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by

Yiannis Karavias and Elias Tzavalis

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A fixed-T Version of Breitung's Panel Data Unit Root Test and its Asymptotic Local Power

Yiannis Karavias[a,*] and Elias Tzavalis[b]

a: School of Economics, University of Nottingham

b: Athens University of Economics & Business

Abstract

We extend Breitung's (2000) large-T panel data unit root test to the case of fixed time dimension while still allowing for heteroscedastic and serially correlated error terms. The analytic local power function of the new test is derived assuming that only the cross section dimension of the panel grows large. It is found that if the errors are serially uncorrelated the test also has trivial power, but, if not, this is no longer the case. Monte Carlo experiments show that the suggested test is more powerful than its large-T, original version when the number of cross section units is moderate or large, regardless of the number of time series observations.

JEL classification: C22, C23

Keywords: Panel unit root; local power function; serial correlation; incidental trends

*Corresponding author. University of Nottingham, Nottingham NG7 2RD, UK. Tel: +441159515480, Email addresses: ioannis.karavias@nottingham.ac.uk (Y. Karavias), etzavalis@aueb.gr (E. Tzavalis).

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1 Introduction

To improve the power performance of panel data unit root tests in the presence of heterogeneous individual trends, Breitung (2000) proposed a statistic for large-T panels based on an orthogonal transformation of the individual series. The test does not require an inconsistency adjustment of the estimator of the autoregressive parameter φ as opposed to other tests in the literature, see e.g., Baltagi (2013) for a survey. Although it was found to be consistent and have superior power in small samples for values of φ not far from unity (e.g., $\varphi = 0.95$), its asymptotic local power in a $T^{-1}N^{-1/2}$ neighbourhood of unity is trivial and equivalent to that of the asymptotically bias corrected tests (see, Moon et al. (2007)).

In this paper, we extend Breitung's (2000) test in two directions. First, we allow the time dimension T of the panel to be finite (fixed) while allowing for heterogeneity, heteroscedasticity, and serial correlation in the error terms. Second, we derive the fixed-T asymptotic local power function of the new test. These extensions make the application of the test valid in cases of short-T panels, often met in practice, and under higher than first order serial correlation. The paper provides a number of interesting results. First, it shows that the fixed-T version of the test can further improve its small sample size and power performance in short panels, compared to its large-T version. Second, the new test also has trivial asymptotic local power in a $N^{-1/2}$ neighbourhood of unity when the error terms are independently distributed over time, which explains analytically Breitung's (2000) findings in his Monte Carlo experiment. Third, when the error terms are serially correlated, the estimator of φ becomes inconsistent and thus, the test needs an inconsistency correction. Fourth, there are forms of serial correlation for which the test has non-trivial local power.

The paper is organized as follows. Section 2 introduces the new test and provides its asymptotic local power function. Section 3 presents the results of our Monte Carlo exercise, while Section 4 concludes the paper. All proofs are relegated to the Appendix.

2 The test statistic and its asymptotic local power

Consider the following first order autoregressive panel data model with individual effects:

$$y_i = \varphi y_{i-1} + (1-\varphi)a_i e + \varphi \beta_i e + (1-\varphi)\beta_i \tau + u_i, \tag{1}$$

where $y_i = (y_{i1}, ..., y_{iT})'$ and $y_i = (y_{i0}, ..., y_{iT-1})'$ are $TX1$ vectors, u_i is the $TX1$ vector of error terms u_{it}, and a_i and β_i are the individual coefficients of the deterministic components of the model. The coefficients a_i reflect the individual effects of the panel, while β_i capture the slopes of individual linear trends, referred to as incidental trends. The $TX1$ vector e has elements $e_t = 1$, for $t = 1...T$, and $\tau_t = t$ is the time trend. Next, define the autoregressive coefficient φ as $\varphi_N = 1 - \frac{c}{\sqrt{N}}$. Then, the null hypothesis of a unit root in φ against its alternative of stationarity (i.e., $\varphi < 1$) can be respectively written as

$$H_0\colon\ c = 0 \ \text{ and } \ H_1\colon\ c > 0,$$

where c is the local to unity parameter. The asymptotic distribution of the extension of Breitung's (2000) test statistic (denoted as UB) is derived by making the following assumption.

