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Testing for Seasonal Unit Roots by Frequency Domain Regression*

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Abstract

This paper develops univariate seasonal unit root tests based on spectral regression estimators. An advantage of the frequency domain approach is that it enables serial correlation to be treated non-parametrically. We demonstrate that our proposed statistics have pivotal limiting distributions under both the null and near seasonally integrated alternatives when we allow for weak dependence in the driving shocks. This is in contrast to the popular seasonal unit root tests of, among others, Hylleberg *et al.* (1990) which treat serial correlation parametrically via lag augmentation of the test regression. Moreover, our analysis allows for (possibly infinite order) moving average behaviour in the shocks, while extant large sample results pertaining to the Hylleberg *et al.* (1990) type tests are based on the assumption of a finite autoregression. The size and power properties of our proposed frequency domain regression-based tests are explored and compared for the case of quarterly data with those of the tests of Hylleberg *et al.* (1990) in simulation experiments.

Keywords: Seasonal unit root tests; moving average; frequency domain regression; spectral density estimator; Brownian motion.

JEL Classification: C22.

1 Introduction

This paper considers testing for *seasonal* unit roots in a univariate time-series process. In the seminal paper in the literature, Hylleberg *et al.* (1990) [HEGY] develop separate regression-based t- and F-tests for unit roots at the zero, Nyquist and annual (harmonic) frequencies in the context of quarterly

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data. Recently, Smith, Taylor and del Bario Castro (2009) have generalised this approach to allow for an arbitrary seasonal aspect, while Rodrigues and Taylor (2007) develop near-efficient versions of the HEGY tests. Other important extensions of the basic HEGY approach appear in, *inter alia*, Ghysels, Lee and Noh (1994), who allow for joint testing across different frequencies, Smith and Taylor (1998), who extend the range of deterministic specifications allowed in HEGY and provide limiting null distributions for the original HEGY statistics, and Rodrigues and Taylor (2004) who develop expressions for the asymptotic local power of the HEGY tests.

These HEGY-type tests are all characterised by the use of parametric lag augmentation, along the lines of the augmented Dickey-Fuller [ADF] test, to allow for weak dependence in the driving shocks. Focusing on the standard assumption made in this literature that the shocks follow a finite-order autoregressive process of order p [$AR(p)$], Burridge and Taylor (2001) and Smith *et al.* (2009) show that such lag augmentation can provide only a partial solution with the limiting null distributions of certain of the harmonic frequency unit root tests still depending, in general, on the parameters of the $AR(p)$ polynomial with the consequence that not all of the HEGY-type tests can be reliably used in practice.

It has been known since the seminal work of Box and Jenkins (1976) that seasonally observed time series tend to display significant moving average behaviour. Indeed, Box and Jenkins (1976) developed the well-known seasonal ARIMA factorisations, the best known example of which being the so-called *airline model*. Allowing for moving average behaviour is, therefore, very important in the context of seasonal unit root testing. While it has been widely conjectured that the results reported above for the case of finite $AR(p)$ shocks would continue to hold in the case where the shocks have an $AR(\infty)$ representation, as would be needed to allow for stationary and invertible autoregressive moving average [$ARMA$] shocks, provided an approach along the lines of that developed in regard of the ADF test by Said and Dickey (1984) was implemented, this has never formally been proved in the literature. Indeed, current practice takes matters a stage further, using data-dependent methods to select the lag augmentation polynomial.

Motivated by these shortcomings of the HEGY-type tests, the purpose of this paper is to develop a new class of regression-based seasonal unit root tests, which are asymptotically valid in the presence of general weak dependence. The key feature that distinguishes our model from this earlier literature is that we explicitly allow for the presence of $ARMA$ shocks. We do so by the use of frequency domain regression [FDR] based test statistics. We consider a variety of possible forms for the deterministic component, proposing tests based on both ordinary least squares [OLS] and quasi-difference [QD] de-trending. We demonstrate that the limiting distributions of all of the resulting HEGY-type t- and F-statistics are pivotal under both the null hypothesis and under near-integrated alternatives, attaining the limiting distributions achieved by their standard HEGY counterparts when the shocks are independent and identically distributed [IID].

Frequency domain analysis has a long history in econometrics, with Granger and Hatanaka (1964) providing an early demonstration of its relevance in the analysis of economic data. Furthermore Granger (1966) observed that many economic time series have considerable power at low frequencies, giving rise to a spectral density that is peaked at the origin and which declines as frequency increases; he described this as the typical spectral shape of an economic variable, and the peak at the origin would nowadays be associated with the variable being integrated of order one. In a seasonal setting these peaks occur at the seasonal frequencies, and our approach is based on a seasonal extension of the unit root tests of Choi and Phillips (1993) which utilise the efficient FDR estimator of Hannan (1963). The main advantage of the FDR approach from our perspective is that, unlike the HEGY approach, it delivers estimators of the parameters corresponding to the seasonal roots whose limiting distributions are free from nuisance parameters, even in the presence of moving average disturbances.

The FDR effectively transforms serial correlation in the disturbances into a form of heteroskedasticity across frequencies that is captured by the spectral density function; the resulting estimators handle this heteroskedasticity by weighting the periodogram ordinates by the inverse of the estimated spectral density. In our implementation of the frequency domain estimator we consider two types of spectral density estimator. The first is a simple weighted periodogram estimator [WPE] that averages a set of periodogram values at frequencies either side of the frequency of interest while the second uses the Berk (1974) autoregressive spectral density estimator [ASDE] derived from an autoregressive approximation to the series of interest. Our use of the ASDE is novel in the sense that we use the autoregressive approximation to obtain an estimator of the spectral density across *all* frequencies. This contrasts with its usual use in unit root testing where it is computed at the fixed frequency of the root being tested; see, e.g., Ng and Perron (1995) for the zero frequency root and Rodrigues and Taylor (2007) for the seasonal frequencies.

The paper is organised as follows. Section 2 outlines the seasonal framework, defines the hypotheses of interest, and briefly reviews the HEGY tests. In Section 3 we introduce our FDR implementations of the HEGY statistics and provide representations for their limiting distributions under both the null and local alternatives, showing these to be pivotal in the presence of weak dependence. An investigation into the relative finite sample performances of the FDR tests and the augmented HEGY tests is provided in Section 5. Section 6 concludes. Proofs are contained in Appendices A and B.

2 The Seasonal Unit Root Framework

2.1 The Seasonal Model

The model we consider for the scalar random variable X_t is given by

$$X_t = Y_t + \mu_t, \qquad t = 1-S, \dots, T, \tag{2.1a}$$

$$a_S(L)Y_t = U_t, \qquad t = 1, \dots, T \tag{2.1b}$$

where $a_S(z) := 1 - \sum_{j=1}^{S} a_j z^j$, S denotes the number of seasons, L denotes the lag operator, and the deterministic component μ_t satisfies

$$\mu_t := \sum_{j=1}^{S} \delta_j D_{jt} + \rho t, \quad t = 1-S, \dots, T, \tag{2.2}$$

where D_{jt} is a seasonal dummy variable such that for $j = 1, \dots, S$, $D_{jt} = 1$ $(t = j \bmod S)$ and $D_{jt} = 0$ otherwise. The initial conditions, $Y_{1-S}, ..., Y_0$, are taken to be of $O_p(T^{\delta})$, $\delta < 0.5$; cf. Rodrigues and Taylor (2007). We assume that the random disturbance U_t in (2.1b) is a mean-zero covariance stationary (linear) process satisfying the following conditions:

Assumption 1 *The random disturbance U_t in (2.1b) admits the moving average representation $U_t = \psi(L)V_t$ where V_t is $IID(0, \sigma^2)$ with finite fourth moments and where the lag polynomial $\psi(z) := 1 + \sum_{i=1}^{\infty} \psi_i z^i$ satisfies: (i) $\psi(\exp\{\pm i2\pi k/S\}) \neq 0$, $k = 0, ..., \lfloor S/2 \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part of its argument and where $i := \sqrt{-1}$, and (ii) $\sum_{j=1}^{\infty} j|\psi_j| < \infty$.*

Remark 1: Assumption 1 ensures that the spectral density function of U_t is bounded and is strictly positive at both the zero and seasonal spectral frequencies, $\omega_k := 2\pi k/S$, $k = 0, ..., \lfloor S/2 \rfloor$.

The model depicted in (2.1)-(2.2) is sufficiently general to enable X_t to be defined in terms of an arbitrary seasonal frequency S and to capture a variety of seasonal intercept and trend effects in the deterministic component $\mu_t := \gamma' d_t$. We shall consider the following five specifications for the deterministic component in which the stated restrictions on δ_j and ρ hold for $j = 1, \dots, S$:

Scheme 1. No intercept, no trend: $\delta_j = \rho = 0$.

Scheme 2. Intercept, no trend: $\delta_j = \delta$, $\rho = 0$; $\gamma := \delta$, $d_t := 1$.

Scheme 3. Seasonal intercepts, no trend: δ_j unrestricted, $\rho = 0$; $\gamma := (\delta_1, ..., \delta_S)'$, $d_t := (D_{1t}, ..., D_{St})'$.

Scheme 4. Intercept, trend: $\delta_j = \delta$, ρ unrestricted $\gamma := (\delta, \rho)'$, $d_t := (1, t)'$.

Scheme 5. Seasonal intercepts, trend: δ_j, ρ unrestricted; $\gamma := (\delta_1, ..., \delta_S, \rho)'$, $d_t := (D_{1t}, ..., D_{St}, t)'$.

Smith *et al.* (2009) also consider the further scheme of seasonal intercepts and seasonal trends,

$$\mu_t := \sum_{j=1}^{S} \delta_j D_{jt} + \sum_{j=1}^{S} \rho_j D_{jt} t, \quad t = 1 - S, \ldots, T, \tag{2.3}$$

with δ_j and ρ_j unrestricted. Here $\gamma := (\delta_1, ..., \delta_S, \rho_1, ..., \rho_S)'$, $d_t := (D_{1t}, ..., D_{St}, D_{1t}t, ..., D_{St}t)'$. We will not explicitly cover this case in what follows (as its empirical relevance is limited) but we will mention how our results carry over to this scheme at appropriate points.

2.2 The Seasonal Unit Root Hypotheses

Our focus is on tests for seasonal unit roots in $a_S(L)$ of (2.1b); that is, the overall null hypothesis of interest is

$$H_0 : a_S(L) = (1 - L^S) =: \Delta_S. \tag{2.4}$$

Under H_0, X_t is a seasonal unit root process, admitting the unit roots $\exp(\pm i2\pi k/S)$, $k = 0, ..., \lfloor S/2 \rfloor$. Following HEGY and Smith *et al.* (2009), the polynomial $a_S(L)$ may be factorised as $a_S(L) = \prod_{k=0}^{\lfloor S/2 \rfloor} \omega_k(L)$, where $\omega_0(L) := (1 - \alpha_0 L)$ associates the parameter α_0 with the zero frequency $\omega_0 := 0$, $\omega_k(L) := [1 - 2(\alpha_k \cos \omega_k - \beta_k \sin \omega_k)L + (\alpha_k^2 + \beta_k^2)L^2]$ corresponds to the conjugate (harmonic) seasonal frequencies $(\omega_k, 2\pi - \omega_k)$, $\omega_k := 2\pi k/S$, with associated parameters α_k and β_k, $k = 1, ..., S^*$, where $S^* := \lfloor (S-1)/2 \rfloor$, and, for S even, $\omega_{S/2}(L) := (1 + \alpha_{S/2} L)$, associates the parameter $\alpha_{S/2}$ with the Nyquist frequency $\omega_{S/2} := \pi$. As a point of notation, throughout the paper where reference is made to the Nyquist frequency this is understood only to apply where S is even.

As discussed in, for example, Smith *et al.* (2009) this factorisation of $a_S(L)$ allows H_0 to be commensurately decomposed into the $(\lfloor S/2 \rfloor + 1)$ frequency-specific unit root null hypotheses

$$H_{0,0} : \alpha_0 = 1, \; H_{0,S/2} : \alpha_{S/2} = 1 \tag{2.5}$$

$$H_{0,k} : \alpha_k = 1, \; \beta_k = 0, \; k = 1, ..., S^*. \tag{2.6}$$

The hypothesis $H_{0,0}$ corresponds to a unit root at the zero-frequency while $H_{0,S/2}$ yields a unit root at the Nyquist frequency. A pair of complex conjugate unit roots at the harmonic seasonal frequency pair $(\omega_k, 2\pi - \omega_k)$ is obtained under $H_{0,k}$, $k = 1, ..., S^*$. Notice that $H_0 = \cap_{k=0}^{\lfloor S/2 \rfloor} H_{0,k}$.

Following Rodrigues and Taylor (2007), the alternative hypotheses of near-integration at the zero and Nyquist frequencies may be stated as,

$$H_{1,\nu_0} : \alpha_0 = \left(1 + \frac{\nu_0}{T}\right),\ H_{1,\nu_{S/2}} : \alpha_{S/2} = \left(1 + \frac{\nu_{S/2}}{T}\right) \tag{2.7}$$

and at the harmonic seasonal frequencies as

$$H_{1,\nu_k} : \alpha_k = \left(1 + \frac{\nu_k}{T}\right),\ \beta_k = 0,\ k = 1, ..., S^*. \tag{2.8}$$

Under H_{1,ν_k}, the process X_t admits either a single root $[k = 0, S/2]$ or a pair of complex conjugate roots $[k = 1, ..., S^*]$ with modulus in the neighbourhood of unity at frequency ω_k. These roots are stable where $\nu_k < 0$. Notice that H_{1,ν_k} reduces to $H_{0,k}$ if $\nu_k = 0$, $k = 0, ..., \lfloor S/2 \rfloor$.

In what follows, let $\nu := (\nu_0, \nu_1, ..., \nu_{\lfloor S/2 \rfloor})'$ be the $(\lfloor S/2 \rfloor + 1)$-vector of non-centrality parameters and denote the lag polynomial $a_S(L)$ under $H_{1,\nu} := \cap_{k=0}^{\lfloor S/2 \rfloor} H_{1,\nu_k}$ as $\Delta_\nu := 1 - \sum_{j=1}^{S} \alpha_j^\nu L^j$.

2.3 HEGY Tests

The regression-based approach to testing for seasonal unit roots in $a_S(L)$ of (2.1b) consists of two steps. In the first step one de-trends the data in order to yield tests which will be exact invariant to the seasonal intercept and trend parameters δ_j and ρ_j, $j = 1, ..., S$, which characterise the deterministic component μ_t of (2.2). This can either be done using OLS de-trending, as in, for example, HEGY and Smith *et al.* (2009), or by QD de-trending as in Rodrigues and Taylor (2007). We define the resulting de-trended data series as x_t. In order to economise on notation we do not at this stage introduce any specific superscripts to distinguish the different de-trending Schemes considered in section 2.1 although we do so later in the characterisation of the limiting distributions of the test statistics.

