0,1)$$
$$+ c \left\{ E\left(\sqrt{B}\right) + E\left(\frac{A_2}{\sqrt{B}}\right) - E\left(\frac{A_1 A_3}{\sqrt{B^3}}\right) \right\} \bigg/ \sqrt{V(\tau^\mu)}$$
$$+ \alpha^2 c \left\{ \frac{1}{48} E\left(\frac{A_1}{\sqrt{B^3}}\right) - \frac{3}{4} E\left(\frac{A_1 A_4^2}{\sqrt{B^5}}\right) \right\} \bigg/ \sqrt{V(\tau^\mu)}$$

This follows directly from combining the results of Lemmas 1-3.

Remark 2. Setting $\alpha = 0$ in Theorem 1 gives the local asymptotic distribution of the IPS test for asymptotically negligible initial conditions; that is, whenever $\xi_i = o(N^{1/4}T^{1/2})$. Note that this corrects the result stated in Breitung and Pesaran (2008), where the offset term in c contains only $E(\sqrt{B})$, thereby incorrectly omitting the other two expectations.

Values for the moments involved in Theorem 1 can be obtained using Monte Carlo simulation. We obtained the values by direct simulation of the distributions, approximating the Wiener processes using i.i.d. $N(0,1)$ random variates, and with the integrals approximated by normalized sums of 1000 steps. Here and throughout the rest of the paper, simulations were programmed in Gauss 7.0 using 50,000 replications. Substitution of these moments into the limit expression in Theorem 1 gives rise to the following

Corallary 1. *Under the conditions of Theorem 1*

$$Z^\mu \overset{T \to \infty, N \to \infty}{\Rightarrow} N(0,1) + (0.282 - 0.135\alpha^2)c. \tag{2}$$

Corollary 1 shows that for a zero (common) initial condition, the limit distribution of Z^μ is $N(0,1) + 0.282c$ and hence the power of the IPS test (since it is left tailed) is monotonically increasing in $c < 0$. However, once we allow for (common) non-zero intial conditions, for a given $c < 0$, the power is monotonically decreasing in $|\alpha|$. In fact, it also follows from (2) that for the IPS test conducted at *any* chosen significance level, asymptotically, its power will fall below nominal size once $0.282 - 0.135\alpha^2 < 0$; that is, once $|\alpha| > 1.445$.

These asymptotic properties of the IPS test are demonstrated graphically in Figure 1, where the nominal signifance level is 5%, $c \in \{-2, -4, -6, -8, -10\}$ and $\alpha \in \{0, 0.1, ..., 4.0\}$. For comparison, also shown is the local asymptotic power of the univariate DF test (for $\rho = 1 + c/T$), as previously studied by Müller and Elliott (2003) and Harvey and Leybourne (2005), when $c = -10$. The contrast in behaviour is completely evident, highlighting the fact that the behaviour of panel unit root tests cannot necessarily be inferred from the corresponding behaviour of their univariate counterparts. While the univariate DF test has power that is monotonically increasing in α, the

IPS test displays the opposite behaviour, with power rapidly decreasing as the magnitude of α increases. By the time $\alpha = 1$, when the initial values are one standard deviation away from the series mean, power has roughly halved relative to the $\alpha = 0$ case; for $\alpha > 1.445$, the above result that IPS power falls below size is clearly observed for all values of c.

Finally, it is important to assess the extent to which the asymptotic behaviour of the IPS test manifests itself when N (and T) is finite. In Figure 2 we report finite sample power simulations for $N \in \{10, 20, 30, 50, 100\}$ when $T = 100$, with $\alpha \in \{0, 0.1, ..., 4.0\}$. These are based on simulation of Z^μ with data generated from the model (1) with $\mu_i = 0$ and the v_{it} generated as i.i.d. $N(0,1)$ independently across i. We set $\rho_i = \rho$ for all i such that $\xi_i = \alpha\sqrt{1/(1-\rho^2)}$ and, to make the comparisons more straightforward, for each value of N, ρ is selected such that finite sample power is equal to 50% when $\alpha = 0$. Results for the univariate DF test are again reported for comparison. We see that as N increases the power curve of the IPS test less and less resembles that of the rising-in-α univariate case and migrates towards that of the decaying-in-α large N case described above. In very broad terms, power is increasing in α when $N < 30$ and decreasing when $N > 30$. Moreover, Z^μ can possess extremely low finite sample power when $N = 50$ or more and α not close to zero. Thus, our large N asymptotics do indeed appear to be a decent predictor of what might occur in finite samples when N is not small. We would therefore suggest that our findings serve a note of caution to those applying the IPS test when there exists uncertainty regarding the magnitude of the initial conditions.