Assumption A

(i) $\{u_i\}$, $i \in \{1, 2, ..., N\}$, are independent random vectors with means $E(u_i) = 0$, heterogeneous variance-covariance matrices $\Gamma_i \equiv E(u_i u_i') \equiv [\gamma_{i,ts}]$, where $\gamma_{i,ts} = E(u_{it}u_{is}) = 0$ for $t < s$ and $s = t + p + 1, ..., T$. The maximum order of serial correlation in u_i is $p = T - 2$. All $4 + \epsilon$ mixed moments are finite.

(ii) $\Gamma = \lim_N \frac{1}{N}\sum_{i=1}^N \Gamma_i$ is a finite, positive definite matrix and $\lim(N\Gamma)^{-1}\Gamma_i$=$\lim\left(\sum_{i=1}^N \Gamma_i\right)^{-1}\Gamma_i = 0$, for all i.

(iii) The random variables y_{io}, a_i and β_i have finite $4 + \epsilon$ moments and are independent from u_{it} for $i = 1, ..., N$ and $t = 1, ..., T$.

Assumption A allows us to derive the distribution of the fixed-T version of Breitung's (2000) large-T panel unit root test statistic under the null hypothesis. Condition (i) determines the order of serial correlation in error terms u_{it} and together with condition (ii) provide the necessary assumptions for the application of the Lindeberg-Feller multivariate CLT. Condition (iii) is needed for the derivation of the asymptotic local power function.

Breitung's (2000) test is based on the forward orthogonal deviations transformation of the individual series y_{it} of model (1). In a first step, the initial observations y_{i0} are subtracted from y_{it}, i.e. $z_{it} = y_{it} - y_{i0}$. Then, define the following $(T-1)XT$ matrices:

$$A = \begin{pmatrix} 0_{1XT} \\ GH \end{pmatrix} \text{ and } B = \begin{pmatrix} 0_{1X(T-2)} & 0 & 0 \\ I_{T-2} & 0_{(T-2)X1} & -\frac{1}{T}\tau_{T-2} \end{pmatrix}, \text{ where}$$

$$G = \begin{pmatrix} \sqrt{\frac{T-2}{T-1}} & & & 0 \\ & \sqrt{\frac{T-3}{T-2}} & & \\ & & \ddots & \\ 0 & & & \sqrt{\frac{1}{2}} \end{pmatrix} \text{ and } H = \begin{pmatrix} 1 & -\frac{1}{T-1} & \cdots & \cdots & \cdots & -\frac{1}{T-1} \\ & \ddots & -\frac{1}{T-2} & & & -\frac{1}{T-2} \\ & & \ddots & & & \vdots \\ & & & 1 & -\frac{1}{2} & -\frac{1}{2} \\ \cdots & \cdots & \cdots & 0 & 1 & -1 \end{pmatrix},$$

with dimensions $(T-2)X(T-2)$ and $(T-2)XT$ respectively, and vector $\tau_{T-2} = (1, 2, ..., T-2)'$. In the case that $u_{it} \sim IIID(0, \sigma^2)$, multiplying Δz_i with matrix A and z_i with matrix

B implies the following orthogonal moment conditions under null hypothesis H_0: $c = 0$:

$$E(z_i' B' A \Delta z_i) = 0. \tag{2}$$

These can be tested based on the following least squares estimator of φ:

$$\hat{\varphi}_{FOD} = 1 + \left(\sum_{i=1}^{N} z_i' B' B z_i \right) \left(\sum_{i=1}^{N} z_i' B' A \Delta z_i \right),$$

which is equal to that of Breitung (2000) plus 1. This estimator is consistent under H_0: $c = 0$, i.e., $p\lim_N \hat{\varphi}_{FOD} = 1$. In the more general case where $\Gamma \neq \sigma^2 I_T$, estimator $\hat{\varphi}_{FOD}$ becomes inconsistent and its asymptotic bias is equal to $p\lim_N (\hat{\varphi}_{FOD} - 1) = \frac{tr((\Lambda+I)' B' A \Gamma)}{tr((\Lambda+I)' B' B (\Lambda+I)\Gamma)}$, where Λ is a TXT matrix which has unities at its lower than its main diagonals, and zero elsewhere, and I is a TXT identity matrix.[1] Thus, to test moment conditions (2), $\hat{\varphi}_{FOD}$ needs to be corrected for its inconsistency (see, e.g., Harris and Tzavalis (1999)).