For OLS de-trending, $x_t := X_t - \hat{\gamma}' d_t$, where $\hat{\gamma}$ is the OLS estimator of γ from regressing X_t onto d_t along $t = 1 - S, ..., T$. Under QD de-trending, as in Rodrigues and Taylor (2007), $x_t := X_t - \tilde{\gamma}' d_t$, where $\tilde{\gamma}$ is the QD estimator of γ obtained from the OLS regression of $\mathbf{x}_\nu$ on $\mathbf{d}_\nu$, where

$$\mathbf{x}_\nu := (x_{1-S}, x_{2-S} - \alpha_1^\nu x_{1-S}, x_{3-S} - \alpha_1^\nu x_{2-S} - \alpha_2^\nu x_{1-S}, ..., x_0 - \alpha_1^\nu x_{-1} - \cdots - \alpha_S^\nu x_{1-S}, \Delta_\nu x_1, ...\Delta_\nu x_T)'$$

$$\mathbf{d}_\nu := (d_{1-S}, d_{2-S} - \alpha_1^\nu d_{1-S}, d_{3-S} - \alpha_1^\nu d_{2-S} - \alpha_2^\nu d_{1-S}, ..., d_0 - \alpha_1^\nu d_1 - \cdots - \alpha_S^\nu d_{1-S}, \Delta_\nu d_1, ..., \Delta_\nu d_T)'$$

for $\nu = \bar{\nu} := (\bar{\nu}_0, \bar{\nu}_1, ..., \bar{\nu}_{\lfloor S/2 \rfloor})'$. The QD de-trending parameters, $\bar{\nu}_k$, $k = 0, ..., \lfloor S/2 \rfloor$, are determined by the significance level that the seasonal unit root tests are to be run at and the de-trending scheme employed; see Rodrigues and Taylor (2007,p.556). For example, under Scheme 3 and for tests run at the 5% level, $\bar{\nu}_0 = \bar{\nu}_{S/2} = -7$ and $\bar{\nu}_k = -3.75$, $k = 1, ..., S^*$. The resulting de-trended series[1] satisfies

$$a_S(L)x_t = u_t, \quad u_t = \psi(L)v_t, \quad t = 1 - S, \ldots, T, \tag{2.9}$$

where u_t is the correspondingly de-trended version of U_t. For example, under Scheme 2, in the case of OLS de-trending, $u_t := U_t - (T+S)^{-1} \sum_{j=1-S}^{T} U_j$. In what follows we assume that μ_t is not estimated under an overly restrictive scheme.

Under the assumption that $\psi(z)$ is invertible[2] with (unique) inverse $\phi(z)$, such that an autoregressive approximation of order say p^* is valid, the second step is to then expand the composite $AR(p^*+S)$ polynomial $\phi^*(z) := a_S(z)\phi(z)$ around the zero and seasonal frequency unit roots $\exp(\pm i 2\pi k/S)$, $k = 0, ..., \lfloor S/2 \rfloor$, to obtain the *augmented* HEGY regression[3]

$$\Delta_S x_t = \beta_0 x_{0,t-1} + \beta_{S/2} x_{S/2,t-1} + \sum_{k=1}^{S^*} \left(\beta_k^c x_{k,t-1}^c + \beta_k^s x_{k,t-1}^s \right) + \sum_{j=1}^{p^*} \phi_j \Delta_S x_{t-j} + u_{t,p^*} \tag{2.10}$$

omitting the term $\beta_{S/2} x_{S/2,t-1}$ where S is odd, and where

$$x_{0,t} = \sum_{j=0}^{S-1} x_{t-j}, \quad x_{S/2,t} = \sum_{j=0}^{S-1} \cos[(j+1)\pi] x_{t-j}, \tag{2.11a}$$

$$x_{k,t}^c = \sum_{j=0}^{S-1} \cos[(j+1)\omega_k] x_{t-j}, \quad x_{k,t}^s = -\sum_{j=0}^{S-1} \sin[(j+1)\omega_k] x_{t-j}, \quad k = 1, \ldots, S^*. \tag{2.11b}$$

Cf. Proposition 1 of Smith *et al.* (2009,p.533).

[1] Under Scheme 1, $x_t := X_t$, by definition, since no de-trending (be it OLS or QD) is performed.

[2] Notice that this is more restrictive than condition (i) of Assumption 1 which only requires invertibility at the zero and seasonal frequencies.

[3] In the case of OLS de-trending, an asymptotically equivalent procedure is to omit the first step and to include the relevant deterministic regressors in the auxiliary regression (2.10).

Unit roots at the zero, Nyquist and harmonic seasonal frequencies imply that $\beta_0 = 0$, $\beta_{S/2} = 0$ (S even) and $\beta_k^c = \beta_k^s = 0$, $k = 1, ..., S^*$, in (2.10) respectively; see Smith *et al.* (2009). Consequently, tests for the presence or otherwise of a unit root at the zero and Nyquist frequencies are conventional lower tailed regression t-tests, denoted t_0^* and $t_{S/2}^*$, for the exclusion of $x_{0,t}$ and $x_{S/2,t}$, respectively, from (2.10). Similarly, the hypothesis of a pair of complex unit roots at the kth harmonic seasonal frequency may be tested by the lower-tailed t_k^{c*} and two-tailed t_k^{s*} regression t-tests from (2.10) for the exclusion of $x_{k,t}^c$ and $x_{k,t}^s$, respectively, or by the (upper-tailed) regression F-test, denoted F_k^*, for the exclusion of both $x_{k,t}^c$ and $x_{k,t}^s$ from (2.10). Ghysels *et al.* (1994) also consider the joint frequency (upper-tail) regression F-tests from (2.10), $F_{1...\lfloor S/2 \rfloor}^*$ for the exclusion of $x_{S/2,t}$ (S even) and $\{x_{k,t}^c, x_{k,t}^s\}_{k=1}^{S^*}$, and $F_{0...\lfloor S/2 \rfloor}^*$ for the exclusion of $x_{0,t}$, $x_{S/2,t}$ (S even) and $\{x_{k,t}^c, x_{k,t}^s\}_{k=1}^{S^*}$. The former tests the null hypothesis of unit roots at all of the seasonal frequencies, whereas the latter tests the overall null, H_0 of (2.4). Implementation of these tests, including relevant critical values, using OLS de-trending has been considered in, *inter alia*, HEGY, Smith *et al.* (2009) and Ghysels *et al.* (1994). Corresponding results for the case of QD de-trending are given in Rodrigues and Taylor (2007).

The limiting null distributions of the OLS de-trended (for each of Schemes 1-5) HEGY statistics are given for the case where $\psi(z) = 1$ in (2.9) and accordingly $p^* = 0$ in (2.10) by Smith and Taylor (1998). In the case where $\phi(z)$ is pth order, $0 \leq p < \infty$, Burridge and Taylor (2001) and Smith *et al.* (2009) show that the limiting null distributions of the OLS de-trended t_0^*, $t_{S/2}^*$ (S even) and F_k^*, $k = 1, ..., S^*$, statistics from (2.10), are as for $p = 0$, provided $p^* \geq p$ in (2.10). They show that this is not true, however, for the t_k^{s*} and t_k^{c*}, $k = 1, ..., S^*$, statistics whose limit distributions depend on functions of the parameters characterising the serial dependence in u_t of (2.9). Representations for the corresponding limiting distributions under near seasonally integrated alternatives are given in Rodrigues and Taylor (2004) and again shown to be free of nuisance parameters with the exception of the t_k^{s*} and t_k^{c*}, $k = 1, ..., S^*$, statistics. Corresponding results for the QD de-trended HEGY-type statistic are given in Rodrigues and Taylor (2007) and here it is also the case that the harmonic frequency t-statistics depend on nuisance parameters arising from the serial correlation in u_t. Where the assumption that $\phi(z)$ is finite is dropped it has been widely conjectured that under suitable assumptions, in particular, if the lag length p^* in (2.10) is such that $1/p^* + (p^*)^3/T \to 0$, as $T \to \infty$, that the limiting distributions of the OLS and QD de-trended HEGY statistics will be as derived for those statistics under finite p. However, this conjecture has not been formally proved.

In this paper we construct regression-based seasonal unit root tests which are both asymptotically valid and have pivotal limiting distributions, under both the null and near-integrated alternatives, in the presence of MA behaviour in the shocks. We do this by carrying out the regression in the frequency domain. Here the dynamics of u_t are handled non-parametrically, via the estimation of its spectral density function. These estimates are used to provide an optimal weighting scheme in a generalised least squares type of spectral regression. We outline our approach in the next section.

3 Frequency Domain Regression HEGY Tests

While the approach outlined in section 2.3 adopts a parametric approach to modelling serial correlation present in u_t of (2.9) in this section we focus on a non-parametric approach. Accordingly, therefore, we use an *un-augmented* HEGY regression; that is, while the first step, in which we de-trend the data, of the two-step HEGY-type procedure remains the same as was outlined in section 3, in the second step we now expand only the polynomial $a_S(z)$ around the zero and seasonal frequency unit roots. Doing so yields the auxiliary regression equation[4]

$$\Delta_S x_t = \beta_0 x_{0,t-1} + \beta_{S/2} x_{S/2,t-1} + \sum_{k=1}^{S^*} \left(\beta_k^c x_{k,t-1}^c + \beta_k^s x_{k,t-1}^s\right) + u_t \tag{3.1}$$

again omitting the term $\beta_{S/2} x_{S/2,t-1}$ where S is odd.[5]

In what follows it will prove convenient to define $y_t := \Delta_S x_t$ together with the $S \times 1$ vectors

$$z_t := \left[x_{0,t-1}, x_{1,t-1}^c, x_{1,t-1}^s, \ldots, x_{S^*,t-1}^c, x_{S^*,t-1}^s, x_{S/2,t-1}\right]' \tag{3.2}$$

$$\beta := [\beta_0, \beta_1^c, \beta_1^s, \ldots, \beta_{S^*}^c, \beta_{S^*}^s, \beta_{S/2}]' \tag{3.3}$$

omitting $x_{S/2,t-1}$ and $\beta_{S/2}$ from (3.2) and (3.3), respectively, if S is odd. The regression model in (3.1) may then be written as

$$y_t = z_t' \beta + u_t, \quad t = 1, \ldots, T. \tag{3.4}$$

Given observations on z_t and y_t, define the discrete Fourier transforms $w_z(\lambda) := \frac{1}{(2\pi T)^{1/2}} \sum_{t=1}^T z_t \exp(it\lambda)$ and $w_y(\lambda) := \frac{1}{(2\pi T)^{1/2}} \sum_{t=1}^T y_t \exp(it\lambda)$, together with the periodogram matrix and vector, $I_{zz}(\lambda) :=$

[4]Again, for the case of OLS de-trending, an asymptotically equivalent procedure is to omit the first step and to include the relevant deterministic regressors in (3.1).

[5]Although we have continued to use the same nomenclature for the focal unit root parameters in (3.1) as in (2.10) they are technically not the same functions of the parameters from (2.9) as they are in (2.10). However, in so far as testing the hypotheses in (2.5)-(2.6) is concerned they have the same interpretation, and so with a small abuse of notation we use the same nomenclature for these parameters in both equations.

$w_z(\lambda)w_z(\lambda)^*$ and $I_{zy}(\lambda) := w_z(\lambda)w_y(\lambda)^*$, respectively, where a * denotes transposition combined with complex conjugation. The FDR estimator we consider is then defined by

$$\hat{\beta} := \left[\sum_{j\in J_T} I_{zz}(\lambda_j)\hat{f}_u(\lambda_j)^{-1}\right]^{-1}\left[\sum_{j\in J_T} I_{zy}(\lambda_j)\hat{f}_u(\lambda_j)^{-1}\right], \tag{3.5}$$

where $J_T := \{j : -\lfloor T/2\rfloor < j \leq \lfloor T/2\rfloor\}$ and $\lambda_j := 2\pi j/T$. In the above definition of $\hat{\beta}$, $\hat{f}_u(\lambda)$ denotes an estimator of the spectral density function of u_t. The estimated covariance matrix of $\hat{\beta}$ is

$$\hat{Q} := \left[\sum_{j\in J_T} I_{zz}(\lambda_j)\hat{f}_u(\lambda_j)^{-1}\right]^{-1} \tag{3.6}$$

the j'th diagonal element of which will be denoted $\hat{q}_j$.

Taken together, (3.5) and (3.6) can be used to construct FDR t- and F-tests for seasonal unit roots in $a_S(L)$ of (2.1b). Specifically, analogous to the t_0^*, $t_{S/2}^*$, t_k^{c*} and t_k^{s*}, $k = 1, ..., S^*$, tests from section 2.3, we may define the corresponding FDR-based t-statistics as follows:

$$t_0 := \frac{\hat{\beta}_0}{\sqrt{\hat{q}_1}}, \quad t_{S/2} := \frac{\hat{\beta}_{S/2}}{\sqrt{\hat{q}_S}}, \quad t_k^c := \frac{\hat{\beta}_k^c}{\sqrt{\hat{q}_{2k}}}, \quad t_k^s := \frac{\hat{\beta}_k^s}{\sqrt{\hat{q}_{2k+1}}}, \quad k = 1, \ldots, S^*,$$

omitting the definition of $t_{S/2}$ where S is odd. As with the decision rules outlined for the standard HEGY tests in section 2.3, lower-tailed tests for the null hypothesis of a unit root at the zero ($H_{0,0}$) and Nyquist ($H_{0,S/2}$) frequencies can be based on t_0 and $t_{S/2}$, respectively, while lower-tailed tests based on t_k^c, and two-tailed tests based on t_k^s, can be used to test $H_{0,k}$, $k = 1, ..., S^*$, the null hypothesis of a complex unit root pair at frequency ω_k.

Hypotheses concerning the joint significance of subsets of the elements of β can again be formed. Analogous to F_k^*, $k = 1, ..., S^*$, $F_{1...\lfloor S/2\rfloor}^*$ and $F_{0...\lfloor S/2\rfloor}^*$ from section 2.3, we can define

$$\begin{aligned} F_k &:= \frac{1}{2}\left\{\hat{\beta}' R_k' \left[R_k \hat{Q} R_k'\right]^{-1} R_k \hat{\beta}\right\}, \quad k = 1, \ldots, S^* \\ F_{1\ldots\lfloor S/2\rfloor} &:= \frac{1}{S-1}\left\{\hat{\beta}' R_{1\ldots\lfloor S/2\rfloor}' \left[R_{1\ldots\lfloor S/2\rfloor}\hat{Q} R_{1\ldots\lfloor S/2\rfloor}'\right]^{-1} R_{1\ldots\lfloor S/2\rfloor}\hat{\beta}\right\} \\ F_{0\ldots\lfloor S/2\rfloor} &:= \frac{1}{S}\left\{\hat{\beta}'\hat{Q}^{-1}\hat{\beta}\right\} \end{aligned}$$

where R_k is a $2 \times S$ matrix of zeros except for ones in column $2k$ of row 1 and column $(2k+1)$ of row 2, these elements picking out $\hat{\beta}_k^c$ and $\hat{\beta}_k^s$ from $\hat{\beta}$ respectively, and $R_{1\ldots\lfloor S/2\rfloor}$ is an $(S-1) \times S$ matrix of ones with the exception of the elements of its first row which are all zero. As with the corresponding tests from section 2.3, right-tailed tests based on these statistics can be used to test $H_{0,k}$, $k = 1, ..., S^*$, $\cap_{k=1}^{\lfloor S/2 \rfloor} H_{0,k}$ and H_0, respectively.

As noted above, construction of our proposed FDR tests requires an estimator of the spectral density function of u_t. A number of possibilities exist for the construction of this spectral density estimator and are described in time series textbooks such as Priestley (1981). Here we shall consider two possible estimators. The first is a weighted periodogram estimator [WPE], and the second is the autoregressive spectral density estimator [ASDE] of Berk (1974).