4 References

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Appendix

Proof of Lemma 1
Using results from Müller and Elliott (2003) and Phillips (1987), we have that as $T \to \infty$

$$t_i^\mu \overset{T \Rightarrow \infty}{} c_i N^{-1/2} \sqrt{\int_0^1 K_{i,c_i}^\mu(r)^2 dr} + \frac{\int_0^1 K_{i,c_i}^\mu(r) dW_i(r)}{\sqrt{\int_0^1 K_{i,c_i}^\mu(r)^2 dr}}$$

where $K_{i,c_i}^{\mu} = K_{i,c_i}(r) - \int_0^1 K_{i,c_i}(s)ds$ and

$$K_{i,c_i}(r) = \alpha(e^{rc_i N^{-1/2}} - 1)(-2c_i N^{-1/2})^{-1/2} + W_i(r) + c_i N^{-1/2}\int_0^r e^{(r-s)c_i N^{-1/2}}W_i(s)ds.$$

Next, via a Taylor series expansion of the form

$$e^{xc_i N^{-1/2}} = 1 + xc_i N^{-1/2} + O(N^{-1})$$

we may write

$$K_{i,c_i}(r) = \alpha rc_i(-2c_i)^{-1/2}N^{-1/4} + W_i(r) + c_i N^{-1/2}\int_0^r W_i(s)ds + O_p(N^{-3/4})$$

$$K_{i,c_i}^{\mu} = \alpha(r - \tfrac{1}{2})c_i(-2c_i)^{-1/2}N^{-1/4} + W_i^{\mu}(r) + c_i N^{-1/2}\left\{\int_0^r W_i(s)ds - \int_0^1\int_0^t W_i(s)dsdt\right\} + O_p(N^{-3/4})$$

where $W_i^{\mu} = W_i(r) - \int_0^1 W_i(s)ds$. Substituting and rearranging, we find

$$t_i^{\mu} \stackrel{T \to \infty}{\Rightarrow} c_i N^{-1/2}\sqrt{B_i + O_p(N^{-1/4})} +$$
$$\frac{A_{1i} + c_i N^{-1/2}A_{2i} + \alpha c_i(-2c_i)^{-1/2}N^{-1/4}A_{5i} + O_p(N^{-3/4})}{\sqrt{B_i + \alpha^2 c_i^2(-2c_i)^{-1}N^{-1/2}/12 + 2c_i N^{-1/2}A_{3i} + 2\alpha c_i(-2c_i)^{-1/2}N^{-1/4}A_{4i} + O_p(N^{-3/4})}} \qquad (3)$$

where the A_{ji}, $j = 1, ..., 5$ and B_i are as defined in the main text. Writing the second term as $F_i(B_i + G_i)^{-1/2}$, where F_i represents the numerator and

$$G_i = \alpha^2 c_i^2(-2c_i)^{-1}N^{-1/2}/12 + 2c_i N^{-1/2}A_{3i} + 2\alpha c_i(-2c_i)^{-1/2}N^{-1/4}A_{4i} + O_p(N^{-3/4})$$