Theorem 1 *Let conditions (i) and (ii) of Assumption A hold and $N \to +\infty$, then under H_0: $c = 0$ we have*

$$UB_T = \sqrt{N} V^{-1/2} \hat{\delta} \left(\hat{\varphi}_{FOD} - 1 - \frac{\hat{b}}{\hat{\delta}} \right) \xrightarrow{d} N(0,1),$$

where $\frac{\hat{b}}{\hat{\delta}} = \frac{tr(\Phi_p \hat{\Gamma})}{\frac{1}{N}\sum_{i=1}^{N} z_i' B' B z_i}$, $\hat{\Gamma} = \frac{1}{N}\sum_{i=1}^{N} \Delta z_i \Delta z_i'$, $\Phi_p = \Psi_p - \frac{e'\Psi_p e}{e' M e} M$ with Ψ_p a TXT matrix having in its diagonals $\{-p, .., 0, ...p\}$ the corresponding elements of matrix $\Xi = (\Lambda + I)' B' A$, and zero elsewhere, M is a TXT selection matrix with elements $m_{ts} = 0$, if $\gamma_{ts} \neq 0$, and $m_{ts} = 1$, if $\gamma_{ts} = 0$, and $V = vec(\Xi' - \Phi_p')' \Theta vec(\Xi' - \Phi_p')$ where $\Theta = \frac{1}{N}\sum_{i=1}^{N} Var(vec(\Delta z_i \Delta z_i'))$.[2]

Implementing test statistic UB_T requires a consistent estimator of variance V, given under H_0 as $\hat{V} = vec(\Xi' - \Phi_p')' \hat{\Theta} vec(\Xi' - \Phi_p')$ where $\hat{\Theta} = \frac{1}{N}\sum_{i=1}^{N} (vec(\Delta z_i \Delta z_i') vec(\Delta z_i \Delta z_i')')$. The main difference between UB_T and Breitung's statistic UB is the replacement of a T-consistent variance estimator of u_i with a N-consistent one.

To study the asymptotic local power of UB_T under H_1: $c > 0$, we will rely on a "slope" parameter, denoted as k, which is defined in local power functions of form $\Phi(z_a + ck)$, where Φ is the standard normal cumulative distribution function and z_a denotes the α*-level* percentile. Since Φ is strictly monotonic, a larger k means greater power for the same value

[1] This happens because $tr((\Lambda + I)' B' A) = 0$ and $tr((\Lambda + I)' B' A \Gamma) \neq 0$.

[2] An alternative specification of UB_T for $u_{it} \sim IID(0, \sigma^2)$ is $UB_{T,2} = \sqrt{N} V_2^{-1/2} (\hat{\varphi}_{FOD} - 1) \xrightarrow{d} N(0,1)$, where $V_2 = vec(\Xi')' \Theta vec(\Xi')$ and $\Xi = (\Lambda + I)' B' A$. If u_{it} are also normally distributed, V_2 becomes $V_2 = \frac{2tr(A_\Xi^2)}{tr((\Lambda+I)' B' B (\Lambda+I))^2}$, with $A_\Xi = \frac{1}{2}(\Xi + \Xi')$.

of c. If $k > 0$, then test statistic UB_T will have non-trivial power. If $k = 0$, it will have trivial power, which is equal to a. Finally, if $k < 0$, it will be biased. In the next theorem, we derive the limiting distribution of UB_T under H_1: $c > 0$.

