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Confidence sets for the date of a break in level and trend when the order of integration is unknown

by

David Harvey and Stephen Leybourne

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Confidence Sets for the Date of a Break in Level and Trend when the Order of Integration is Unknown*

David I. Harvey and Stephen J. Leybourne
School of Economics, University of Nottingham

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Abstract

We propose methods for constructing confidence sets for the timing of a break in level and/or trend that have asymptotically correct coverage for both I(0) and I(1) processes. These are based on inverting a sequence of tests for the break location, evaluated across all possible break dates. We separately derive locally best invariant tests for the I(0) and I(1) cases; under their respective assumptions, the resulting confidence sets provide correct asymptotic coverage regardless of the magnitude of the break. We suggest use of a pre-test procedure to select between the I(0)- and I(1)-based confidence sets, and Monte Carlo evidence demonstrates that our recommended procedure achieves good finite sample properties in terms of coverage and length across both I(0) and I(1) environments. An application using US macroeconomic data is provided which further evinces the value of these procedures.

Keywords: Level break; Trend break; Stationary; Unit root; Locally best invariant test; Confidence sets.
JEL Classification: C22.

1 Introduction

It has now been widely established that structural change in the time series properties of macroeconomic and financial time series is commonplace (see, *inter alia*, Stock and Watson (1996)), and much work has been devoted to this area of research in the literature. Focusing on the underlying trend function of a series, the primary issues to be resolved when considering the possibility of structural change are whether a break is present, and, if so, when the break occurred. The focus of this paper concerns the latter issue regarding the timing of the break, and is therefore complementary to procedures that focus on break detection. A proper understanding of the likely timing of a break in the trend function is crucial for modelling and forecasting efforts, and is also of clear importance when attempting to

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Correspondence to: David Harvey, School of Economics, University of Nottingham, Nottingham, NG7 2RD, UK. E-mail: dave.harvey@nottingham.ac.uk

gain economic insight into the cause and impact of a break. While a number of procedures exist to determine a point estimate of a break in level and/or trend, this paper concentrates on ascertaining the degree of uncertainty surrounding break date estimation by developing procedures for calculating a confidence set for the break date, allowing practitioners to identify a valid set of possible break points with a specified degree of confidence.

The methodology of Bai (1994) allows construction of a confidence set for a break in level in a time series, extended in Bai (1997) to allow for a break in trend, with the confidence set comprised of a confidence interval surrounding an estimated break point, with the interval derived from the asymptotic distribution of the break date estimator. However, as Elliott and Müller (2007) [EM] argue, the asymptotic theory employed in this approach relies on the break magnitude being in some sense "large", in that the magnitude can be asymptotically shrinking only at a rate sufficiently slow to permit break detection procedures to have power close to one, so that although the magnitude is asymptotically vanishing, the break is still large enough to be readily detectable. EM argue that in many practical applications it is "small" breaks (for which detection is somewhat uncertain) that are typically encountered, and these authors go on to demonstrate that for smaller magnitude breaks, the Bai approach results in confidence sets that suffer from coverage rates substantially below the nominal level, with the true break date being excluded from the confidence set much too frequently. EM suggest an alternative approach to deriving confidence sets that achieve asymptotic validity, based on inverting a sequence of tests of the null that the break occurs at a maintained date, with the resulting confidence set comprised of all maintained dates for which the corresponding test did not reject. By deriving a locally best invariant test that is invariant to the magnitude of the break under the null, the EM confidence sets have asymptotically correct coverage, regardless of the magnitude of the break (and therefore regardless of whether the magnitude is treated as fixed or asymptotically vanishing).

The EM model and assumptions pertain to a break in a linear time series regression, of which a break in level is a special case. They do not, however, consider the case of a break in linear trend, hence our first contribution is to develop an EM-type methodology for calculating asymptotically valid confidence sets for the date of a break in trend (and/or level). As in their approach, we derive a locally best invariant test of the null that the break occurs at a maintained date, and make an expedient choice for the probability measure used in deriving the test so as to render the resulting test statistic asymptotically invariant to the break timing.