then a Taylor series expansion around $G_i = 0$ gives

$$\begin{aligned}
(B_i + G_i)^{-1/2} &= \frac{1}{\sqrt{B_i}} - \frac{G_i}{2\sqrt{B_i^3}} + \frac{3G_i^2}{8\sqrt{B_i^5}} + O_p(N^{-3/4}) \\
&= \frac{1}{\sqrt{B_i}} - \frac{\alpha^2 c_i^2(-2c_i)^{-1}N^{-1/2}/12}{2\sqrt{B_i^3}} - \frac{\alpha c_i(-2c_i)^{-1/2}N^{-1/4}A_{4i}}{\sqrt{B_i^3}} \\
&\quad - \frac{c_i N^{-1/2}A_{3i}}{\sqrt{B_i^3}} + \frac{3\alpha^2 c_i^2(-2c_i)^{-1}N^{-1/2}A_{4i}^2}{2\sqrt{B_i^5}} + O_p(N^{-3/4}). \qquad (4)
\end{aligned}$$

On combining (3) and (4), together with $\sqrt{B_i + O_p(N^{-1/4})} = \sqrt{B_i} + O_p(N^{-1/4})$, we have

$$\begin{aligned}
t_i^{\mu} \stackrel{T \to \infty}{\Rightarrow} &\ \frac{A_{1i}}{\sqrt{B_i}} + c_i N^{-1/2}\sqrt{B_i} + c_i N^{-1/2}\frac{A_{2i}}{\sqrt{B_i}} - c_i N^{-1/2}\frac{A_{1i}A_{3i}}{\sqrt{B_i^3}} \\
&+ \frac{1}{48}\alpha^2 c_i N^{-1/2}\frac{A_{1i}}{\sqrt{B_i^3}} - \frac{3}{4}\alpha^2 c_i N^{-1/2}\frac{A_{1i}A_{4i}^2}{\sqrt{B_i^5}} \\
&+ \frac{1}{\sqrt{2}}\alpha(-c_i)^{1/2}N^{-14}\frac{A_{1i}A_{4i}}{\sqrt{B_i^3}} - \frac{1}{\sqrt{2}}\alpha(-c_i)^{1/2}N^{-1/4}\frac{A_{5i}}{\sqrt{B_i}} \\
&+ \frac{1}{2}\alpha^2 c_i N^{-1/2}\frac{A_{4i}A_{5i}}{\sqrt{B_i^3}} + O_p(N^{-3/4})
\end{aligned}$$

and the result of the lemma then follows by considering $\sqrt{V(t_0^{\mu})}Z^{\mu} = N^{1/2}\{N^{-1}\sum_{i=1}^N t_i^{\mu} - E(t_0^{\mu})\}$.

Proof of Lemma 2
This follows from application of a standard central limit theorem in N for *i.i.d.* random variables with bounded variance.

Proof of Lemma 3

(i) Write

$$
\begin{aligned}
N^{-1}\sum_{i=1}^{N} c_i \sqrt{B_i} &= N^{-1}\sum_{i=1}^{N} c_i E(\sqrt{B}) + N^{-1}\sum_{i=1}^{N} c_i\{\sqrt{B_i} - E(\sqrt{B})\} \\
&= cE(\sqrt{B}) + O_p(N^{-1/2}) \\
&\overset{N\to\infty}{\Rightarrow} c(E\sqrt{B})
\end{aligned}
$$

using a standard weak law of large numbers in N for i.i.d. random variables, which applies since $E(B_i) < \infty$. Results *(ii)-(v)* follow similarly.

Proof of Lemma 4

We will show *(i)*, since *(ii)* and *(iii)* follow similarly. This involves showing

$$
E\left(\frac{A_{1i}A_{4i}}{\sqrt{B_i^3}}\right) = 0, \tag{5}
$$

and

$$
E\left(\frac{A_{1i}^2 A_{4i}^2}{B_i^3}\right) < \infty. \tag{6}
$$

Then by independence we will have

$$
E\left[\left(N^{-1}\sum_{i=1}^{N}(-c_i)^{1/2}\frac{A_{1i}A_{4i}}{\sqrt{B_i^3}}\right)^2\right] = N^{-2}\sum_{i=1}^{N}|c_i|\,E\left(\frac{A_{1i}^2 A_{4i}^2}{B_i^3}\right) = O\left(N^{-1}\right)
$$

hence proving *(i)*.