Theorem 2 *Under Assumption A and H_1: $c > 0$, we have*

$$UB_T = \sqrt{N}V^{-1/2}\hat{\delta}\left(\hat{\varphi}_{FOD} - 1 - \frac{\hat{b}}{\hat{\delta}}\right) \xrightarrow{d} N(-ck, 1), \tag{3}$$

as $N \rightarrow \infty$, where

$$k = \frac{tr(\Lambda' B' A \Lambda \Gamma) + tr(B' A \Lambda \Gamma) + tr(\Lambda' B' A \Gamma) + tr(F' B' A \Gamma) - tr(\Lambda' \Phi_p \Gamma) - tr(\Phi_p \Lambda \Gamma)}{\sqrt{V}}, \tag{4}$$

where F is defined in the Appendix.

The result of Theorem 2 implies that UB_T can have non-trivial power, as k can be positive. Power becomes trivial if u_{it} are serially uncorrelated. Then, UB_T will suffer from the problem of zero asymptotic local power due to incidental trends, noted by Moon et al. (2007) for large-T panel unit root tests.[3] This explains Breitung's (2000) Monte Carlo findings. Note that this power also depends on the moments of nuisance parameters β_i, entered in the denominator of k through the variance function V. For instance, if u_{it} and β_i are zero-mean normally distributed random variables, then V is given as $V = 2tr((A_{FOD}\Gamma + E(\beta_i^2)A_{FOD}ee')^2)$, where $A_{FOD} = \frac{1}{2}(\Xi + \Xi' - \Phi_p - \Phi_p')$ (see proof of Theorem 1).

3 Simulation Results

The aim of our simulation study is twofold: first, to examine if the size and power performance of the fixed-T test statistic UB_T in small samples is satisfactory compared to its large-T version and, second, to investigate how well the asymptotic local power function can approximate the actual power of the test. In our analysis, we assume that error terms u_{it} are generated from the MA(1) process $u_{it} = \varepsilon_{it} + \theta\varepsilon_{it-1}$, with innovations $\varepsilon_{it} \sim NIID(0, 1)$ and values of $\theta \in \{-0.8, -0.4, 0, 0.4, 0.8\}$. We set $y_{i0} = 0$ and $a_i = 0$, without loss of generality as these parameters do not appear in the local power function. For β_i, we consider $\beta_i = 0$ or $\beta_i \sim NIID(0, 1)$. Finally, $\varphi \in \{1, 0.95\}$, $N \in \{20, 50, 100\}$ and $T \in \{7, 10, 15, 20, 50\}$.

[3] The limiting distribution of $UB_{T,2}$ under H_1:$c > 0$ becomes $UB_{T,2} = \sqrt{N}V_2^{-1/2}(\hat{\varphi}_{FOD} - 1) \xrightarrow{d} N(-ck_2, 1)$, where $k_2 = 0$, which means that the test has trivial power.

Rejection frequencies are computed from 10000 replications at the 5% significance level.

	N	20				50				100			
	φ/T	10	20	50	100	10	20	50	100	10	20	50	100
UB_T	1	0.093	0.104	0.110	0.114	0.077	0.087	0.088	0.093	0.070	0.070	0.073	0.080
	0.95	0.061	0.083	0.267	0.840	0.065	0.106	0.582	0.997	0.074	0.158	0.854	1
UB	1	0.082	0.074	0.063	0.061	0.079	0.069	0.066	0.057	0.075	0.066	0.059	0.057
	0.95	0.055	0.069	0.291	0.886	0.059	0.101	0.547	0.998	0.064	0.138	0.823	1

Table 1: Size and size-adjusted power of test statistics UB_T and UB, for $\theta = 0$.

Table 1 presents the size and the size-adjusted power of UB_T and Breitung's statistic UB. This is done for $\theta = 0$ and $\beta_i = 0$, for all i (see also Breuitung (2000)). The results of the table clearly indicate that both the size and power of UB_T are satisfactory (see De Blander and Dhaene (2012)). The size of the test is very close to its nominal 5% level. Its power increases with N or T, but faster with T than N. For small N (i.e., $N = 20$) and large T, statistic UB has better size and more power than UB_T. However, as N increases UB_T improves its size and is more powerful than the UB test irrespective of T. This qualifies application of UB_T also in cases where both dimensions N and T of the panel are large.