When attempting to specify the deterministic component of an economic time series in practice, a critical consideration is the order of integration of the stochastic element of the process. Given the prevalence of integrated data, it is important to develop methods that are valid in the presence of I(1) shocks. Moreover, since there is very often a large degree of uncertainty regarding the order of integration in any given series, it is extremely useful to have available techniques that are robust to the order of integration, dealing with the potential for either stationary or unit root behaviour at the same time as specifying the deterministic component. A body of work has developed in recent years focusing on such concerns, developing order of integration-robust tests for a linear trend (e.g. Vogelsang (1998), Bunzel and Vogelsang (2005), Harvey *et al.* (2007), Perron and Yabu (2009a)), tests for a break in trend (e.g. Harvey *et al.* (2009), Perron and Yabu (2009b), Saygınsoy and Vogelsang (2011)), and tests for multiple breaks in level (e.g. Harvey *et al.* (2010)). Most recently, Harvey and Leybourne (2013) have proposed methods for estimating the date of a break in level and trend that performs well for both I(0) and I(1) shocks.

In the current context, it is clear that reliable specification of confidence sets for the date of a break in level/trend will be dependent on the order of integration of the data under consideration. Perron and Zhu (2005) extend the results of Bai (1994, 1997) to allow for I(1), as well as I(0), processes when estimating the timing of a break in trend or level and trend, and different distributional results are obtained under I(0) and I(1) assumptions. Similarly, and as would be expected, we show that the EM procedure for calculating confidence sets, which is appropriate for I(0) shocks, does not result in sets with asymptotically correct coverage when the driving shocks are actually I(1). However, extension to the I(1) case is possible via a modified approach applied to the first differences of the data, whereby the level break and trend break are transformed into an outlier and a level break, respectively. This development comprises the second main contribution of our paper. Since there is typically uncertainty surrounding the integration order in practice, we propose a unit root pre-test-based procedure for calculating confidence sets that are asymptotically valid regardless of the order of integration of the data. We find the new procedure allows construction of confidence sets with correct asymptotic coverage under both I(0) and I(1) shocks (irrespective of the magnitude of the break). We also examine the performance of our procedure under local-to-I(1) shocks, and find that it displays asymptotic over-coverage (i.e. coverage rates above the nominal level), hence the confidence sets are asymptotically conservative in such situations, including the true date in the confidence set at least as frequently as the nominal rate would suggest. Monte Carlo simulations demonstrate that our recommended procedure performs well in finite samples, in terms of both coverage and length (the number of dates included in the confidence set as a proportion of the sample size).

The paper is structured as follows. Section 2 sets out the level/trend break model. Section 3 derives the locally best invariant tests for a break at a maintained date in both the stationary and unit root environments. The large sample properties under the null of correct break placement are established when correct and incorrect orders of integration are assumed, with the implications discussed for the corresponding confidence sets based on these tests. The properties of feasible variants of these tests, and corresponding confidence sets, are subsequently investigated. In section 4 we propose use of a unit root pre-test to select between I(0) and I(1) confidence sets when the order of integration is not known. The finite sample behaviour of the various procedures is examined in section 5. Here we also consider trimming as a means of potentially shortening the confidence sets. Section 6 provides empirical illustrations of our proposed procedure using US macroeconomic data, while section 7 concludes.

The following notation is also used: '$\lfloor \cdot \rfloor$' denotes the integer part, '$\Rightarrow$' denotes weak convergence, and '$1(\cdot)$' denotes the indicator function.

2 The model and confidence sets

We consider the following model which allows for a level and/or a trend break in either a stationary or unit root process. The DGP for an observed series y_t we assume is given by

$$y_t = \beta_1 + \beta_2 t + \delta_1 1(t > \lfloor \tau_0 T \rfloor) + \delta_2 (t - \lfloor \tau_0 T \rfloor) 1(t > \lfloor \tau_0 T \rfloor) + \varepsilon_t, \quad t = 1, ..., T \tag{1}$$

$$\varepsilon_t = \rho \varepsilon_{t-1} + u_t, \quad t = 2, ..., T, \quad \varepsilon_1 = u_1 \tag{2}$$

with $\lfloor \tau_0 T \rfloor \in \{2, ..., T-2\} \equiv \Lambda_T$ the level and/or trend break point with associated break fraction τ_0. In (1), a level break occurs at time $\lfloor \tau_0 T \rfloor$ when $\delta_1 \neq 0$; likewise, a trend break occurs if $\delta_2 \neq 0$. The parameters β_1, β_2, δ_1 and δ_2 are unknown, as is the break point $\lfloor \tau_0 T \rfloor$, inference on which is the central focus of our analysis. Our generic specification for ε_t is given by (2) assuming that $-1 < \rho \leq 1$ and that u_t is I(0).