To show (5), the numerator can be written

$$
\begin{aligned}
A_1 A_4 &= \int_0^1\left(r - \tfrac{1}{2}\right)W^\mu(r)\,dr\int_0^1 W^\mu(r)\,dW(r) \\
&= \int_0^1\left(r - \tfrac{1}{2}\right)W(r)\,dr\left\{\int_0^1 W(r)\,dW(r) - \int_0^1 W(r)\,dr\int_0^1 dW(r)\right\} \\
&= \tfrac{1}{2}\int_0^1\left(s - s^2\right)dW(s)\left[\tfrac{1}{2}\left\{\int_0^1 dW(r)\right\}^2 - \tfrac{1}{2} - \int_0^1(1-r)\,dW(r)\int_0^1 dW(r)\right] \\
&= \tfrac{1}{2}\int_0^1\left(s - s^2\right)dW(s)\int_0^1 dW(r)\int_0^1\left(r - \tfrac{1}{2}\right)dW(r) - \tfrac{1}{4}\int_0^1\left(s - s^2\right)dW(s) \\
&\equiv \tfrac{1}{2}X_1 X_2 X_3 - \tfrac{1}{4}X_1.
\end{aligned}
$$

Now X_1, X_2, X_3 are jointly normal with mean zero, and we find that

$$
E(X_1^2) = \int_0^1\left(s - s^2\right)^2 ds = \tfrac{1}{30}, \qquad E(X_2^2) = 1,
$$

$$
E(X_3^2) = \int_0^1\left(r - \tfrac{1}{2}\right)^2 dr = \tfrac{1}{12}, \qquad E(X_1 X_2) = \int_0^1\left(s - s^2\right)ds = \tfrac{1}{6},
$$

$$
E(X_1 X_3) = \int_0^1\left(s - s^2\right)\left(s - \tfrac{1}{2}\right)ds = 0, \qquad E(X_2 X_3) = \int_0^1\left(r - \tfrac{1}{2}\right)dr = 0.
$$

That is

$$
\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/30 & 1/6 & 0 \\ 1/6 & 1 & 0 \\ 0 & 0 & 1/12 \end{pmatrix}\right).
$$

Since X_3 is independent of both X_1 and X_2, it follows that

$$
E(X_1 X_2 X_3) = E(X_1 X_2)E(X_3) = 0
$$

and hence
$$E\left(\tfrac{1}{2}X_1X_2X_3 - \tfrac{1}{4}X_1\right) = 0$$
so that $A_1 A_4$ has zero mean. Now consider the skewness of $A_1 A_4$

$$E\left\{\left(\tfrac{1}{2}X_1X_2X_3 - \tfrac{1}{4}X_1\right)^3\right\} = \tfrac{1}{8}E(X_1^3X_2^3X_3^3) - \tfrac{3}{16}E(X_1^3X_2^2X_3^2) + \tfrac{3}{32}E(X_1^3X_2X_3) - \tfrac{1}{64}E(X_1^3).$$

The first and third terms are obviously zero since X_3 is independent of both X_1 and X_2, and X_3 is symmetric due to normality. The fourth term is also zero due to the normality of X_1. The second term can be written
$$E(X_1^3X_2^2X_3^2) = E(X_1^3X_2^2)E(X_3^2).$$

Next consider a population regression of X_2 on X_1. Since $E(X_1X_2)/E(X_1^2) = 5$ we can write

$$X_2 = 5X_1 + X_2^*$$

where X_1 and X_2^* are independent and jointly normal with mean zero. Substituting $X_2 = 5X_1 + X_2^*$ into $E(X_1^3X_2^2)$ gives

$$\begin{aligned}
E(X_1^3X_2^2) &= E\{X_1^3(25X_1^2 + X_2^{*2} + 5X_1X_2^*)\} \\
&= 25E(X_1^5) + E(X_1^3X_2^{*2}) + 5E(X_1^4X_2^*) \\
&= 25E(X_1^5) + E(X_1^3)E(X_2^{*2}) + 5E(X_1^4)E(X_2^*) \\
&= 0.
\end{aligned}$$