The WPE is based on the residuals, $\hat{u}_t := y_t - z_t' \hat{\beta}_{OLS}$, obtained from a time domain regression of y_t on z_t, $t = 1, ..., T$, $\hat{\beta}_{OLS}$ denoting the OLS estimator of β. The WPE is then defined as

$$\hat{f}_u(\lambda) := \frac{1}{2m+1} \sum_{k=-m}^{m} I_{\hat{u}\hat{u}}(\lambda + \lambda_k), \tag{3.7}$$

where $I_{\hat{u}\hat{u}}(\lambda)$ denotes the periodogram constructed from the residuals $\hat{u}_t$. The parameter m is a positive bandwidth whose rate of increase with T is as prescribed in the following assumption.

Assumption 2 *As $T \to \infty$, $m^{-1} + mT^{-1} \to 0$.*

Remark 2: Assumption 2 imposes that m increases at a slower rate than T, i.e. $m = o(T)$, and ensures, in particular, that $\hat{f}_u(\lambda)$ in (3.7) is a uniformly consistent estimator of $f_u(\lambda)$ as $T \to \infty$.

The ASDE is constructed from the estimated augmented HEGY regression, (2.10). Let the OLS residual variance estimator and the fitted augmentation polynomial from (2.10) be denoted by $\hat{\sigma}^2$ and $\hat{\phi}(z) := (1 - \hat{\phi}_{1,p^*} z - \cdots - \hat{\phi}_{p^*,p^*} z^{p^*})$, respectively, where $\hat{\phi}_{j,p^*}$ denotes the OLS estimator of ϕ_j, $j = 1, ..., p^*$, from (2.10). Then, following Berk (1974), the ASDE is given by

$$\hat{f}_u(\lambda) := \frac{\hat{\sigma}^2}{2\pi} \left[\hat{c}_{p^*}(\lambda)^2 + \hat{s}_{p^*}(\lambda)^2 \right]^{-1} \tag{3.8}$$

where $\hat{c}_{p^*}(\lambda) := 1 - \sum_{j=1}^{p^*} \hat{\phi}_{j,p^*} \cos(j\lambda)$ and $\hat{s}_{p^*}(\lambda) := -\sum_{j=1}^{p^*} \hat{\phi}_{j,p^*} \sin(j\lambda)$. Where the ASDE is concerned, we replace Assumption 2 with the following assumption; cf. Berk (1974).

Assumption 3 *(i) As $T \to \infty$, $(1/p^*) + (p^*)^3/T \to 0$. (ii) The lag polynomial $\psi(z)$ is invertible with (unique) inverse $\phi(z)$.*

Remark 3: Part (i) of Assumption 3 controls the lag truncation parameter p^* in (2.10) to increase at a slower rate than $T^{1/3}$, i.e. $p^* = o(T^{1/3})$, while part (ii) imposes the condition that the spectral density of u_t is positive for all λ. The latter condition, not required for the WPE in (3.7), ensures that the autoregressive approximation to $\psi(L)$ embodied in (2.10) is valid. Taken together, these conditions ensure that $\hat{f}_u(\lambda)$ of (3.8) is a uniformly consistent estimator of $f_u(\lambda)$.

We now derive representations for the limiting distributions of the FDR estimator from (3.5) and the associated test statistics for the cases of both OLS and QD de-trending. These representations are indexed by the parameter τ whose value is determined by which of Schemes 1-5 of μ_t of (2.2) holds and the frequency under test. For the zero frequency ω_0 tests: Scheme 1: $\tau = 0$; Schemes 2 and 3: $\tau = 1$; Schemes 4 and 5: $\tau = 2$. For the seasonal frequency ω_k, $k = 1, ..., \lfloor S/2 \rfloor$, tests: Schemes 1, 2, and 4: $\tau = 0$; Schemes 3 and 5: $\tau = 1$.[6] All of the large sample results which follow hold regardless of whether the WPE or ASDE of $f_u(\lambda)$ is used.

In Theorem 1 we first present results for the limiting distributions of the elements of $T\hat{\beta}$. Throughout this paper the notation "$\Rightarrow$" is used to denote weak convergence as $T \to \infty$.

Theorem 1 *Let X_t be generated by* (2.1)-(2.2) *under Assumption 1. If the estimator $\hat{\beta}$ in* (3.5) *is constructed using the WPE estimator from* (3.7) *let Assumption 2 hold. Alternatively, if $\hat{\beta}$ is constructed using the ASDE estimator from* (3.8) *let Assumption 3 hold. Then, the normalised elements of $\hat{\beta}$ under $H_{1,\nu} : \nu = (\nu_0, \nu_1, ..., \nu_{\lfloor S/2 \rfloor})'$ are such that:*

$$T\hat{\beta}_j \Rightarrow \nu_j + \frac{\int_0^1 J_{j,\nu_j}^{\tau}(r) dW_j(r)}{\int_0^1 J_{j,\nu_j}^{\tau}(r)^2 dr}, \quad j = 0, S/2 \tag{3.9a}$$

$$T\hat{\beta}_k^c \Rightarrow \nu_k + \frac{2\int_0^1 \left(J_{k,\nu_k,c}^{\tau}(r) dW_{k,c}(r) + J_{k,\nu_k,s}^{\tau}(r) dW_{k,s}(r) \right)}{\int_0^1 \left(J_{k,\nu_k,c}^{\tau}(r)^2 + J_{k,\nu_k,s}^{\tau}(r)^2 \right) dr}, \quad k = 1, \dots, S^*, \tag{3.9b}$$

$$T\hat{\beta}_k^s \Rightarrow \frac{2\int_0^1 \left(J_{k,\nu_k,s}^{\tau}(r) dW_{k,c}(r) - J_{k,\nu_k,c}^{\tau}(r) dW_{k,s}(r) \right)}{\int_0^1 \left(J_{k,\nu_k,c}^{\tau}(r)^2 + J_{k,\nu_k,s}^{\tau}(r)^2 \right) dr}, \quad k = 1, \dots, S^*, \tag{3.9c}$$

where W_0, $W_{S/2}$, $W_{k,c}$ and $W_{k,s}$, $k = 1, ..., S^*$, are mutually independent standard Brownian motions, while J_{0,ν_0}^{τ}, $J_{S/2,\nu_{S/2}}^{\tau}$, $J_{k,\nu_k,c}^{\tau}$ and $J_{k,\nu_k,s}^{\tau}$, $k = 1, ..., S^*$, are mutually independent functionals of these Brownian motions whose precise form depends on the de-trending index τ and on whether x_t is formed using OLS de-trending or QD de-trending. In the case of OLS de-trending: for $\tau = 0$ these are standard

[6] So, for example, substituting $\tau = 0$ into the expression given in (3.9a) of Theorem 1 for t_0 gives the limiting representation for the t_0 statistic under Scheme 1, whereas $\tau = 1$ gives the limiting representation which obtains under Schemes 2 and 3. Similarly, substituting $\tau = 1$ into (3.9b) and (3.9c) gives the limiting representations for the t_k^c and t_k^s statistics, respectively, under Schemes 3, 4 and 5.

Ornstein-Uhlenbeck [OU] processes, viz.,

$$J_{j,\nu_j}^0(r) := \int_0^r \exp(\nu_j(r-\lambda))dW_j(\lambda), \quad j = 0, S/2 \tag{3.10a}$$

$$J_{k,\nu_k,c}^0(r) := \int_0^r \exp(\nu_k(r-\lambda))dW_{k,c}(\lambda), \quad k = 1, ..., S^* \tag{3.10b}$$

$$J_{k,\nu_k,s}^0(r) := \int_0^r \exp(\nu_k(r-\lambda))dW_{k,s}(\lambda), \quad k = 1, ..., S^*, \tag{3.10c}$$

for $\tau = 1$ these are de-meaned standard OU processes, so that for example $J_{0,\nu_0}^1(r) := J_{0,\nu_0}^0(r) - \int_0^1 J_{0,\nu_0}^0(\lambda)d\lambda$, and for $\tau = 2$, J_{0,ν_0}^2 is the de-meaned and de-trended standard OU process, $J_{0,\nu_0}^2(r) := J_{0,\nu_0}^1(r) - 12\left(r - \frac{1}{2}\right)\int_0^1 \left(\lambda - \frac{1}{2}\right) J_{0,\nu_0}^1(\lambda)d\lambda$. For QD de-trending they are standard OU processes for both $\tau = 0$ and $\tau = 1$, as given above, while for $\tau = 2$,

$$J_{0,\nu_0}^2(r) := J_{0,\nu_0}^0(r) - r\left\{\frac{(1-\bar{\nu}_0)J_{0,\nu_0}^0(1) + \bar{\nu}_0^2\int_0^1 \lambda J_{0,\nu_0}(\lambda)d\lambda}{1 - \bar{\nu}_0 + \bar{\nu}_0^2/3}\right\}. \tag{3.11}$$

Remark 4: As Theorem 1 shows, the normalised FDR estimators possess pivotal limiting null distributions and asymptotic local power functions which, for a given value of τ (the de-trending index), depend only on the non-centrality parameter(s) being tested. For the case of OLS de-trending the representation in (3.9a) for $j = 0, S/2$, is equivalent to that given in Phillips (1987) for the non-seasonal ($S = 1$) case. For $\tau = 0, 1, 2$, asymptotic null critical values from these distributions are provided in Fuller (1996), while the associated power functions are graphed in Figures 1, 2 and 3, respectively, of Elliott *et al.* (1996).

Remark 5: Under the seasonal intercepts plus seasonal trends scheme of (2.3), the representation for the limiting distribution of $T\hat{\beta}_0$ is of the same form as given in (3.9a) for $\tau = 2$. The same is true of the limiting representation for $T\hat{\beta}_{S/2}$ where $J_{S/2,\nu_{S/2}}^2$ is now defined analogously to (3.11) in the case of QD de-trending, while it is the de-meaned and de-trended counterpart of $J_{S/2,\nu_{S/2}}^0$ under OLS de-trending. Under OLS de-trending the normalised harmonic frequency coefficient estimates, $T\hat{\beta}_k^c$ and $T\hat{\beta}_k^s$ converge to the form given in (3.9b) and (3.9c), with $J_{k,\nu_k,s}^2$ and $J_{k,\nu_k,s}^2$, respectively, the de-meaned and de-trended counterparts of $J_{k,\nu_k,s}^0$ and $J_{k,\nu_k,s}^0$, $k = 1, ..., S^*$.

Remark 6: It is seen from the results in Theorem 1 that the FDR method eradicates the dependence of the limiting distributions of the normalised coefficient estimates from the unaugmented HEGY regression in 3.1 on nuisance parameters characterising the dynamics of u_t. That the frequency domain estimator results in asymptotically pivotal distributions is a significant advantage over regression methods based on purely autoregressive representations, such as the HEGY procedure, which can be shown to admit limiting representations which depend on these serial correlation nuisance parameters; cf. Equation [17.7.35] of Hamilton (1994,p.523) for the case of the zero frequency OLS de-trended Dickey-Fuller normalised coefficient under the non-seasonal unit root null hypothesis.

An immediate consequence of Remark 6 is that the normalised vector $T\hat{\beta}$ could be used directly to test the hypotheses of interest. However, we choose to focus instead on the t- and F-type statistics that are more commonly employed. We now give the limiting distributions of these in Theorem 2.

Theorem 2 *Let the conditions of Theorem 1 hold. Then, under $H_{1,\nu} : \nu = (\nu_0, \nu_1, ..., \nu_{\lfloor S/2 \rfloor})'$:*

$$t_0 \Rightarrow \nu_j \left(\int_0^1 J_{j,\nu_j}^{\tau}(r)^2 dr \right)^{1/2} + \frac{\int_0^1 J_{j,\nu_j}^{\tau}(r) dW_j(r)}{\left(\int_0^1 J_{j,\nu_j}^{\tau}(r)^2 dr \right)^{1/2}} =: \Xi_{j,\nu_j}, \quad j = 0, S/2 \tag{3.12a}$$

$$t_k^c \Rightarrow \frac{\nu_k}{2} \mathcal{D}_k^{1/2} + \frac{\int_0^1 \left(J_{k,\nu_k,c}^{\tau}(r) dW_{k,c}(r) + J_{k,\nu_k,s}^{\tau}(r) dW_{k,s}(r) \right)}{\mathcal{D}_k^{1/2}} =: \Xi_{k,\nu_k}, \quad k = 1, \ldots, S^* \tag{3.12b}$$

$$t_k^s \Rightarrow \frac{\int_0^1 \left(J_{k,\nu_k,s}^{\tau}(r) dW_{k,c}(r) + J_{k,\nu_k,c}^{\tau}(r) dW_{k,s}(r) \right)}{\mathcal{D}_k^{1/2}} =: \Xi_{k,\nu_k,s}, \quad k = 1, \ldots, S^* \tag{3.12c}$$

where $\mathcal{D}_k := \int_0^1 \left(J_{k,\nu_k,c}^{\tau}(r)^2 + J_{k,\nu_k,s}^{\tau}(r)^2 \right) dr$, and where W_0, $W_{S/2}$, J_{0,ν_0}^{τ}, $J_{S/2,\nu_{S/2}}^{\tau}$ and $(W_{k,c}, W_{k,s}, J_{k,\nu_k,c}^{\tau}, J_{k,\nu_k,s}^{\tau})$, $k = 1, ..., S^*$, are as defined in Theorem 1. Moreover,

$$F_k \Rightarrow \frac{1}{2} \left\{ \Xi_{k,\nu_k}^2 + \Xi_{k,\nu_k,s}^2 \right\}, \quad k = 1, \ldots, S^* \tag{3.13a}$$

$$F_{1 \ldots \lfloor S/2 \rfloor} \Rightarrow \frac{1}{S-1} \left\{ \sum_{j=1}^{\lfloor S/2 \rfloor} \Xi_{j,\nu_j}^2 + \sum_{k=1}^{S^*} \Xi_{k,\nu_k,s}^2 \right\} \tag{3.13b}$$

$$F_{0 \ldots \lfloor S/2 \rfloor} \Rightarrow \frac{1}{S} \left\{ \sum_{j=0}^{\lfloor S/2 \rfloor} \Xi_{j,\nu_j}^2 + \sum_{k=1}^{S^*} \Xi_{k,\nu_k,s}^2 \right\} \tag{3.13c}$$

Remark 7: The results in Theorem 2 show that, as with the results in Theorem 1, all of the FDR statistics from section 3 possess pivotal limiting null distributions and asymptotic local power functions which, for a given value of τ (the de-trending index), depend only on the non-centrality parameter(s) being tested. This holds for both OLS de-trended data and QD de-trended data. These tests therefore provide a suitable basis for (asymptotic) inference. Unlike the corresponding lag-augmented standard HEGY tests from (2.10) which are derived under the assumption that the shocks follow an $AR(p)$ process with p finite, this holds under general weak dependence of the form given in Assumption 1 (the additional requirement of invertibility needed when using the ASDE).