	N	20				50				100			
θ	φ/T	10	20	50	100	10	20	50	100	10	20	50	100
-0.8	1	0.054	0.054	0.057	0.054	0.051	0.057	0.054	0.057	0.053	0.054	0.056	0.051
	0.95	0.061	0.073	0.099	0.114	0.070	0.086	0.128	0.164	0.075	0.105	0.173	0.246
-0.4	1	0.051	0.059	0.066	0.080	0.050	0.055	0.066	0.071	0.054	0.054	0.059	0.061
	0.95	0.062	0.091	0.252	0.711	0.074	0.115	0.435	0.695	0.077	0.138	0.656	0.998
0.4	1	0.079	0.096	0.113	0.111	0.070	0.084	0.082	0.089	0.061	0.069	0.082	0.079
	0.95	0.092	0.161	0.489	0.950	0.093	0.181	0.728	0.999	0.093	0.216	0.924	1.00
0.8	1	0.074	0.097	0.111	0.122	0.068	0.078	0.090	0.090	0.064	0.073	0.080	0.078
	0.95	0.095	0.168	0.496	0.958	0.090	0.185	0.747	0.999	0.100	0.219	0.927	1.00

Table 2: Size and power of the fixed-T panel root test statistic UB_T when $\theta \neq 0$.

Regarding the effects of serial correlation on the test, the results of Table 2, which presents size and power of statistic UB_T for non-zero θ, indicate that positive serial correlation ($\theta > 0$) in the errors u_{it} increases considerably the power of UB_T, even for very small values of T and N. Also, the size performance of UB_T is unaffected when error terms u_{it} are negatively correlated ($\theta < 0$). This result is in contrast to that of single time series unit root tests which are critically oversized for $\theta < 0$ (see, e.g., Schwert (1989)).

To see how well the asymptotic theory approximates the local power of UB_T in the neighbourhood of unity, Table 3 presents power values when $\varphi = 1 - c/\sqrt{N}$, for $c = 1$, $N \in \{50, 100, 300, 1000\}$, $T = 10$ and two cases of β_i: $\beta_i = 0$ and $\beta_i \sim NIID(0, 1)$. The results of Table 3 indicate that the estimates of the power obtained by our Monte Carlo experiment tend to approximate their theoretical values (TV). For $\theta < 0$, the test has non-trivial local power while for $\theta > 0$, it is biased. The non-trivial local power of the test for $\theta < 0$ can be attributed to the fact that the individual series of the panel y_{it} become close to those of a panel data autoregressive model with a common trend, for all i. In this case, the incidental trends problem does not apply (see Moon et al. (2007). Finally, the power losses for $\beta_i \sim NIID(0, 1)$ are not very large. They become minimal for $\theta = 0$, where β_i does not affect the local power function.

$\beta_i = 0$, $i = 1, ..., N$						$\beta_i \sim N(0, 1)$, $i = 1, ..., N$					
$\theta \backslash N$	50	100	300	1000	TV	$\theta \backslash N$	50	100	300	1000	TV
-0.8	0.125	0.123	0.113	0.096	0.067	-0.8	0.091	0.086	0.084	0.076	0.059
-0.4	0.142	0.132	0.109	0.099	0.059	-0.4	0.089	0.086	0.075	0.068	0.054
0	0.222	0.182	0.115	0.086	0.050	0	0.203	0.154	0.105	0.081	0.050
0.4	0.286	0.213	0.132	0.088	0.045	0.4	0.173	0.138	0.102	0.077	0.047
0.8	0.308	0.233	0.147	0.096	0.044	0.8	0.191	0.154	0.111	0.079	0.046

Table 3: Local power values of statistic UB_T for $T = 10$, when $u_{it} = \varepsilon_{it} + \theta \varepsilon_{it-1}$.

4 Conclusions

This paper extends Breitung's (2000) panel unit root test to the case of fixed-T time dimension and derives its asymptotic local power. It shows that the new test can further improve its small sample size and power performance in short panels, compared to its large-T version. In addition to this, allowing for serial correlation in error terms leads to a test which can have non-trivial local asymptotic power in the presence of incidental trends. Monte Carlo analysis confirms the asymptotic results provided by the paper.