Thus
$$E\left\{\left(\tfrac{1}{2}X_1X_2X_3 - \tfrac{1}{4}X_1\right)^3\right\} = 0$$

and so the distribution of $A_1 A_4$ is symmetric about zero. Now it follows that

$$\begin{aligned}
E\left(\frac{A_{1i}A_{4i}}{\sqrt{B_i^3}}\right) &= E\left(\operatorname{sgn}(A_{1i}A_{4i})\sqrt{\frac{A_{1i}^2A_{4i}^2}{B_i^3}}\right) \\
&= E\left(\operatorname{sgn}(A_{1i}A_{4i})\right)E\left(\sqrt{\frac{A_{1i}^2A_{4i}^2}{B_i^3}}\right) \\
&= 0
\end{aligned}$$

provided (6) holds.

To show (6) we apply the Cauchy Schwarz inequality

$$E\left(\frac{A_{1i}^2A_{4i}^2}{B_i^3}\right) \le E\left(A_{1i}^4A_{4i}^4\right)^{1/2}E\left(B_i^{-6}\right)^{1/2}$$

and since it is clear that $A_{1,i}$ and $A_{4,i}$ have moments of all orders, we just need to verify $E\left(B_i^{-6}\right) < \infty$. From equation (5.7) of Evans and Savin (1981), we can check the existence of the right hand side of

$$E\left(B^{-r}\right) = \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}E\left(e^{-tB}\right)dt, \; r > 0.$$

From equation (4.12) of Tanaka (1996) we can deduce the mgf of B to be $E\left(e^{tB}\right) = \left((2t)^{-1/2}\sin\sqrt{2t}\right)^{-1/2}$ and hence for $t > 0$ it follows that $E\left(e^{-tB}\right) = \left((2t)^{-1/2}\sinh\sqrt{2t}\right)^{-1/2}$. By the change of variable $u = \sqrt{2t}$ we arrive at the integral

$$E\left(B^{-r}\right) = \frac{1}{2^{r-1}\Gamma(r)}\int_0^\infty \frac{u^{2r-1/2}}{(\sinh u)^{1/2}}du$$

which exists for any $r > 0$.

If it is needed to verify this existence, we can write

$$\int_0^\infty \frac{u^{2r-1/2}}{(\sinh u)^{1/2}} du = \int_0^1 \frac{u^{2r-1/2}}{(\sinh u)^{1/2}} du + \int_1^\infty \frac{u^{2r-1/2}}{(\sinh u)^{1/2}} du.$$

In the first term we use $\sinh u \geq u$ to write

$$\int_0^1 \frac{u^{2r-1/2}}{(\sinh u)^{1/2}} du \leq \int_0^1 u^{2r-1} dr$$

which exists and is equal to $(2r)^{-1}$ for $r > 0$. In the second term we use $\sinh(u) = \frac{1}{2} e^u \left(1 - e^{-2u}\right) \geq \frac{1}{2} e^u \left(1 - e^{-2}\right)$ on $[1, \infty)$, so