Remark 8: In the case of OLS de-trended data, the limiting representations for the t_0, $t_{S/2}$, F_k, $k = 1, ..., S^*$, $F_{1...\lfloor S/2 \rfloor}$ and $F_{1...\lfloor S/2 \rfloor}$ statistics given in Theorem 2 coincide with the representations given for the corresponding lag-augmented HEGY statistics from (2.10) given in Rodrigues and Taylor (2004), albeit noting that Rodrigues and Taylor (2004) constrain the near-integration parameter to be common across the zero and seasonal frequencies while we do not, and that those representations are derived under the assumption that the shocks follow a finite $AR(p)$. Consequently, the discussion concerning those representations given in Remarks 3.1, 3.2, 3.5, 3.6 and 3.7 apply equally well to the FDR test statistics from section 3, including the reference to relevant asymptotic critical values given there. However, the representations for the limiting distributions of the lag-augmented t_k^{c*} and t_k^{s*}, $k = 1, ..., S^*$, HEGY statistics from (2.10) depend in general on the lag coefficients characterising the aforementioned $AR(p)$ polynomial; see equations (3.2) and (3.3) of Rodrigues and Taylor (2007,p.653). The use of asymptotic critical values relevant to the case where the shocks are IID cannot therefore deliver (asymptotically) valid inference for tests based on these statistics, while it can for the analogous frequency domain tests based on t_k^c and t_k^s, $k = 1, ..., S^*$.

Remark 9: Similar comments to those made in Remark 8 also apply to the frequency domain tests when based on QD de-trended data and, in particular, the asymptotic critical values given in Table 1 and footnote 5 of Rodrigues and Taylor (2007,pp.557 and 561) for the corresponding tests from (2.10) may be used. Remarks 5.2 and 5.3 from Rodrigues and Taylor (2007,pp.560-61) are also germane to the frequency domain approach of section 3. Tabulations of the asymptotic local power functions for t_k^c, F_k, $F_{1...\lfloor S/2 \rfloor}$ and $F_{1...\lfloor S/2 \rfloor}$ under QD de-trending are also, therefore, as given for the corresponding tests from (2.10) given in Table 5 of Rodrigues and Taylor (2007,p.565).

Remark 10: The results in Theorem 2 can again be generalised to the seasonal intercepts plus seasonal trends deterministic scheme of (2.3). Here one simply re-defines the limiting processes given in the representations in Theorem 2 in the same way as outlined for Theorem 1 in Remark 5. Moreover, under QD de-trending the harmonic frequency t-statistics, t_k^c and t_k^s in this case converge to the rather involved limiting functionals $A_{\tau,k,\nu_k,\bar{\nu}_k}$ and $B_{\tau,k,\nu_k,\bar{\nu}_k}$, respectively, given in Theorem 5.1 of Rodrigues and Taylor (2007,p.560), for $k = 1, ..., S^*$.

4 Numerical Results

In this Section we use Monte Carlo simulation methods to investigate the small sample properties (size under autocorrelated errors and power under stationary alternatives) of the FDR HEGY-type tests from section 3 for the case of quarterly data, $S = 4$, comparing these with the conventional HEGY tests from section 2.3.

In assessing the finite-sample size and power properties of these tests we report results for $N = 50$ and $N = 100$ for Scheme 3, where $\tau = 1$ for all reported tests. All tests were run at the nominal 0.05 level using finite sample critical values generated under the quarterly seasonal random walk null. The remaining deterministic cases and other nominal levels were also considered, as were the corresponding tests for other values of S, but in each case yielded qualitatively similar results to those reported. The reported simulations were programmed using the `rndnKMn` function of Gauss 9 with $50,000$ replications for each experiment. These programs are available on request.

For the FDR HEGY tests which employ the WPE, an appropriate bandwidth m must be chosen. One method is to keep m fixed at $[T^{\delta}]$, and in this case we let $\delta = 0.5$. Alternatively, automatic bandwidth selection methods can be used and here the methods of Lee (1997) and Ombao, Raz, Strawderman and von Sachs (2001) are considered. Both involve selecting m based on minimising a risk criterion. Lee (1997) is based on unbiased risk estimation and Ombao *et al.* (2001) uses a generalised cross-validation method. In each case we set the maximum possible bandwidth to be 40.

For the conventional HEGY tests and the corresponding FDR HEGY tests based on the ASDE, the lag augmentation polynomial in (2.10) was chosen *via* a data-dependent rule. As is commonly done in practice, we followed the *general-to-specific* approach outlined in Beaulieu and Miron (1993,pp.318-19), starting with a maximum lag order of $p^* = \lfloor p_{\max}(T/100)^{1/4} \rfloor$ in (2.10) and progressively deleting those lags which are insignificant at the 0.10 level, with the final fitted lag polynomial denoted $\hat{\phi}(z)$. Results are reported for $p_{\max} = 4$ and $p_{\max} = 12$. The resulting estimates were then used in constructing the ASDE in (3.8), as detailed in section 3. Both the WPE and ASDE were constructed using QD de-trended data. All tests based on OLS de-trended data were run using indirect de-trending, as in footnotes 3 and 4.

4.1 Size Properties

Table 1 reports empirical rejection frequencies for the standard HEGY tests, and Tables 2 and 3 the corresponding results for the WPE- and ASDE-based FDR tests, respectively, under the DGP,

$$X_t = X_{t-S} + U_t, \ t = 1, ..., T \tag{4.1a}$$

$$(1 - \phi L)U_t = (1 + \theta L^2)V_t \sim IN(0, 1), \quad t = -100, \ldots, T, \tag{4.1b}$$

with $X_j = 0$, $j = 1-S, \ldots, 0$. We consider the effects of $\phi = 0.9$, holding $\theta = 0$, and $\theta = \pm 0.6$, holding $\phi = 0$.[7] The first case allows for a large peak in the spectrum of $\{v_{4n+s}\}$ at the zero frequency, while the second induces a *near cancellation of roots* at both the zero and Nyquist frequencies for $\theta = -0.6$, and at the harmonic seasonal frequency for $\theta = 0.6$.

A comparison of the results in Table 1 with those in Tables 2 and 3 shows that in most (but not all) cases the conventional HEGY tests display superior finite sample size control than their FDR analogues in both the OLS and QD de-trending environments. Moreover, while the ASDE-based FDR tests display size patterns which are mostly not too dissimilar to the standard HEGY tests, the WPE-based FDR tests display very poor size control throughout, pretty much regardless of which of the three bandwidth selection methods is used. Indeed for this reason we will not consider the WPE-based FDR tests any further. In contrast, the maximum lag order, $p_{\max}$, used in connection with the standard HEGY and ASDE-based FDR HEGY tests, can have a marked impact on the size properties of both of these tests. In the case of MA errors the size properties of these tests are considerably improved for $p_{\max} = 12$ *vis-à-vis* $p_{\max} = 4$, as might be expected. For these tests, the observed size distortions are also smaller, other things equal, for $N = 100$ than for $N = 50$. Distortions also appear to be generally smaller, other things being equal, for the QD de-trended tests than for their OLS de-trended counterparts.

4.2 Empirical Power

We now compare the finite sample power properties of the conventional HEGY tests and their ASDE-based FDR analogues against the near-seasonally integrated DGP:

$$\left[1 - \left(1 + \frac{\nu_0}{T}\right) L\right] \left[1 + \left(1 + \frac{\nu_2}{T}\right) L\right] \left[1 + \left(1 + \frac{\nu_1}{T}\right)^2 L^2\right] X_t = U_t \sim IN(0, 1), \tag{4.2}$$

[7] Other parameter values were considered but qualitatively did not add to or contradict what is reported.

$t = 1, ..., T$ with $X_j = U_j = 0$, $j \leq 0$. We investigate the effects of varying the non-centrality parameters ν_k among $\nu_k \in \{-3, -5, -7, -11, -15, -19\}$, $k = 0, 1, 2$, in our experiments. Results for the conventional HEGY tests are reported in Table 4 and the corresponding results for the ASDE-based FDR tests are reported in Table 5. These results pertain to the case where, when moving a particular non-centrality parameter v_k, $k = 0, 1, 2$, away from unity, the remaining non-centrality parameters are all held at zero.[8] The shocks, U_t, are set to be serially uncorrelated so that one can compare the powers of the two approaches from a common base of exact 5% sized tests. This also implies that the t_1^s tests should not reject with probability in excess of the nominal level in the limit, regardless of the values of the ν_k, $k = 0, 1, 2$; see, for example, Rodrigues and Taylor (2004,2007).

A comparison of the results in Tables 4 and 5 shows a clear picture. Under OLS de-trending, in almost all cases the ASDE-based FDR HEGY tests display significantly higher power than their conventional HEGY test analogues. In the case of QD de-trending there is little to choose between the tests, mirroring the smaller differences seen between the sizes of the two approaches noted in section 4.1. In most cases power is lower for $p_{\max} = 12$ *vis-à-vis* $p_{\max} = 4$, although the losses tend to be rather moderate. Power is higher, often substantially so, for the QD de-trended variants of the tests than the OLS de-trended variants, as might be expected from the results in Rodrigues and Taylor (2007). The relative power performance of the standard HEGY tests and their ASDE-based FDR counterparts, taken together with their relative size performance reported in section 4.1 suggests that a very useful finite sample size-power trade-off exists between these classes of tests. While the former tend to display better size control for a given value of $p_{\max}$, the latter tend to display better power properties, again for a given value of $p_{\max}$. This trade-off is most pronounced for the case of tests based on OLS de-trended data but also exists, albeit it to a far lesser extent, in the case of QD de-trended data.

5 Conclusions

In this paper we have proposed new regression-based tests for seasonal unit roots based on spectral (frequency domain) regression estimation methods. A key aspect of this approach is that any serial correlation present in the shocks is treated non-parametrically, rather than parametrically as in the corresponding lag-augmented tests of Hylleberg *et al.* (1990), *inter alia*. We have shown that all of our proposed statistics retain pivotal limiting distributions under both the null and near seasonally

[8] Allowing the other non-centrality parameters to simultaneously deviate from zero had little effect.

integrated alternatives in the presence of weakly dependent (linear process) shocks. This contrasts with the lag-augmented HEGY tests not all of which retain pivotal limit distributions under weak dependence and, moreover, are only asymptotically valid in the case where the data are generated according to a finite-order autoregression. We have used Monte Carlo methods to compare the size and power properties of our proposed frequency domain regression-based tests, using either a weighted periodogram or an autorgressive spectral density estimator of the spectrum of the shocks, with those of the lag-augmented tests of Hylleberg *et al.* (1990). These simulations suggested that the weighted periodogram-based tests display very poor size control, but highlighted an interesting size-power trade-off between the autoregressive spectral density variant of the frequency domain regression HEGY tests and their conventional analogues, with the former tending to display slightly worse size control in general, but not always, than the latter but significantly better power properties overall. This trade-off was most pronounced in the case of OLS de-trending.

A Appendix A

It is convenient to define some additional notation and some representations that form the basis of the proofs. Under the hypotheses H_{1,ν_k} $(k = 0, 1, \ldots, S^*, S/2)$ the coefficients $\alpha_k = (1 + \nu_k/T)$ will be replaced by $\alpha_k = e^{\nu_k/T}$ as in Phillips (1987) for the $k = 0$ case; the asymptotics remain the same by noting that $e^{\nu_k/T} = 1 + \nu_k/T + O(T^{-2})$. The following partial sum processes play a prominent role in the development of the asymptotics:

$$P_{0,t} := \sum_{j=1}^{t} e^{\nu_0(t-j)/T} u_j, \quad P_{S/2,t} := \sum_{j=1}^{t} e^{\nu_{S/2}(t-j)/T} (-1)^j u_j, \tag{A.1a}$$

$$P_{k,t} := P_{k,t}^c + iP_{k,t}^s = \sum_{j=1}^{t} e^{\nu_k(t-j)/T} e^{ij\omega_k} u_j, \quad k = 1, \ldots, S^*. \tag{A.1b}$$

In particular the variables in the HEGY and FDRs can be represented as:

$$x_{0,t} = P_{0,t} + e^{\nu_0 t/T} x_{0,0}, \quad x_{S/2,t} = P_{S/2,t} + e^{\nu_{S/2} t/T} (-1)^t x_{S/2,0}, \tag{A.2a}$$

$$x_{k,t} = x_{k,t}^c + ix_{k,t}^s = e^{-i(t+1)\omega_k} \left[P_{k,t} + e^{\nu_k t/T} x_{k,0} \right], \quad k = 1, \ldots, S^*. \tag{A.2b}$$

In order to save on notation the superscript τ relating to Schemes 1–5 and the frequency under test in the limiting O-U and Wiener processes will be omitted.

Lemma 1. *Under Assumption 1,*

$$\frac{1}{\sqrt{T}}P_{k,\lfloor Tr\rfloor} \Rightarrow \begin{cases} \sigma c(1)J_{0,\nu_0}(r), & k=0, \\ \frac{\sigma}{\sqrt{2}}c(e^{i\omega_k})J_{k,\nu_k}(r), & k=1,\ldots,S^*, \\ \sigma c(-1)J_{S/2,\nu_{S/2}}(r), & k=S/2, \end{cases}$$

where $r\in[0,1]$, $J_{k,\nu_k}(r) := J_{k,\nu_k,c}(r)+iJ_{k,\nu_k,s}(r)$, *and* $J_{0,\nu_0}(r)$, $J_{S/2,\nu_{S/2}}(r)$, $J_{k,\nu_k,c}(r)$ *and* $J_{k,\nu_k,s}(r)$ *are as defined in Theorem 1.*

Proof of Lemma 1. The $k=0$ case follows immediately from Lemma 1 of Phillips (1987). For $k=1,\ldots,S^*$ define the (complex-valued) random variables $\xi_{k,t} := e^{it\omega_k}u_t$ so that $P_{k,t} = \sum_{j=1}^{t} e^{\nu_k(t-j)/T}\xi_{k,j}$. Note that $\xi_{k,t}$ has the representation $\xi_{k,t} = c_k(L)\epsilon_{k,t}$ where $c_k(z) = \sum_{j=0}^{\infty} c_{k,j}$, $c_{k,j} = c_j e^{ij\omega_k}$ and $\epsilon_{k,t} = e^{it\omega_k}\epsilon_t$. The random variable $\epsilon_{k,t}$ satisfies $E(\epsilon_{k,t})=0$, $E(\epsilon_{k,t}\overline{\epsilon_{k,t}})=\sigma^2$ and $E(\epsilon_{k,t}\overline{\epsilon_{k,s}})=0$ for $t\neq s$; the long-run variance of $\xi_{k,t}$ is equal to $\sigma^2|c_k(1)|^2 = \sigma^2|c(e^{i\omega_k})|^2$. It follows that $P_{k,\lfloor Tr\rfloor}$ satisfies the stated invariance principle, where $J_{k,\nu_k}(r) := \int_0^r e^{\nu_k(r-q)}dW_k(q)$ and $W_k(q) := W_{k,c}(q)+iW_{k,s}(q)$ is a complex-valued Wiener process with $E(W_k(q)\overline{W_k(q)})=2$. The result for $k=S/2$ follows immediately by taking the real part with $\omega_k=\pi$ and noting that $e^{i\pi}=-1$. □