For an assumed break point $\lfloor \tau T \rfloor \in \Lambda_T$, our interest centres on testing whether or not $\lfloor \tau_0 T \rfloor$ and $\lfloor \tau T \rfloor$ coincide, which we can write in hypothesis testing terms as a test of the null hypothesis $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ against the alternative $H_1 : \lfloor \tau_0 T \rfloor \neq \lfloor \tau T \rfloor$. Then, following EM, a $(1-\alpha)$-level confidence set for τ_0 is constructed by inverting a sequence of α-level tests of $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ for $\lfloor \tau T \rfloor \in \Lambda_T$, with the resulting confidence set comprised of all $\lfloor \tau T \rfloor$ for which H_0 is not rejected. Provided the test of $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ has size α for all $\lfloor \tau T \rfloor$, the confidence set will have correct coverage, since the probability of excluding τ_0 from the confidence set (via a spurious rejection of H_0) is α. In terms of confidence set length, a shorter than $(1-\alpha)$-level confidence set arises whenever the tests of $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ reject with probability greater than α under the alternative $H_1 : \lfloor \tau_0 T \rfloor \neq \lfloor \tau T \rfloor$ across $\lfloor \tau T \rfloor$. Other things equal, the more powerful a test is in distinguishing between H_0 and H_1, the shorter this confidence set should be. Note that this approach to constructing confidence sets does not guarantee that the set is comprised of contiguous sample dates, cf. EM (p. 1207).

In the next section, we consider construction of powerful tests of H_0 against H_1, deriving locally best invariant tests along the lines of EM when $\rho = 0$ and when $\rho = 1$, under a Gaussianity assumption for u_t. The large sample properties of these tests are subsequently established under weaker conditions for ρ and u_t.

3 Locally best invariant tests

For the purposes of constructing locally best invariant tests, we make the standard assumption that $u_t \sim NIID(0, \sigma_u^2)$, and we suppose that ρ in (2) is restricted to taking the two values $\rho = 0$ or $\rho = 1$. In the case of $\rho = 0$, we find that (1) reduces to

$$y_t = \beta_1 + \beta_2 t + \delta_1 1(t > \lfloor \tau_0 T \rfloor) + \delta_2 (t - \lfloor \tau_0 T \rfloor) 1(t > \lfloor \tau_0 T \rfloor) + u_t, \quad t = 1, ..., T \tag{3}$$

while for $\rho = 1$, (1) can be written as

$$\Delta y_t = \beta_2 + \delta_1 1(t = \lfloor \tau_0 T \rfloor + 1) + \delta_2 1(t > \lfloor \tau_0 T \rfloor) + u_t, \quad t = 2, ..., T. \tag{4}$$

Now write either of the models (3) or (4), for an arbitrary break point $\lfloor \tau T \rfloor$, in the generic form

$$z_t = d_t'\beta + d_{\tau,t}'\delta + u_t \tag{5}$$

where $\delta = [\delta_1\ \delta_2]'$, and, under (3), $z_t = y_t$, $d_t = [1\ t]'$, $\beta = [\beta_1\ \beta_2]'$, $d_{\tau,t} = [1(t > \lfloor \tau T \rfloor)\ (t - \lfloor \tau T \rfloor)1(t > \lfloor \tau T \rfloor)]'$; while under (4), $z_t = \Delta y_t$, $d_t = 1$, $\beta = \beta_2$, $d_{\tau,t} = [1(t = \lfloor \tau T \rfloor + 1)\ 1(t > \lfloor \tau T \rfloor)]'$. In an obvious matrix form, (5) can be expressed as