$$\int_1^\infty \frac{u^{2r-1/2}}{(\sinh u)^{1/2}} du \leq \sqrt{\frac{2}{1 - e^{-2}}} \int_1^\infty u^{2r-1/2} e^{-u/2} du \leq \frac{2^{2r+1}}{\sqrt{1 - e^{-2}}} \Gamma\left(2r - \frac{1}{2}\right) < \infty.$$
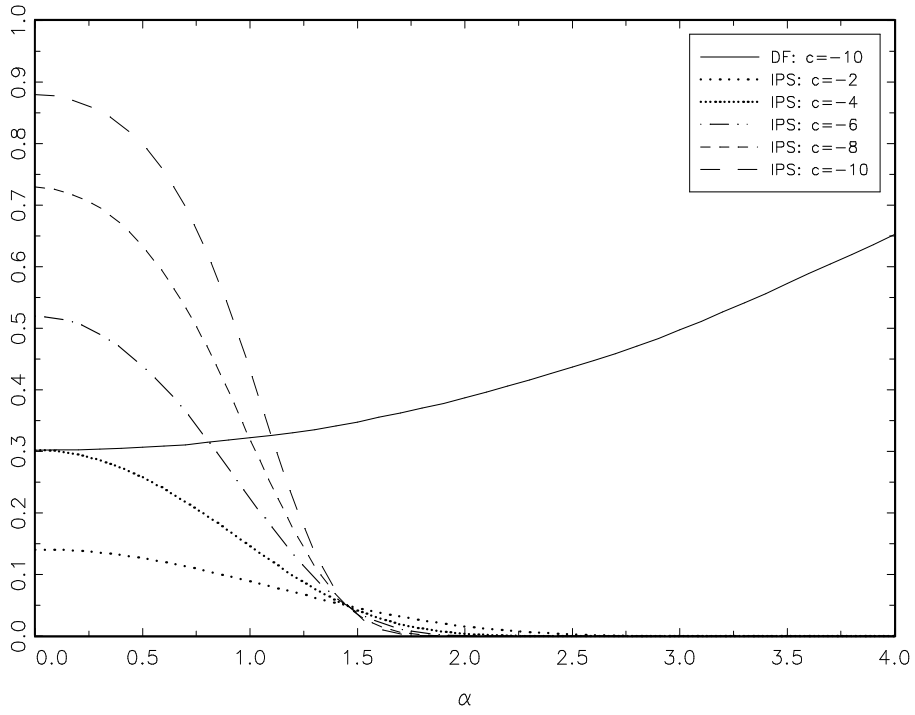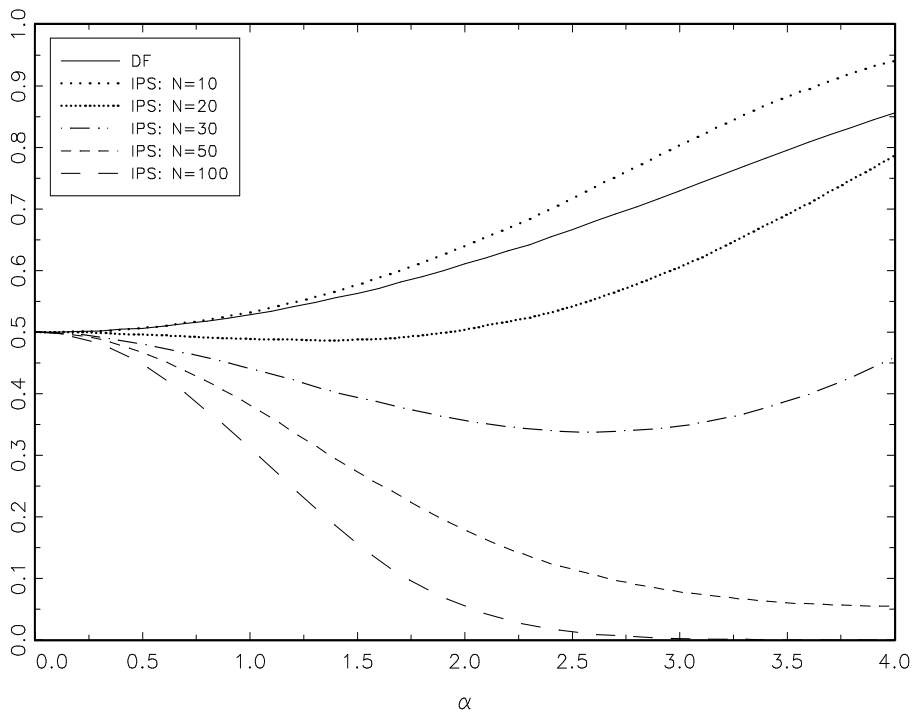
Figure 1. Asymptotic local power of DF and IPS tests

Figure 2. Finite sample power of DF and IPS tests, $T = 100$