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5 Appendix

Theorem 1: Under H_0: $c = 0$, we have $z_i = z_{i-1} + \beta_i e + u_i$ and $z_{i-1} = \Lambda e \beta_i + \Lambda u_i$. Then, the denominator of $\hat{\varphi}_{FOD} - 1$, denoted as $\hat{\delta}$, is $\frac{1}{N}\sum_{i=1}^{N} z_i' B' A \Delta z_i = \frac{1}{N}\sum_{i=1}^{N}(z_{i-1}' + \beta_i e' + u_i) B' A(\beta_i e + u_i) = \frac{1}{N}\sum_{i=1}^{N}(u_i'(\Lambda' + I_T) + \beta_i \tau') B' A(\beta_i e + u_i) = \frac{1}{N}\sum_{i=1}^{N} u_i'(\Lambda' + I_T) B' A u_i$, since $(\Lambda + I_T)e = \tau$ and $\tau' B' = 0_{1XT}$, $B' A e = 0_{TX1}$ by construction. By Khinchine's Weak Law of Large Numbers:

$$\frac{1}{N}\sum_{i=1}^{N} u_i'(\Lambda' + I_T) B' A u_i = \frac{1}{N}\sum_{i=1}^{N} u_i' \Xi u_i \xrightarrow{p} tr(\Xi\Gamma).$$

Similarly, it can be shown that the denominator of $\hat{\varphi}_{FOD}-1$ has the following limit:

$$\frac{1}{N}\sum_{i=1}^{N} z_i' B' B z_i = \frac{1}{N}\sum_{i=1}^{N} u_i'(\Lambda' + I_T)B'B(\Lambda + I_T)u_i \xrightarrow{p} tr((\Lambda' + I_T)B'B(\Lambda + I_T)\Gamma).$$

The last two relationships imply that the inconsistency of $\hat{\varphi}_{FOD}$ is given as $p\lim_N(\hat{\varphi}_{FOD} - 1) = \frac{tr(\Xi\Gamma)}{tr((\Lambda'+I_T)B'B(\Lambda+I_T)\Gamma)}$. Thus $\hat{\varphi}_{FOD}$ becomes unbiased, if $tr(\Xi\Gamma) = 0$ i.e. $\Gamma = \sigma^2 I_T$. Combining the above, the limiting distribution of UB_T can be derived as follows:

$$\begin{aligned} \sqrt{N}\hat{\delta}\left(\hat{\varphi}_{FOD} - 1 - \hat{b}/\hat{\delta}\right) &= \\ \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N} u_i'(\Lambda' + I_T)B'Au_i - \frac{1}{N}\sum_{i=1}^{N}\Delta z_i'\Phi_p\Delta z_i\right) &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\Delta z_i'(\Xi - \Phi_p)\Delta z_i \end{aligned}$$

since $\Delta z_i'\Xi\Delta z_i = u_i'\Xi u_i$, where $E(\Delta z_i'(\Xi-\Phi_p)\Delta z_i) = 0$ by construction of Φ_p and $Var(\Delta z_i'(\Xi-\Phi_p)\Delta z_i) = vec(\Xi - \Phi_p)'Var(vec(\Delta z_i \Delta z_i'))vec(\Xi - \Phi_p)$. The result follows by applying the Lindeberg-Feller CLT. If u_i and β_i are zero-mean normally distributed random variables, then Δz_i is also normal and $Var(\Delta z_i'(\Xi - \Phi_p)\Delta z_i) = 2tr(\left(A_{FOD}\left(\Gamma + E(\beta_i^2)ee'\right)\right)^2)$.

Theorem 2: To prove the theorem, we will employ following relationships:

$$\begin{aligned} z_i &= \varphi_N z_{i-1} + X\zeta_i + u_i, \quad i = 1, 2, ..., N &(5)\\ z_{i-1} &= \Omega X\zeta_i + \Omega u_i + (w - e)y_{i0}, &(6)\\ \text{and} \quad \Delta z_i &= (\varphi_N - 1)z_{i-1} + X\zeta_i + u_i, &(7) \end{aligned}$$

where $\zeta_i = \begin{pmatrix} (1-\varphi_N)(a_i - y_{i0}) + \varphi\beta_i \\ (1-\varphi_N)\beta_i \end{pmatrix}$, $X = (e, \tau)$, $w = (1, \varphi_N, \varphi_N^2, ..., \varphi_N^{T-1})'$ and