Lemma 2. *Define* $z_t := \left[x_{0,t-1}, x^c_{1,t-1}, x^s_{1,t-1}, \ldots, x^c_{S^*,t-1}, x^s_{S^*,t-1}, x_{S/2,t-1}\right]'$ *and* $C_{zz}(n) := T^{-1}\sum_{1\le t,t+n\le T} z_t z'_{t+n}$. *Then, under Assumption 1,* $T^{-1}C_{zz}(n) \Rightarrow G(n)$, *where* $G(n) := \mathrm{diag}[G_0(n), G_1(n), ..., G_{S^*}(n), G_{S/2}(n)]$, *with* $G_0(n) := \sigma^2 c(1)^2\int_0^1 J_{0,\nu_0}(r)^2dr$, $G_{S/2}(n) := (-1)^n\sigma^2c(-1)^2\int_0^1 J_{S/2,\nu_{S/2}}(r)^2dr$, *and, for* $k = 1,\ldots,S^*$,

$$G_k(n) := \frac{1}{4}\sigma^2\left|c(e^{i\omega_k})\right|^2\int_0^1\left(J_{k,\nu_k,c}(r)^2+J_{k,\nu_k,s}(r)^2\right)dr\begin{bmatrix}\cos n\omega_k & -\sin n\omega_k\\ \sin n\omega_k & \cos n\omega_k\end{bmatrix}.$$

Proof of Lemma 2. For $n=0$ the fact that $T^{-2}\sum_t x^2_{0,t-1}\Rightarrow G_0(0)$ follows from Lemma 1 of Phillips (1987). For $n\neq 0$ note that $\frac{1}{T^2}\sum_t x_{0,t-1}x_{0,t+n-1} = e^{\nu_0 n/T}\frac{1}{T^2}\sum_t x^2_{0,t-1}+o_p(1)$, and the result follows because $e^{\nu_0 n/T}\to 1$. For $k=1,\ldots,S^*$ note that

$$x_{k,t+n} = e^{-i(t+n+1)\omega_k}P_{k,t+n} + e^{-i(t+n+1)\omega_k}e^{\nu_k(t+n)/T}x_{k,0} \tag{A.3}$$

while $P_{k,t+n} = e^{\nu_k n/T}P_{k,t} + \sum_{j=t+1}^{t+n} e^{\nu_k(t+n-j)/T}\xi_{k,j}$. Combining these expressions yields $x_{k,t+n} =$

$e^{-in\omega_k} e^{\nu_k n/T} x_{k,t} + e^{-i(t+n+1)\omega_k} \sum_{j=t+1}^{t+n} e^{\nu_k (t+n-j)/T} \xi_{k,j}$, from which it follows that

$$\begin{aligned}\frac{1}{T^2}\sum_t x_{k,t-1}x_{k,t+n-1} &= e^{-in\omega_k}e^{\nu_k n/T}\frac{1}{T^2}\sum_t x_{k,t-1}^2 + o_p(1),\\ \frac{1}{T^2}\sum_t x_{k,t-1}\overline{x_{k,t+n-1}} &= e^{in\omega_k}e^{\nu_k n/T}\frac{1}{T^2}\sum_t x_{k,t-1}\overline{x_{k,t-1}} + o_p(1).\end{aligned}$$

Setting $n=0$ and lagging by one period in (A.3) we find that

$$\begin{aligned}\frac{1}{T^2}\sum_t x_{k,t-1}^2 &= \frac{1}{T^2}\sum_t e^{-2it\omega_k}P_{k,t-1}^2 + 2x_{k,0}\frac{1}{T^2}\sum_t e^{-2it\omega_k}e^{\nu_k t/T}P_{k,t-1}\\ &\quad + x_{k,0}^2\frac{1}{T^2}\sum_t e^{-2it\omega_k}e^{2\nu_k t/T} = \frac{1}{T^2}\sum_t e^{-2it\omega_k}P_{k,t-1}^2 + o_p(1) \Rightarrow 0\end{aligned}$$

from an extension of Lemma 3.3.6 of Chan and Wei (1988) and Lemma A.1 of Gregoir (2006). Similarly,

$$\begin{aligned}\frac{1}{T^2}\sum_t x_{k,t-1}\overline{x_{k,t-1}} &= \frac{1}{T^2}\sum_t P_{k,t-1}\overline{P_{k,t-1}} + x_{k,0}\frac{1}{T^2}\sum_t e^{\nu_k t/T}\overline{P_{k,t-1}}\\ &\quad + \frac{1}{T^2}\sum_t e^{\nu_k t/T}P_{k,t-1}\overline{x_{k,0}} + \frac{1}{T^2}\sum_t e^{2\nu_k t/T}x_{k,0}\overline{x_{k,0}}\\ &= \frac{1}{T^2}\sum_t P_{k,t-1}\overline{P_{k,t-1}} + o_p(1) \Rightarrow \frac{1}{2}\sigma^2\left|c(e^{i\omega_k})\right|^2\int_0^1 J_{k,\nu_k}(r)\overline{J_{k,\nu_k}(r)}dr.\end{aligned}$$

It follows that $T^{-2}\sum_t x_{k,t-1}x_{k,t+n-1} \Rightarrow 0$ while

$$\frac{1}{T^2}\sum_t x_{k,t-1}\overline{x_{k,t+n-1}} \Rightarrow e^{in\omega_k}\frac{1}{2}\sigma^2\left|c(e^{i\omega_k})\right|^2\int_0^1 J_{k,\nu_k}(r)\overline{J_{k,\nu_k}(r)}dr =: \Phi_k := \Phi_{k,c} + i\Phi_{k,s}.$$

Using these two results and the fact that

$$\begin{aligned}x_{k,t-1}\overline{x_{k,t+n-1}} &= x_{k,t-1}^c x_{k,t+n-1}^c + x_{k,t-1}^s x_{k,t+n-1}^s + i(x_{k,t-1}^s x_{k,t+n-1}^c - x_{k,t-1}^c x_{k,t+n-1}^s),\\ x_{k,t-1}x_{k,t+n-1} &= x_{k,t-1}^c x_{k,t+n-1}^c - x_{k,t-1}^s x_{k,t+n-1}^s + i(x_{k,t-1}^s x_{k,t+n-1}^c + x_{k,t-1}^c x_{k,t+n-1}^s),\end{aligned}$$

we find that $T^{-2}\sum_t x_{kt}^s x_{k,t+n}^s \Rightarrow \frac{1}{2}\Phi_k^c$, $T^{-2}\sum_t x_{kt}^c x_{k,t+n}^c \Rightarrow \frac{1}{2}\Phi_k^c$, $T^{-2}\sum_t x_{kt}^s x_{k,t+n}^c \Rightarrow \frac{1}{2}\Phi_k^s$, and $T^{-2}\sum_t x_{kt}^c x_{k,t+n}^s \Rightarrow -\frac{1}{2}\Phi_k^s$. The stated result then follows immediately. Finally, the properties involving $x_{S/2,t}$ are a special case of the above obtained by taking the real part of the limiting distribution, setting $\omega_k = \pi$ and noting that $\cos n\pi = (-1)^n$. □

Lemma 3. *Let* $C_{zu}(n) := T^{-1}\sum_{1\le t,t+n\le T} z_t u_{t+n}$ *and define* $\gamma_j := E(u_0 u_j)$ *and* $e_i := (1,i)'$. *Then,* *under Assumption 1,* $C_{zu}(n) \Rightarrow g(n)$*, where* $g(n) := \left[g_0(n), g_1(n)', \ldots, g_{S^*}(n)', g_{S/2}(n)\right]'$,

$$
\begin{aligned}
g_0(n) &:= \sigma^2 c(1)^2 \int_0^1 J_{0,\nu_0}(r) dW_0(r) + \sum_{j=n+1}^{\infty} \gamma_j, \\
e_i' g_k(n) &:= e^{in\omega_k} \frac{\sigma^2 |c(e^{i\omega_k})|^2}{2} \int_0^1 J_{k,\nu_k}(r) d\overline{W_k(r)} + \sum_{j=n+1}^{\infty} e^{i(n-j)\omega_k} \gamma_j, \quad k = 1, \ldots, S^*, \\
g_{S/2}(n) &:= \cos(n\pi) \sigma^2 c(-1)^2 \int_0^1 J_{S/2,\nu_{S/2}}(r) dW_{S/2}(r) + \sum_{j=n+1}^{\infty} \cos[(n-j)\pi] \gamma_j.
\end{aligned}
$$

Proof of Lemma 3. From Lemma 1 of Phillips (1987) $T^{-1} \sum_t x_{0,t-1} u_t \Rightarrow g_0(0)$. Now $x_{0,t} = e^{\nu_0/T} x_{0,t-1} + u_t$ so that $x_{0,t+n-1} = e^{n\nu_0/T} x_{0,t-1} + \sum_{\ell=0}^{n-1} e^{\ell\nu_0/T} u_{t+n-1-\ell}$ for $n > 0$. Solving this expression for $x_{0,t-1}$ it follows that

$$
\begin{aligned}
\frac{1}{T} \sum_t x_{0,t-1} u_{t+n} &= e^{-n\nu_0/T} \left\{ \frac{1}{T} \sum_t x_{0,t+n-1} u_{t+n} - \sum_{\ell=0}^{n-1} e^{\ell\nu_0/T} \frac{1}{T} \sum_t u_{t+n-1-\ell} u_{t+n} \right\} \\
&\Rightarrow \sigma^2 c(1)^2 \int_0^1 J_{0,\nu_0}(r) dW_0(r) + \sum_{j=1}^{\infty} \gamma_j - \sum_{\ell=0}^{n-1} \gamma_{\ell+1} = g_0(n).
\end{aligned}
$$

For $k = 1, \ldots, S^*$, usual arguments (e.g. extending Lemma A.1 of Gregoir, 2006) establish that $T^{-1} \sum_t x_{k,t-1} u_t \Rightarrow g_k(0)$. Noting that $x_{k,t-1} = e^{-n\nu_k/T} \left\{ e^{in\omega_k} x_{k,t+n-1} - e^{-it\omega_k} \sum_{\ell=t}^{t+n-1} e^{\nu_k(t+n-1-\ell)/T} \xi_{k,\ell} \right\}$, we obtain that

$$
\frac{1}{T} \sum_t x_{k,t-1} u_{t+n} = e^{-n\nu_0/T} \left\{ e^{in\omega_k} \frac{1}{T} \sum_t x_{k,t+n-1} u_{t+n} - \frac{1}{T} \sum_t e^{-it\omega_k} \sum_{\ell=t}^{t+n-1} e^{\nu_k(t+n-1-\ell)/T} \xi_{k,\ell} u_{t+n} \right\}.
$$

Recalling that $\xi_{k,t} = e^{it\omega_k} u_t$, the second term in parentheses above can be written as

$$
-\sum_{j=1}^{n} e^{\nu_k(j-1)/T} e^{i(n-j)\omega_k} \frac{1}{T} \sum_t u_{t+n-j} u_{t+n}
$$

and converges in probability to $-\sum_{j=1}^{n} e^{i(n-j)\omega_k} \gamma_j$; hence $T^{-1} \sum_t x_{k,t-1} u_{t+n} \Rightarrow g_k(n)$ as stated. Picking out the real and imaginary parts this result can also be written in the form

$$
\begin{aligned}
\frac{1}{T}\sum_t \begin{pmatrix} x_{k,t-1}^c \\ x_{k,t-1}^s \end{pmatrix} u_{t+n} &= \frac{\sigma^2|c(e^{i\omega_k}|^2}{2}\begin{bmatrix} \cos n\omega_k & -\sin n\omega_k \\ \sin n\omega_k & \cos n\omega_k \end{bmatrix} \\
&\times \begin{bmatrix} \int_0^1 (J_{k,\nu_k,c}(r)dW_{k,c}(r) + J_{k,\nu_k,s}(r)dW_{k,s}(r)) \\ \int_0^1 (J_{k,\nu_k,s}(r)dW_{k,c}(r) - J_{k,\nu_k,c}(r)dW_{k,s}(r)) \end{bmatrix} \\
&+ \sum_{j=n+1}^{\infty} \begin{pmatrix} \cos(n-j)\omega_k \\ \sin(n-j)\omega_k \end{pmatrix} \gamma_j.
\end{aligned}
$$

Finally, the result for $g_{S/2}(n)$ follows the same lines as above by setting $\omega_k = \pi$. Analogous arguments apply when $n < 0$. □

Lemma 4. *Let* $H_T := \sum_{j\in J_T} I_{zz}(\lambda_j) f_u(\lambda_j)^{-1}$. *Then, under Assumption 1,* $T^{-2}H_T \Rightarrow H$, *where* $H :=$ diag$[H_0, H_1, ..., H_{S^*}, H_{S/2}]$, *with* $H_j := \int_0^1 J_{j,\nu_j}(r)^2 dr$, $j = 0, S/2$, *and* $H_k := \frac{1}{4} I_2 \int_0^1 \left(J_{k,\nu_k,c}(r)^2 + J_{k,\nu_k,s}(r)^2\right) dr$, $k = 1, \ldots, S^*$, *and where* I_2 *denotes the* 2×2 *identity matrix.*

Proof of Lemma 4. Let $\phi(\lambda) := f_u(\lambda)^{-1}$ and consider its Nth first-order Cesàro mean $\phi_N(\lambda) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\mu) F_N(\lambda - \mu) d\mu$, where $F_N(\lambda) := N^{-1}\left|\sum_{n=0}^{N-1} e^{in\lambda}\right|^2 = \sum_{|n|<N}(1 - |n|N^{-1})e^{in\lambda}$ is Fejér's kernel. Then, for N sufficiently large, $\sup_\lambda |\phi(\lambda) - \phi_N(\lambda)| < \epsilon$. Consider $H_T(\phi) := \sum_{j\in J_T} I_{zz}(\lambda_j)\phi(\lambda_j)$. Then

$$
\begin{aligned}
\left\|T^{-2}H_T(\phi) - T^{-2}H_T(\phi_N)\right\| &= \left\| T^{-2}\sum_{j\in J_T} I_{zz}(\lambda_j)\left[\phi(\lambda_j) - \phi_N(\lambda_j)\right]\right\| \\
&\leq \epsilon T^{-2}\sum_{j\in J_T}\|I_{zz}(\lambda_j)\| \leq \epsilon(2\pi T)^{-1}\mathrm{tr}\left(T^{-1}\sum_{t=1}^{T} z_t z_t'\right) = O_p(\epsilon),
\end{aligned}
$$

using an inequality of Robinson (1972, p.764) and Lemma 2. Since ϵ is arbitrary, we can replace ϕ by ϕ_N. Then, following Robinson (1976, pp.231–232),

$$
\begin{aligned}
T^{-2}H_T(\phi_N) &= \frac{1}{T^2}\sum_{j\in J_T} I_{zz}(\lambda_j)\phi_N(\lambda_j) \\
&= \frac{1}{T^2}\sum_{j\in J_T}\left(\frac{1}{2\pi}\sum_{l=-T+1}^{T-1} C_{zz}(l)e^{-il\lambda_j}\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(\mu)\sum_{|n|<N}\left(1 - \frac{|n|}{N}\right)e^{in(\lambda_j-\mu)}d\mu\right)
\end{aligned}
$$

$$= \left(\frac{1}{2\pi}\right)^2 \frac{1}{T^2} \sum_{j\in J_T} \sum_{l=-T+1}^{T-1} \sum_{|n|<N} \int_{-\pi}^{\pi} C_{zz}(l) \left(1 - \frac{|n|}{N}\right) \phi(\mu) e^{i(n-l)\lambda_j - in\mu} d\mu$$

$$= \left(\frac{1}{2\pi}\right)^2 \frac{1}{T} \sum_{|n|<N} C_{zz}(n) \int_{-\pi}^{\pi} \left(1 - \frac{|n|}{N}\right) \phi(\mu) e^{-in\mu} d\mu + o_p(1).$$

From Lemma 2 we know that $T^{-1}C_{zz}(n) \Rightarrow G(n)$. Let $H_{T,k}(\phi_N)$ $(k = 0, 1, \ldots, S^*, S/2)$ denote the elements and 2×2 sub-matrices of $H_T(\phi_N)$ corresponding to the non-zero terms of $G(n)$. Taking each in turn we find that:

$$\begin{aligned} T^{-2}H_{T,0}(\phi_N) &\Rightarrow \sigma^2 c(1)^2 \int_0^1 J_{0,\nu_0}(r)^2 dr \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \phi(\mu) \sum_{|n|<N} \left(1 - \frac{|n|}{N}\right) e^{-in\mu} d\mu \\ &= \sigma^2 c(1)^2 \int_0^1 J_{0,\nu_0}(r)^2 dr \frac{1}{2\pi} \phi_N(0). \end{aligned}$$

But $\left|\sigma^2 c(1)^2 \int_0^1 J_{0,\nu_0}(r)^2 dr (1/2\pi) \left[\phi_N(0) - f_u(0)^{-1}\right]\right| \leq (\epsilon/2\pi)\sigma^2 c(1)^2 \int_0^1 J_{0,\nu_0}(r)^2 dr = O_p(\epsilon)$, so we can replace $\phi_N(0)$ by $f_u(0)^{-1}$. Noting that $(1/2\pi) f_u(0)^{-1} = [\sigma^2 c(1)^2]^{-1}$ yields the limit $\int_0^1 J_{0,\nu_0}(r)^2 dr$. Next, following the same arguments as above and noting that $(-1)^n = e^{in\pi}$, we find that $T^{-2}H_{T,S/2}(\phi_N) \Rightarrow \sigma^2 c(-1)^2 \int_0^1 J_{S/2,\nu_{S/2}}(r)^2 dr (1/2\pi) \phi_N(\pi)$. But

$$\left|\sigma^2 c(-1)^2 \int_0^1 J_{S/2,\nu_{S/2}}(r)^2 dr \frac{1}{2\pi} \left[\phi_N(\pi) - f_u(\pi)^{-1}\right]\right| \leq \frac{\epsilon}{2\pi} \sigma^2 c(-1)^2 \int_0^1 J_{S/2,\nu_{S/2}}(r)^2 dr = O_p(\epsilon),$$

so we can replace $\phi_N(\pi)$ by $f_u(\pi)^{-1}$. Noting that $(1/2\pi) f_u(\pi)^{-1} = [\sigma^2 c(-1)^2]^{-1}$ yields the required limit. Finally, for the remaining non-zero elements $(k = 1, \ldots, S^*)$ we find that

$$\begin{aligned} T^{-2}H_{T,k}(\phi_N) &\Rightarrow \frac{\sigma^2}{4} \left|c(e^{i\omega_j})\right|^2 \int_0^1 \left(J_{k,\nu_k,c}(r)^2 + J_{k,\nu_k,s}(r)^2\right) dr \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \phi(\mu) \\ &\quad \times \sum_{|n|<N} \left(1 - \frac{|n|}{N}\right) e^{-in\mu} d\mu \begin{bmatrix} \cos n\omega_k & -\sin n\omega_k \\ \sin n\omega_k & \cos n\omega_k \end{bmatrix}. \end{aligned}$$

Note that $\cos(n\omega_k)e^{-in\mu} = \mathrm{Re}\{e^{in(\omega_k - \mu)}\}$, $\sin(n\omega_k)e^{-in\mu} = \mathrm{Im}\{e^{in(\omega_k-\mu)}\}$ and, furthermore that, $(1/2\pi) \int_{-\pi}^{\pi} \phi(\mu) \sum_{|n|<N} (1 - |n|N^{-1}) e^{in(\omega_k - \mu)} d\mu = \phi_N(\omega_k)$ so that, by similar arguments to above, we can replace $\phi_N(\omega_k)$ by $f_u(\omega_k)^{-1}$, whose imaginary part is zero. But the real part is $(2\pi/\sigma^2)|c(e^{i\omega_k})|^{-2}$, and so $T^{-2}H_{T,k}(\phi_N) \Rightarrow \frac{1}{4} I_2 \int_0^1 \left(J_{k,\nu_k,c}(r)^2 + J_{k,\nu_k,s}(r)^2\right) dr$, as required. □

Lemma 5. *Let* $h_T := \sum_{j\in J_T} I_{zu} f_u(\lambda_j)^{-1}$. *Then, under Assumption 1,* $T^{-1}h_T \Rightarrow h$, *where* $h :=$

$\left[h_0, h_1', \ldots, h_{S^*}', h_{S/2}\right]'$, $h_j := \int_0^1 J_{j,\nu_j}(r)dW_j(r)$ $(j = 0, S/2)$, *and*

$$h_k := \left[\begin{array}{c} \frac{1}{2}\int_0^1 \left(J_{k,\nu_k,c}(r)dW_{k,c}(r) + J_{k,\nu_k,s}(r)dW_{k,s}(r)\right) \\ \frac{1}{2}\int_0^1 \left(J_{k,\nu_k,s}(r)dW_{k,c}(r) - J_{k,\nu_k,c}(r)dW_{k,s}(r)\right) \end{array} \right], \quad k = 1, \ldots, S^*.$$

Proof of Lemma 5. Let $h_T(\phi) := \sum_{j\in J_T} I_{zu}(\lambda_j)\phi(\lambda_j)$ where $\phi(\lambda) := f_u(\lambda)^{-1}$. Then, with $\phi_N(\lambda)$ denoting the Nth first-order Cesàro mean of $\phi(\lambda)$, as in the proof of Lemma 4,

$$\left\|T^{-1}h_T(\phi) - T^{-1}h_T(\phi_N)\right\| \leq \epsilon T^{-1}\sum_{j\in J_T}\|I_{zu}(\lambda_j)\| = O_p(\epsilon)$$

by Lemma A of Chambers and McCrorie (2007). Hence, because ϵ is arbitrary, we can replace ϕ by ϕ_N, obtaining

$$\begin{aligned} T^{-1}h_T(\phi_N) &= \frac{1}{T}\sum_{j\in J_T} I_{zu}(\lambda_j)\phi_N(\lambda_j) \\ &= \frac{1}{T}\sum_{j\in J_T}\left(\frac{1}{2\pi}\sum_{l=-T+1}^{T-1} C_{zu}(l)e^{-il\lambda_j}\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(\mu)\sum_{|n|<N}\left(1-\frac{|n|}{N}\right)e^{in(\lambda_j-\mu)}d\mu\right) \\ &= \left(\frac{1}{2\pi}\right)^2\frac{1}{T}\sum_{j\in J_T}\sum_{l=-T+1}^{T-1}\sum_{|n|<N}\int_{-\pi}^{\pi} C_{zu}(l)\left(1-\frac{|n|}{N}\right)\phi(\mu)e^{i(n-l)\lambda_j - in\mu}d\mu \\ &= \left(\frac{1}{2\pi}\right)^2\sum_{|n|<N} C_{zu}(n)\int_{-\pi}^{\pi}\left(1-\frac{|n|}{N}\right)\phi(\mu)e^{-in\mu}d\mu + o_p(1). \end{aligned}$$

From Lemma 3 we know that $C_{zu}(n) \Rightarrow g(n)$. Defining $h_{T,k}(\phi_N)$ $(k = 0, 1, \ldots, S^*, S/2)$ to be the corresponding elements and sub-vectors of $h_T(\phi_N)$ and taking each in turn we find, first, that

$$\begin{aligned} T^{-1}h_{T,0}(\phi_N) \quad &\Rightarrow \quad \sigma^2 c(1)^2\int_0^1 J_{0,\nu_0}(r)dW_0(r)\left(\frac{1}{2\pi}\right)^2\int_{-\pi}^{\pi}\phi(\mu)\sum_{|n|<N}\left(1-\frac{|n|}{N}\right)e^{-in\mu}d\mu \\ &\quad + \left(\frac{1}{2\pi}\right)^2\int_{-\pi}^{\pi}\phi(\mu)\sum_{|n|<N}\left(1-\frac{|n|}{N}\right)e^{-in\mu}\sum_{j=n+1}^{\infty}\gamma_j d\mu = p^0(\phi_N) + q^0(\phi_N), \end{aligned}$$

where $p^0(\phi_N) := \sigma^2 c(1)^2\int_0^1 J_{0,\nu_0}(r)dW_0(r)(1/2\pi)\phi_N(0)$ and, defining $a_n := 1 - |n|N^{-1}$, $q^0(\phi_N) := (1/2\pi)^{-2}\int_{-\pi}^{\pi}\phi(\mu)\sum_{|n|<N} a_n e^{-in\mu}\sum_{l=0}^{\infty}\gamma_{l+n+1}d\mu$. Now $\left|p^0(\phi_N) - p^0(\phi)\right| \leq \left(\frac{\epsilon}{2\pi}\right)\sigma^2 c(1)^2\left|\int_0^1 J_{0,\nu_0}(r)dW_0(r)\right|$ $= O_p(\epsilon)$, and so we can replace $\phi_N(0)$ by $\phi(0) = f_u(0)^{-1}$ in $p^0(\cdot)$. But $f_u(0) = (\sigma^2/2\pi)c(1)^2$ and hence

$p^0(\phi) = \int_0^1 J_{0,\nu_0}(r)dW_0(r)$. Turning to $q^0(\phi_N)$ we can write

$$q^0(\phi_N) = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \phi(\mu) \sum_{|n|<N} a_n e^{-in\mu} \sum_{l=0}^{\infty} \int_{-\pi}^{\pi} e^{i(l+n+1)\lambda} f_u(\lambda) d\lambda d\mu = \sum_{l=0}^{\infty} q_l^0(\phi_N),$$

where $q_l^0(\phi_N) := (1/2\pi) \int_{-\pi}^{\pi} \phi_N(\lambda) f_u(\lambda) e^{i(l+1)\lambda} d\lambda$. Now let $q^0(\phi_N) := q_{1N}^0(\phi_N) + q_{2N}^0(\phi_N)$ where $q_{1N}^0(\phi_N) := \sum_{l=0}^{N} q_l^0(\phi_N)$ and $q_{2N}^0(\phi_N) := \sum_{l=N+1}^{\infty} q_l^0(\phi_N)$. For sufficiently large but finite N:

$$\left|q_{1N}^0(\phi_N) - q_{1N}^0(\phi)\right| = \left|\sum_{l=0}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\phi_N(\lambda) - \phi(\lambda)\right] f_u(\lambda) e^{i(l+1)\lambda} d\lambda\right| \leq \frac{\epsilon(N+1)}{2\pi} \int_{-\pi}^{\pi} |f_u(\lambda)|\, d\lambda.$$

Replacing ϕ_N by ϕ in $q_{1N}^0(\cdot)$ it follows that $q_{1N}^0(\phi) = (1/2\pi) \sum_{l=0}^{N} \int_{-\pi}^{\pi} e^{i(l+1)\lambda} d\lambda$ in view of $\phi(\lambda) f_u(\lambda) = 1$. But, for $l \neq -1$, $\int_{-\pi}^{\pi} e^{i(l+1)\lambda} d\lambda = 0$ and hence $q_{1N}^0(\phi) = 0$. As $N \to \infty$ we find that $q_{2N}^0(\phi_N) \to 0$ and hence $q^0(\phi_N) \to 0$, from which we can conclude that $T^{-1} h_{T,0}(\phi) \Rightarrow \int_0^1 J_{0,\nu_0}(r) dW_0(r)$. Turning to $h_{T,S/2}(\phi_N)$ we proceed as before by using the decomposition $T^{-1} h_{T,S/2}(\phi_N) \Rightarrow p^{S/2}(\phi_N) + q^{S/2}(\phi_N)$, where it follows from Lemma 3 that $p^{S/2}(\phi_N) = \sigma^2 c(-1)^2 \int_0^1 J_{S/2,\nu_{S/2}}(r) dW_{S/2}(r)(1/2\pi)\phi_N(\pi)$ and $q^{S/2}(\phi_N) = \sum_{l=0}^{\infty} q_l^{S/2}(\phi_N)$ where we have defined $q_l^{S/2}(\phi_N) := e^{-(l+1)\pi}(1/2\pi) \int_{-\pi}^{\pi} \phi_N(\lambda) f_u(\lambda) e^{i(l+1)\lambda} d\lambda$. But $\left|p^{S/2}(\phi_N) - p^{S/2}(\phi)\right| \leq \left(\frac{\epsilon}{2\pi}\right) \sigma^2 c(-1)^2 \left|\int_0^1 J_{S/2,\nu_{S/2}}(r) dW_{S/2}(r)\right| = O_p(\epsilon)$, and so we can replace $\phi_N(\pi)$ by $\phi(\pi) = f_u(\pi)^{-1}$, resulting in $p^{S/2}(\phi) = \int_0^1 J_{S/2,\nu_{S/2}}(r) dW_{S/2}(r)$. Let $q^{S/2}(\phi_N) := q_{1N}^{S/2}(\phi_N) + q_{2N}^{S/2}(\phi_N)$, $q_{1N}^{S/2}(\phi_N) := \sum_{l=0}^{N} q_l^{S/2}(\phi_N)$ and $q_{2N}^{S/2}(\phi_N) := \sum_{l=N+1}^{\infty} q_l^{S/2}(\phi_N)$. For sufficiently large, finite N it can be shown that $|q_{1N}^{S/2}(\phi_N) - q_{1N}^{S/2}(\phi)| = O(\epsilon)$ and, replacing ϕ_N by ϕ we find that $q_{1N}^{S/2}(\phi) = 0$. As $N \to \infty$, $q_{2N}^{S/2}(\phi_N) \to 0$ and hence it follows that $T^{-1} h_{T,S/2}(\phi) \Rightarrow \int_0^1 J_{S/2,\nu_{S/2}}(r) dW_{S/2}(r)$. For the remaining terms it is convenient to work with the complex random variables

$$\eta_{T,k}(\phi_N) := e_i' h_{T,k}(\phi_N) := h_{T,k1}(\phi_N) + i h_{T,k2}(\phi_N), \quad k = 1, \ldots, S^*,$$

where $h_{T,k}(\phi_N) := [h_{T,k1}(\phi_N), h_{T,k2}(\phi_N)]'$. Proceeding as before, from Lemma 3 we obtain $T^{-1}\eta_{T,k}(\phi_N) \Rightarrow p^k(\phi_N) + q^k(\phi_N)$ with $p^k(\phi_N) := (\sigma^2 |c(e^{i\omega_k})|^2/2) \int_0^1 J_{k,\nu_k}(r) d\overline{W_k(r)}(1/2\pi)\phi_N(\omega_k)$ and $q^k(\phi_N) := \sum_{l=0}^{\infty} q_l^k(\phi_N)$ with $q_l^k(\phi_N) := e^{-(l+1)\omega_k}(1/2\pi) \int_{-\pi}^{\pi} \phi_N(\lambda) f_u(\lambda) e^{i(l+1)\lambda} d\lambda$. Now

$$\left|p^k(\phi_N) - p^k(\phi)\right| \leq \left(\frac{\epsilon}{4\pi}\right) \sigma^2 |c(e^{i\omega_k})|^2 \left|\int_0^1 J_{k,\nu_k}(r) d\overline{W_k(r)}\right| = O_p(\epsilon),$$

and replacing ϕ_N by ϕ we obtain $p^k(\phi) = (1/2) \int_0^1 J_{k,\nu_k}(r) d\overline{W_k(r)}$. Using the decomposition $q^k(\phi_N) := q_{1N}^k(\phi_N) + q_{2N}^k(\phi_N)$ where $q_{1N}^k(\phi_N) := \sum_{l=0}^{N} q_l^k(\phi_N)$ and $q_{2N}^k(\phi_N) := \sum_{l=N+1}^{\infty} q_l^k(\phi_N)$ we find that for sufficiently large but finite N, $|q_{1N}^k(\phi_N) - q_{1N}^k(\phi)| = O(\epsilon)$. Replacing ϕ_N by ϕ as before we find that $q_{1N}^k(\phi) = 0$ while $q_{2N}(\phi_N) \to 0$ as $N \to \infty$, the result being that $T^{-1}\eta_{T,k}(\phi) \Rightarrow (1/2)\int_0^1 J_{k,\nu_k}(r)d\overline{W_k(r)}$, from which the real and imaginary components are easily extracted. □