$$\Omega = \begin{pmatrix} 0 & . & . & . & . & . & 0 \\ 1 & 0 & & & & & . \\ \varphi_N & 1 & . & & & & . \\ \varphi_N^2 & \varphi_N & . & . & & & . \\ . & & . & . & . & & . \\ . & & & . & 1 & 0 & . \\ \varphi_N^{T-2} & \varphi_N^{T-3} & . & . & \varphi_N & 1 & 0 \end{pmatrix}.$$ Note that, for $\varphi_N = 1$, we have $\Omega \equiv \Lambda$. The first order Taylor expansions of Ω and w yield

$$\Omega = \Lambda + F(\varphi_N - 1) + o(1) \text{ and } w = e + f(\varphi_N - 1) + o(1), \tag{8}$$

respectively, where $F = \frac{d\Omega}{d\varphi_N}\mid_{\varphi_N=1}$ and $f = \frac{dw}{d\varphi_N}\mid_{\varphi_N=1}$(see also Madsen (2010)). ζ_i can be written in more compact form as

$$\zeta_i = \frac{c}{\sqrt{N}}\mu_i + \beta_i e_2, \tag{9}$$

where $\frac{c}{\sqrt{N}} = (1-\varphi_N)$, $\mu_i = (a_i - y_{i0} - \beta_i, \beta_i)'$ and $e_2 = (1,0)'$. The following equalities also hold:

$$\begin{aligned}
tr(\Xi) &= 0 \text{ and } tr(\Lambda' B' A) = -tr(B'A), \\
e'\Xi &= 0_{1XT} \text{ and } \Xi e = 0_{TX1}, \\
B'AXe_2 &= 0_{TX1}, && (10) \\
e_2' X'\Lambda' B' A\Lambda X e_2 &= e_2' X'\Lambda' B' AX\tilde{e}, && (11) \\
e_2' X' B' A\Lambda X e_2 &= e_2' X' B' AX\tilde{e}, \\
e_2' X'\Phi_p \Lambda X e_2 &= e_2' X'\Phi_p X\tilde{e}, && (12) \\
e_2' X'\Lambda'\Phi_p X e_2 &= \tilde{e}' X'\Phi_p X e_2, && (13)
\end{aligned}$$

where $\tilde{e} = (-1,1)'$. Consider the following formula of test statistic UB_T:

$$\sqrt{N}\hat{\delta}\left(\hat{\varphi}_{FOD} - \varphi_N - \frac{\hat{b}}{\hat{\delta}}\right) = \tag{14}$$

$$= \sqrt{N}\hat{\delta}\left(1 + \frac{\frac{1}{N}\sum_{i=1}^N z_i' B' A\Delta z_i}{\frac{1}{N}\sum_{i=1}^N z_i' B' B z_i} - \varphi_N - \frac{\frac{1}{N}\sum_{i=1}^N \Delta z_i'\Phi_p \Delta z_i}{\frac{1}{N}\sum_{i=1}^N z_i' B' B z_i}\right) \tag{15}$$

$$= \frac{c}{N}\sum_{i=1}^N z_i' B' B z_i + \frac{1}{\sqrt{N}}\sum_{i=1}^N z_i' B' A\Delta z_i - \frac{1}{\sqrt{N}}\sum_{i=1}^N \Delta z_i'\Phi_p \Delta z_i = (I) + (II) + (III).$$

The limiting distribution of the above statistic is derived by taking limits of (I), (II) and (III), for $N \to \infty$. To derive the limit of (I), we will employ (5). Then, (I) can be written as $\frac{c}{N}\sum_{i=1}^N z_i' B' B z_i = \frac{c}{N}\sum_{i=1}^N \phi_N^2 z_{i-1}' B' B z_{i-1} + \phi_N z_{i-1}' B' B X\zeta_i + \phi_N z_{i-1}' B' B u_i + \phi_N \zeta_i' X' B' B z_{i-1} + \zeta_i' X' B' B X\zeta_i + \zeta_i' X' B' B u_i + \phi_N u_i' B' B z_{i-1} + u_i' B B' X\zeta_i + u_i' B' B u_i$. Using (6) and (8) and (9), the first term of the last relationship can be written as $\frac{c}{N}\sum_{i=1}^N \phi_N^2 z_{i-1}' B' B z_{i-1} = \frac{c}{N}\sum_{i=1}^N z_{i-1}' B' B z_{i-1} + o_p(1) = \frac{c}{N}\sum_{i=1}^N (\beta_i e_2' X'\Lambda' + u_i'\Lambda') B' B(\Lambda X e_2 \beta_i + \Lambda u_i) + o_p(1)$. Since the sum is multiplied by $\frac{1}{N}$, any summand coming from the expansion of it which is also multiplied by $\frac{1}{N}$, or $\frac{1}{\sqrt{N}}$, will be asymptotically negligible, $o_p(1)$. By KWLLN and standard results on quadratic forms (see Schott (1996)), we can show that $\frac{c}{N}\sum_{i=1}^N (\beta_i e_2' X'\Lambda' + u_i'\Lambda') B' B(\Lambda X e_2 \beta_i + \Lambda u_i) \xrightarrow{p} c\left[E(\beta_i^2) e_2' X'\Lambda' B' B\Lambda X e_2 + tr(\Lambda' B' B\Lambda\Gamma)\right]$. Following analogous arguments to the above, it can be shown that

$$(I): \frac{c}{N}\sum_{i=1}^{N} z_i' B' B z_i \xrightarrow{p} c \begin{bmatrix} tr(\Lambda' B' B \Lambda \Gamma) + tr(\Lambda' B' B \Gamma) + tr(B' B \Lambda \Gamma) + tr(B' B \Gamma) \\ +E(\beta_i^2) e_2' X' \Lambda' B' B \Lambda X e_2 + E(\beta_i^2) e_2' X' \Lambda' B' B X e_2 \\ +E(\beta_i^2) e_2' X' B' B \Lambda X e_2 + E(\beta_i^2) e_2' X' B' B X e_2 \end{bmatrix} \quad (16)$$

Similarly, we can show

$$(II): \frac{1}{\sqrt{N}}\sum_{i=1}^{N} z_i' B' A \Delta z_i \xrightarrow{p} N(c\mu_1, V_{(II)}) \quad (17)$$

where $\mu_1 = c \begin{bmatrix} -tr(\Lambda' B' A \Lambda \Gamma) - tr(\Lambda' B' A \Gamma) - tr(B' A \Lambda \Gamma) - tr(F' B' A \Gamma) \\ -E(\beta_i^2) e_2' X' \Lambda' B' A \Lambda X e_2 + E(\beta_i^2) \tilde{e}' X' \Lambda' B' A X \tilde{e} + \\ -E(\beta_i^2) e_2' X' B' A \Lambda X e_2 + E(\beta_i^2) e_2' X' B' A X \tilde{e} \end{bmatrix} + tr(\Lambda' B' A \Gamma) +$ $tr(B' A \Gamma)$ and

$$(III): -\frac{1}{\sqrt{N}}\sum_{i=1}^{N} \Delta z_i' \Phi_p \Delta z_i \xrightarrow{p} N(c\mu_2, V_{(III)}) \quad (18)$$

where $\mu_2 = c \begin{bmatrix} tr(\Lambda' \Phi_p \Gamma) + tr(\Phi_p \Lambda \Gamma) \\ +E(\beta_i^2) e_2' X' \Lambda' \Phi_p X e_2 + E(\beta_i^2) e_2' X' \Phi_p \Lambda X e_2 \\ -E(\beta_i^2) e_2' X' \Phi_p X \tilde{e} - E(\beta_i^2) \tilde{e}' X' \Phi_p X e_2 \end{bmatrix} - tr(\Phi_p \Gamma)$. Summing up the results in (16), (17) and (18) and using the results of equations (13), we can prove the result of Theorem 2. Note that the variance functions of the limiting distributions of quantities (I) and (II): $V_{(II)}$ and $V_{(III)}$, as well as their covariance do not need to be calculated, given that they are equal to variance V of the test statistic UB_T, under H_0: $c = 0$. This happens because these functions are independent of c (see also Breitung (2000)).