B Appendix B

Proof of Theorem 1. Detailed proofs are given for the WPE under Assumptions 1 and 2; suitable modifications need to be made for the ASDE under Assumptions 1 and 3. Using (3.5) and (3.4),

$$T(\hat{\beta} - \beta) = \left[T^{-2}\sum_{j \in J_T} I_{zz}(\lambda_j)\hat{f}_{\hat{u}}(\lambda_j)^{-1}\right]^{-1}\left[\sum_{j \in J_T} I_{zu}(\lambda_j)\hat{f}_{\hat{u}}(\lambda_j)^{-1}\right], \tag{B.1}$$

where $I_{zu}(\lambda_j)$ denotes the cross-periodogram between z_t and u_t. In order to save on notation we define $I_{zz,j} := I_{zz}(\lambda_j)$, $I_{zu,j} := I_{zu}(\lambda_j)$, $\hat{f}_{\hat{u},j} := \hat{f}_{\hat{u}}(\lambda_j)$, $\hat{f}_{u,j} := \hat{f}_u(\lambda_j)$, and $f_{u,j} := f_u(\lambda_j)$, so that the first term of interest may be written

$$T^{-2}\sum_j I_{zz,j}\hat{f}_{\hat{u},j}^{-1} = T^{-2}\sum_j I_{zz,j}\left(\hat{f}_{\hat{u},j}^{-1} - \hat{f}_{u,j}^{-1}\right) + T^{-2}\sum_j I_{zz,j}\left(\hat{f}_{u,j}^{-1} - f_{u,j}^{-1}\right) + T^{-2}H_T, \tag{B.2}$$

where H_T is defined in Lemma 4 where its limiting distribution is obtained. We make use of the fact that $\hat{u}_t = y_t - z_t'\hat{\beta}_{OLS} = u_t - z_t'(\hat{\beta}_{OLS} - \beta)$ from which it follows that $w_{\hat{u}}(\lambda) = w_u(\lambda) - (\hat{\beta}_{OLS} - \beta)'w_z(\lambda)$. The periodogram of $\hat{u}_t$ can then be written

$$\begin{aligned} I_{\hat{u}\hat{u}}(\lambda) &= \left[w_u(\lambda) - (\hat{\beta}_{OLS} - \beta)'w_z(\lambda)\right]\left[w_u(\lambda) - (\hat{\beta}_{OLS} - \beta)'w_z(\lambda)\right]^* \\ &= I_{uu}(\lambda) - 2(\hat{\beta}_{OLS} - \beta)'\mathrm{Re}\left\{I_{zu}(\lambda)\right\} + (\hat{\beta}_{OLS} - \beta)'I_{zz}(\lambda)(\hat{\beta}_{OLS} - \beta). \end{aligned} \tag{B.3}$$

Now, in view of $\hat{f}_x(\lambda) = (2m+1)^{-1}\sum_{k=-m}^{m} I_{xx}(\lambda + \lambda_k)$ for a variable x, we obtain

$$\begin{aligned} \hat{f}_{\hat{u},j} - \hat{f}_{u,j} &= \frac{1}{2m+1}\sum_{k=-m}^{m}\left[I_{\hat{u}\hat{u}}(\lambda_j + \lambda_k) - I_{uu}(\lambda_j + \lambda_k)\right] \\ &= -2(\hat{\beta}_{OLS} - \beta)'\phi_j + (\hat{\beta}_{OLS} - \beta)'\Phi_j(\hat{\beta}_{OLS} - \beta) \end{aligned} \tag{B.4}$$

where (noting that $\lambda_j + \lambda_k = \lambda_{j+k}$)

$$\phi_j := \frac{1}{2m+1}\sum_{k=-m}^{m}\mathrm{Re}\left\{I_{zu}(\lambda_{j+k})\right\}, \quad \Phi_j := \frac{1}{2m+1}\sum_{k=-m}^{m} I_{zz}(\lambda_{j+k}). \tag{B.5}$$

By the Cauchy-Schwarz inequality we have

$$\left\| \frac{1}{T^2} \sum_j I_{zz,j} (\hat{f}_{\hat{u},j}^{-1} - \hat{f}_{u,j}^{-1}) \right\|^2 \leq \left(\sum_j \left| \hat{f}_{\hat{u},j} - \hat{f}_{u,j} \right|^2 \left| \hat{f}_{\hat{u},j} \hat{f}_{u,j} \right|^2 \right) \left(\frac{1}{T^4} \sum_j \|I_j\|^2 \right). \tag{B.6}$$

The first term in the right member of (B.6) is bounded by $\sup_j \left| \hat{f}_{\hat{u},j} \hat{f}_{u,j} \right|^{-2} \sum_j \left| \hat{f}_{u,j} - \hat{f}_{\hat{u},j} \right|^2$. Because $\hat{f}_{\hat{u},j} \geq K > 0$ and $\hat{f}_{u,j} \geq K > 0$ with probability approaching 1 as $T \to \infty$ (see, for example, Hannan, 1963 or p.489 of Hannan, 1970) we have $\sup_j |\hat{f}_{\hat{u},j} \hat{f}_{u,j}|^{-2} < \infty$. Furthermore, from (B.4), we find that, by Minkowski's inequality,

$$\sum_j \left| \hat{f}_{u,j} - \hat{f}_{\hat{u},j} \right|^2 \leq \left\{ \left[4\|\hat{\beta}_{OLS} - \beta\|^2 \sum_j \|\phi_j\|^2 \right]^{\frac{1}{2}} + \left[\|\hat{\beta}_{OLS} - \beta\|^4 \sum_j \|\Phi_j\|^2 \right]^{\frac{1}{2}} \right\}^2. \tag{B.7}$$

Now, because of the nature of v_t and u_t, $\hat{\beta}_{OLS} - \beta = O_p(T^{-1})$. From an extension of the proof of Lemma A of Chambers and McCrorie (2007) it follows that $E\|\phi_j\|^2 = O(1)$, and from the Markov inequality we can always find an M_ϵ such that $\Pr\left(T^{-1} \sum_j \|\phi_j\|^2 \geq M_\epsilon\right) \leq \frac{\sum_j E\|\phi_j\|^2}{TM_\epsilon} < \epsilon$, implying that $\sum_j \|\phi_j\|^2 = O_p(T)$ and hence that the first term in the right member of (B.7) is $O_p(T^{-1})$. Turning to the second term we note that the expression in square brackets is bounded by $\|\hat{\beta}_{OLS} - \beta\|^4 \sup_j \|\Phi_j\| \sum_j \|\Phi_j\|$. The first component is $O_p(T^{-4})$ while the last satisfies

$$\begin{aligned}
\sum_j \|\Phi_j\| = \frac{1}{2m+1} \sum_j \left\| \sum_k I_{zz}(\lambda_{j+k}) \right\| &\leq \frac{1}{2m+1} \sum_j \sum_k \|I_{zz}(\lambda_{j+k})\| \\
&= \sum_j \|I_{zz}(\lambda_j)\| \quad \text{by periodicity} \\
&= \mathrm{tr} \sum_j I_{zz}(\lambda_j) \quad \text{because } \|I_{zz}(\lambda)\| = \mathrm{tr} I_{zz}(\lambda) \\
&= \frac{1}{2\pi} \mathrm{tr} \sum_t z_t z_t' = O_p(T^2),
\end{aligned} \tag{B.8}$$

the last line following because $\sum_j I_{zz}(\lambda_j) = (2\pi T)^{-1} \sum_j \sum_t \sum_s z_t z_s' e^{i(t-s)\lambda_j} = (1/2\pi) \sum_t z_t z_t'$ due to $\sum_j e^{i(t-s)\lambda_j} = T$ if $t = s$ and 0 otherwise. Similarly note that

$$\begin{aligned}
\|\Phi_j\| = \frac{1}{2m+1} \left\| \sum_k I_{zz}(\lambda_{j+k}) \right\| &\leq \frac{1}{2m+1} \sum_k \|I_{zz}(\lambda_{j+k})\| \\
&\leq \frac{1}{2m+1} \sum_j \|I_{zz}(\lambda_j)\| = O_p\left(\frac{T^2}{m}\right)
\end{aligned} \tag{B.9}$$

uniformly in j. Hence the contribution of the second term on the right hand side of (B.7) is $O_p(m^{-1})$ and so $\sum_j \left|\hat{f}_{u,j} - \hat{f}_{\hat{u},j}\right|^2 = o_p(1)$, thereby implying that the first term on the right hand side of (B.6) is also $o_p(1)$. The second term in (B.6) satisfies $T^{-4}\sum_j \|I_{zz,j}\|^2 \leq T^{-2}\sup_j \|I_{zz,j}\| T^{-2}\sum_j \|I_{zz,j}\|$. By the arguments leading to (B.9) and (B.8), respectively, $\sup_j \|I_{zz,j}\| = O_p(T^2)$ and $\sum_j \|I_{zz,j}\| = O_p(T^2)$. Hence (B.6) is $o_p(1)$ and we are led to consider the next term in (B.2), which satisfies

$$\begin{aligned}\left\|T^{-2}\sum_j I_{zz,j}\left(\hat{f}_{u,j}^{-1} - f_{u,j}^{-1}\right)\right\| &= \left\|T^{-2}\sum_j I_{zz,j}\left(f_{u,j} - \hat{f}_{u,j}\right)\hat{f}_{u,j}^{-1} f_{u,j}^{-1}\right\| \\ &\leq \sup_j \left|\hat{f}_{u,j}^{-1} f_{u,j}^{-1}\right| \sup_j \left|f_{u,j} - \hat{f}_{u,j}\right| T^{-2}\sum_j \|I_{zz,j}\|. \qquad \text{(B.10)}\end{aligned}$$

As before we have $\sup_j \left|\hat{f}_{u,j}^{-1} f_{u,j}^{-1}\right| \leq K$ and $T^{-2}\sum_j \|I_{zz,j}\| = O_p(1)$. Furthermore, under Assumptions 1 and 2, we have $E|\hat{f}_{u,j} - f_{u,j}|^2 = o(1)$ uniformly in j (see Brockwell and Davis, 1991, p.353) implying (by Markov's inequality) that $|\hat{f}_{u,j} - f_{u,j}| = o_p(1)$ uniformly in j. Hence the right member of (B.10) is $o_p(1)$ and we are led to consider the final term in (B.2) whose limit was established in Lemma 4.

Turning to the cross-product term and proceeding in a similar fashion we obtain

$$T^{-1}\sum_j I_{zu,j}\hat{f}_{\hat{u},j}^{-1} = T^{-1}\sum_j I_{zu,j}\left(\hat{f}_{\hat{u},j}^{-1} - \hat{f}_{u,j}^{-1}\right) + T^{-1}\sum_j I_{zu,j}\left(\hat{f}_{u,j}^{-1} - f_{u,j}^{-1}\right) + T^{-1}h_T, \qquad \text{(B.11)}$$

where h_T is defined in Lemma 5 where its limiting distribution is obtained. The squared modulus of the first term on the right hand side of (B.11) is bounded by

$$\left(\sup_j \left|\hat{f}_{\hat{u},j}\hat{f}_{u,j}\right|^{-2}\sum_j \left|\hat{f}_{u,j} - \hat{f}_{\hat{u},j}\right|^2\right)\left(T^{-2}\sum_j \|I_{zu,j}\|^2\right).$$

The first component has already been shown to be $o_p(1)$ while our earlier examination of $\sum_j \|\phi_j\|^2$ can be used to show that the second component is $O_p(1)$. Turning to the second term on the right hand side of (B.11) we note that its modulus is bounded by

$$\sup_j \left|\hat{f}_{u,j}^{-1} f_{u,j}^{-1}\right| \sup_j \left|f_{u,j} - \hat{f}_{u,j}\right| T^{-1}\sum_j \|I_{zu,j}\|. \qquad \text{(B.12)}$$

The arguments following (B.10) are also relevant here and the fact that $T^{-1}\sum_j \|I_{zu,j}\| = O_p(1)$ establishes that (B.12) is $o_p(1)$. Hence the limiting distribution of (B.11) is determined by the third term on the right hand side which has been given in Lemma 5. The limiting distribution of $T(\hat{\beta} - \beta)$

then follows directly from the limit of the product $(T^{-2}H_T)^{-1}T^{-1}h_T$ using Lemmas 4 and 5 and the continuous mapping theorem. □

Proof of Theorem 2. All of the statistics of interest can be written in terms of (elements of) the normalised vector $T\hat{\beta}$ and matrix $T^2\hat{Q}$. Theorem 1 describes the limiting behaviour of $T\hat{\beta}$ under the null by setting $\beta = 0$, while the proof of Theorem 1, following the decomposition (B.2), and Lemma 4 establish that $T^2\hat{Q} \Rightarrow H^{-1}$, where H is the diagonal matrix of random variables defined in Lemma 4. The limiting distributions of the statistics of interest then follow straightforwardly by picking out the relevant elements from these limits. □

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Table 1: Empirical Null Rejection Frequencies: DGP (4.1).
Conventional OLS and GLS De-trended HEGY Tests - Scheme 3.

			OLS De-trended Tests							GLS De-trended Tests						
ϕ	θ	N	t_0^*	t_2^*	t_1^{c*}	t_1^{s*}	F_1^*	F_{12}^*	F_{012}^*	t_0^*	t_2^*	t_1^{c*}	t_1^{s*}	F_1^*	F_{12}^*	F_{012}^*
Panel A: $p_{\max} = 4$																
0.9	0.0	50	5.4	4.6	2.8	16.5	4.9	4.6	5.1	4.7	4.6	4.4	5.1	4.8	4.7	4.6
0.0	-0.6		21.1	21.9	3.9	6.5	4.7	14.3	19.1	21.6	22.3	3.1	6.6	5.2	15.6	21.0
0.0	0.6		4.1	4.3	22.6	1.2	18.8	16.0	14.2	3.2	3.5	21.7	1.1	13.9	11.3	9.3
0.9	0.0	100	5.2	4.8	2.8	17.7	4.8	4.7	4.8	5.2	5.0	4.9	5.5	5.1	5.3	5.3
0.0	-0.6		21.0	20.9	3.3	6.9	4.6	12.8	18.2	20.6	20.5	3.0	7.1	5.4	14.7	20.9
0.0	0.6		3.5	3.5	23.2	0.8	18.3	14.4	11.7	3.1	3.1	21.8	0.8	12.5	10.3	8.7
Panel B: $p_{\max} = 12$																
0.9	0.0	50	6.0	4.3	3.1	15.5	4.8	4.5	5.3	5.6	4.4	4.5	5.1	4.7	4.5	5.0
0.0	-0.6		14.4	14.7	5.2	5.5	5.5	11.2	13.6	12.7	13.0	4.8	5.4	5.5	10.1	12.4
0.0	0.6		5.4	5.7	13.4	2.5	11.3	10.9	10.6	5.4	5.6	11.0	2.5	7.1	7.2	7.1
0.9	0.0	100	5.3	4.8	2.9	15.6	5.0	5.0	5.2	5.1	4.6	4.4	5.1	4.8	4.5	4.8
0.0	-0.6		9.6	10.3	4.9	5.1	5.2	8.1	8.9	8.9	9.0	4.7	5.2	5.0	6.8	8.1
0.0	0.6		5.0	5.7	9.0	2.9	7.8	7.6	7.3	5.1	5.2	8.6	3.0	5.6	5.5	5.5

Table 2: Empirical Null Rejection Frequencies: DGP (4.1).
OLS and GLS De-trended Frequency Domain HEGY (ASDE) Tests - Scheme 3.

			OLS De-trended Tests							GLS De-trended Tests						
ϕ	θ	N	t_0	t_2	t_1^c	t_1^s	F_1	F_{12}	F_{012}	t_0	t_2	t_1^c	t_1^s	F_1	F_{12}	F_{012}
Panel A: $p_{\max} = 4$																
0.9	0.0	50	8.9	7.1	7.5	5.1	7.4	8.1	10.5	8.4	6.5	6.7	5.0	6.2	7.1	11.8
0.0	-0.6		24.0	24.9	2.9	8.1	4.5	15.5	25.7	24.5	25.0	2.6	8.5	6.3	19.1	28.9
0.0	0.6		3.4	3.7	29.2	1.0	23.5	18.5	15.3	3.0	3.3	26.3	0.9	16.9	13.5	11.3
0.9	0.0	100	9.7	5.9	6.2	4.9	6.0	6.2	9.1	7.7	5.9	6.2	4.9	5.5	6.2	10.2
0.0	-0.6		22.4	22.4	3.0	7.9	4.5	13.9	23.1	21.7	21.6	2.8	8.1	6.3	16.5	24.7
0.0	0.6		3.4	3.3	27.2	0.7	21.5	16.9	13.8	3.1	3.1	23.7	0.7	13.7	11.5	9.8
Panel B: $p_{\max} = 12$																
0.9	0.0	50	9.7	7.2	7.4	5.0	7.1	8.0	11.1	8.9	6.6	6.7	5.1	5.9	6.9	12.1
0.0	-0.6		16.5	17.1	3.7	7.2	5.0	11.8	17.6	16.0	16.3	4.1	7.6	6.7	13.5	19.0
0.0	0.6		4.9	5.0	16.4	2.3	12.7	11.0	10.1	5.3	5.5	15.3	2.3	9.0	8.4	8.1
0.9	0.0	100	10.1	5.8	6.4	5.0	6.2	6.4	9.0	7.5	5.4	5.4	4.9	5.3	5.4	9.5
0.0	-0.6		10.5	10.8	4.3	6.1	5.0	8.2	10.6	10.1	10.1	4.6	6.4	6.0	8.6	11.0
0.0	0.6		5.0	5.1	11.4	2.9	9.1	8.2	7.5	5.5	5.5	10.4	2.7	6.5	6.3	6.3

Table 3: Empirical Null Rejection Frequencies: DGP (4.1).
OLS and GLS De-trended Frequency Domain HEGY (WPE) Tests - Scheme 3.

			OLS De-trended Tests							GLS De-trended Tests						
ϕ	θ	N	t_0	t_2	t_1^c	t_1^s	F_1	F_{12}	F_{012}	t_0	t_2	t_1^c	t_1^s	F_1	F_{12}	F_{012}
Panel A: Fixed Bandwidth, $m = [T^{0.5}]$																
0.9	0.0	50	5.0	53.0	27.1	22.3	33.4	62.7	60.1	0.1	56.3	32.0	18.1	31.3	63.4	61.9
0.0	-0.6		47.9	49.3	2.4	7.3	3.9	35.3	56.1	47.8	48.6	1.8	7.5	5.2	38.6	59.1
0.0	0.6		1.9	2.1	51.2	0.1	43.3	35.8	30.4	1.3	1.4	45.0	0.1	32.4	27.2	23.5
0.9	0.0	100	5.6	55.6	33.9	13.7	33.9	64.4	61.4	0.2	54.8	34.5	11.4	28.7	60.1	59.5
0.0	-0.6		40.2	40.5	2.5	6.6	3.4	28.5	47.2	38.0	38.2	2.4	6.8	4.9	30.3	47.4
0.0	0.6		2.5	2.4	39.6	0.2	33.0	27.0	23.2	2.1	2.1	34.6	0.2	23.7	20.2	17.8
Panel B: Automatic Bandwidth Selection, Lee method																
0.9	0.0	50	7.9	72.7	58.3	48.1	64.8	85.1	84.2	5.0	76.3	63.3	43.3	61.8	85.0	84.6
0.0	-0.6		55.0	56.2	2.3	7.9	3.9	41.7	63.9	55.6	56.2	1.6	8.3	5.9	45.3	66.8
0.0	0.6		1.8	2.0	61.4	0.1	53.4	44.8	38.3	1.0	1.2	54.1	0.0	40.0	33.9	29.8
0.9	0.0	100	8.9	70.6	58.7	36.1	59.8	81.9	80.6	4.4	70.7	58.8	32.7	53.4	78.7	78.8
0.0	-0.6		46.8	47.0	2.5	6.8	3.5	34.9	54.8	44.2	44.0	2.3	7.1	5.4	35.9	53.6
0.0	0.6		2.5	2.3	48.1	0.1	41.3	34.7	30.2	2.0	1.8	41.6	0.1	29.2	25.4	22.8
Panel B: Automatic Bandwidth Selection, Ombao *et al* method																
0.9	0.0	50	9.4	38.1	23.0	29.5	33.4	50.8	52.8	0.0	43.6	28.2	24.0	32.0	53.0	56.8
0.0	-0.6		68.5	69.8	1.8	11.6	5.0	52.0	79.3	69.4	70.4	0.7	12.2	8.5	59.1	83.5
0.0	0.6		1.8	1.8	83.6	0.0	76.7	67.6	59.5	0.6	0.6	75.9	0.0	61.0	52.1	45.0
0.9	0.0	100	4.9	30.3	7.1	7.9	10.1	28.0	26.5	0.0	32.6	12.3	6.2	11.8	32.0	36.1
0.0	-0.6		40.0	40.4	1.7	7.6	3.0	26.0	45.1	37.9	38.0	1.5	7.8	5.6	29.3	46.5
0.0	0.6		1.9	1.8	44.6	0.1	36.5	28.8	23.6	1.5	1.4	38.2	0.1	25.4	20.5	17.4

Table 4: Empirical Power: DGP (4.2).
OLS and GLS De-trended Conventional HEGY Tests - Scheme 3.

		OLS De-trended Tests							GLS De-trended Tests						
N	v_k	t_0^*	t_2^*	t_1^{c*}	t_1^{s*}	F_1^*	F_{12}^*	F_{012}^*	t_0^*	t_2^*	t_1^{c*}	t_1^{s*}	F_1^*	F_{12}^*	F_{012}^*
Panel A: $p_{\max} = 4$															
50	-3	7.9	8.5	11.9	4.7	10.7	13.2	14.9	16.0	16.6	36.1	4.2	23.6	32.9	39.9
	-5	11.9	12.3	20.4	4.2	18.0	23.4	28.2	28.0	29.1	67.8	4.1	47.8	63.5	74.1
	-7	17.4	18.2	34.3	3.9	29.1	39.4	48.4	42.3	44.0	88.5	4.1	71.6	86.6	93.5
	-11	35.1	35.7	68.4	3.2	59.7	76.2	86.4	71.5	72.5	99.2	3.9	95.8	99.2	99.8
	-15	56.3	57.9	90.7	3.0	84.9	95.2	98.3	89.6	90.6	99.9	3.7	99.6	100.0	100.0
	-19	76.3	77.3	97.8	2.9	95.9	99.3	99.7	96.9	97.5	100.0	3.8	99.9	100.0	100.0
100	-3	8.2	7.5	11.4	5.0	10.6	12.4	14.4	16.9	16.8	37.3	5.0	23.9	31.9	40.5
	-5	12.0	11.5	19.8	4.2	17.3	22.7	28.2	30.2	30.6	70.2	4.8	48.7	64.4	77.0
	-7	18.1	17.3	33.9	3.7	28.6	39.0	49.1	46.8	46.8	90.9	4.7	73.3	88.1	95.3
	-11	35.8	34.3	69.6	3.4	60.1	77.1	87.7	77.2	77.7	99.7	4.7	97.1	99.6	100.0
	-15	58.4	56.4	92.2	3.2	86.4	96.2	99.0	93.7	93.8	100.0	4.8	99.9	100.0	100.0
	-19	78.1	76.5	98.7	3.1	97.0	99.6	99.9	98.7	98.8	100.0	4.8	100.0	100.0	100.0
Panel B: $p_{\max} = 12$															
50	-3	8.7	8.3	11.7	4.9	10.2	12.4	14.4	16.4	15.9	33.7	4.1	20.8	28.7	36.1
	-5	12.6	11.9	19.4	4.4	16.0	20.9	26.1	27.6	27.6	61.6	4.1	40.3	55.8	68.2
	-7	17.9	17.4	31.3	3.8	25.3	34.5	43.4	40.5	41.4	82.3	4.0	62.0	79.7	89.3
	-11	32.4	31.9	60.5	3.3	50.4	67.7	79.2	65.5	67.7	97.4	3.9	89.8	97.6	99.2
	-15	50.6	49.3	83.3	3.1	74.9	89.4	95.8	83.9	86.1	99.5	3.9	97.9	99.7	99.9
	-19	67.6	67.4	94.0	3.0	89.8	97.5	99.0	93.1	95.1	99.9	3.9	99.5	100.0	100.0
100	-3	8.3	8.3	11.6	4.8	10.5	12.8	14.4	16.9	16.4	36.0	4.7	21.8	29.1	36.3
	-5	12.0	11.9	20.5	4.3	17.3	22.3	27.4	29.6	28.8	66.7	4.7	44.3	59.2	71.5
	-7	17.7	17.9	33.1	3.7	27.2	37.9	46.5	45.3	44.5	87.2	4.5	67.2	83.5	92.4
	-11	34.0	33.5	65.0	3.5	55.7	73.2	84.2	73.4	73.1	99.1	4.8	94.3	98.9	99.7
	-15	53.3	53.7	87.9	3.1	80.7	93.7	97.7	90.3	90.5	99.9	4.7	99.3	99.9	100.0
	-19	71.9	71.5	96.7	3.1	93.9	99.0	99.7	97.0	97.1	100.0	4.7	99.9	100.0	100.0

Table 5: Empirical Power: DGP (4.2).
OLS and GLS De-trended Frequency Domain HEGY (ASDE) Tests - Scheme 3.

		OLS De-trended Tests							GLS De-trended Tests						
N	v_k	t_0	t_2	t_1^c	t_1^s	F_1	F_{12}	F_{012}	t_0	t_2	t_1^c	t_1^s	F_1	F_{12}	F_{012}
Panel A: $p_{\max} = 4$															
50	-3	8.4	8.9	13.0	5.2	12.0	15.4	18.5	16.1	16.6	36.3	4.7	23.5	32.3	39.5
	-5	12.9	13.6	22.9	4.9	20.9	28.4	35.8	27.9	29.1	68.2	4.8	47.6	62.8	73.5
	-7	19.1	20.3	38.8	4.9	34.6	47.0	58.7	42.2	44.0	89.0	5.0	71.6	86.2	93.2
	-11	38.8	39.4	74.0	4.4	67.2	83.3	92.1	71.5	72.5	99.3	5.0	95.9	99.3	99.8
	-15	60.8	62.3	93.9	4.5	90.0	97.6	99.4	89.8	90.6	100.0	5.3	99.7	100.0	100.0
	-19	80.1	81.5	98.9	4.9	98.0	99.8	100.0	97.1	97.5	100.0	5.7	100.0	100.0	100.0
100	-3	8.4	8.1	12.0	5.4	11.6	13.8	16.7	16.9	16.8	37.3	5.2	23.9	31.7	40.4
	-5	12.6	12.3	21.4	4.9	19.3	26.0	33.1	30.3	30.6	70.6	5.3	48.6	64.3	76.8
	-7	19.2	18.7	36.9	4.6	32.3	44.4	56.1	46.7	46.8	91.2	5.5	73.4	88.1	95.4
	-11	38.1	37.1	73.4	4.4	65.8	82.3	91.7	77.1	77.7	99.7	5.8	97.2	99.6	100.0
	-15	61.6	60.1	94.1	4.5	89.9	97.8	99.6	93.8	93.8	100.0	6.1	99.9	100.0	100.0
	-19	80.8	79.8	99.2	4.6	98.2	99.8	100.0	98.7	98.8	100.0	6.5	100.0	100.0	100.0
Panel B: $p_{\max} = 12$															
50	-3	9.3	9.1	12.6	6.1	11.9	15.1	19.0	16.6	15.9	34.0	5.0	19.9	28.0	35.2
	-5	14.3	14.2	22.5	6.2	20.8	28.3	36.3	28.2	27.6	62.8	5.6	39.9	55.2	66.9
	-7	21.1	21.5	38.1	5.9	33.7	46.2	58.1	41.6	41.4	84.0	5.7	61.7	79.4	88.9
	-11	39.5	40.3	70.9	5.9	64.3	81.2	90.5	68.1	67.7	98.2	6.1	90.6	97.8	99.4
	-15	61.3	60.9	90.9	6.0	86.7	96.2	99.0	86.4	86.1	99.8	6.5	98.4	99.9	100.0
	-19	78.3	78.9	97.5	6.2	96.2	99.4	99.9	95.1	95.1	100.0	6.8	99.7	100.0	100.0
100	-3	8.6	8.6	11.6	5.7	11.6	14.0	16.9	17.0	16.4	36.3	5.5	21.7	28.6	36.1
	-5	13.2	13.1	21.3	5.5	19.8	26.2	33.6	29.7	28.8	67.5	5.9	44.5	58.6	71.1
	-7	20.5	20.2	36.3	5.4	32.5	45.0	56.1	45.7	44.5	88.2	6.1	67.6	83.4	92.4
	-11	39.3	39.0	70.7	5.8	64.4	81.2	90.9	74.2	73.1	99.3	7.2	94.6	99.1	99.8
	-15	60.8	60.8	91.7	5.7	87.7	97.0	99.2	91.1	90.5	100.0	7.4	99.5	100.0	100.0
	-19	78.9	78.6	98.3	5.7	97.0	99.7	100.0	97.6	97.1	100.0	7.7	99.9	100.0	100.0