$$z = D\beta + D_\tau\delta + u. \tag{6}$$

We consider tests based on $\hat{u}$, the vector of OLS residuals from the regression (6), that is, $\hat{u} = M_\tau z$, where $M_\tau = I - C_\tau(C_\tau' C_\tau)^{-1}C_\tau$ with $C_\tau = [D : D_\tau]$. Such tests are by construction invariant to the unknown parameters β and δ under H_0. The likelihood ratio statistic for testing H_0 against H_1 can then be derived as follows. Let k^* and T^* denote the number of regressors and the effective sample size, respectively, in the regression (6). Also, let B_τ be the $T^* \times (T^* - k^*)$ matrix defined such that $B_\tau' B_\tau = I_{T^*-k^*}$ and $B_\tau B_\tau' = M_\tau$. Since $B_\tau' z = B_\tau' \hat{u}$ is invariant to β, it follows that, on setting $\beta = 0$ without loss of generality, $B_\tau' z \sim N(B_\tau' D_{\tau_0}\delta, \sigma_u^2 I_{T^*-k^*})$ under H_1. Under H_0, $B_\tau' z = B_{\tau_0}' z$ is also invariant to δ, hence, on setting $\delta = \beta = 0$ without loss of generality, $B_\tau' z \sim N(0, \sigma_u^2 I_{T^*-k^*})$. The likelihood ratio statistic is then

$$\begin{aligned}
LR(\tau, \delta, \tau_0) &= \frac{(2\pi\sigma_u^2)^{-(T^*-k^*)/2} \exp\{-(2\sigma_u^2)^{-1}(B_\tau' z - B_\tau' D_{\tau_0}\delta)'(B_\tau' z - B_\tau' D_{\tau_0}\delta)\}}{(2\pi\sigma_u^2)^{-(T^*-k^*)/2} \exp\{-(2\sigma_u^2)^{-1}(B_\tau' z)' B_\tau' z\}} \\
&= \exp[-(2\sigma_u^2)^{-1}\{(B_\tau' z - B_\tau' D_{\tau_0}\delta)'(B_\tau' z - B_\tau' D_{\tau_0}\delta) - (B_\tau' z)' B_\tau' z\}] \\
&= \exp\{\sigma_u^{-2} z' B_\tau B_\tau' D_{\tau_0}\delta - \tfrac{1}{2}\sigma_u^{-2}\delta' D_{\tau_0}' B_\tau B_\tau' D_{\tau_0}\delta\} \\
&= \exp(\sigma_u^{-2}\hat{u}' D_{\tau_0}\delta - \tfrac{1}{2}\sigma_u^{-2}\delta' D_{\tau_0}' M_\tau D_{\tau_0}\delta).
\end{aligned}$$

Following the approach of Andrews and Ploberger (1994), to remove the dependence of the statistic on the parameters δ and τ_0, we consider tests that maximize the weighted average power criterion

$$\sum_{\substack{\lfloor \eta T \rfloor \in \Lambda_T, \\ \lfloor \eta T \rfloor \neq \lfloor \tau T \rfloor}} \lambda_{\lfloor \eta T \rfloor} \int P(\text{test rejects} |\ \lfloor \tau_0 T \rfloor = \lfloor \eta T \rfloor, \delta = \delta^*) dv_{\lfloor \eta T \rfloor}(\delta^*)$$

over all tests that satisfy $P(\text{test rejects} |\ \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor) = \alpha$, where the weights $\{\lambda_t\}$ are non-negative real numbers and $\{v_t(\delta^*)\}$ is a sequence of non-negative measures on $\mathbb{R}^2$. This yields a test of the form

$$LR(\tau) = \sum_{\substack{\lfloor \eta T \rfloor \in \Lambda_T, \\ \lfloor \eta T \rfloor \neq \lfloor \tau T \rfloor}} \lambda_{\lfloor \eta T \rfloor} \int LR(\tau, f, \eta) dv_{\lfloor \eta T \rfloor}(f).$$

As in EM, we set $\lambda_{\lfloor \eta T \rfloor} = 1$, such that equal weights are placed on alternative break dates, and take $v_{\lfloor \eta T \rfloor}(f)$ to be a probability measure of $N(0, b^2 H_{\lfloor \eta T \rfloor})$. We then obtain (after some algebra)

$$LR(\tau) = \sum_{\substack{\lfloor \eta T \rfloor \in \Lambda_T, \\ \lfloor \eta T \rfloor \neq \lfloor \tau T \rfloor}} \left| I + b^2\sigma_u^{-2} H_{\lfloor \eta T \rfloor} D_\eta' M_\tau D_\eta \right|^{-1/2} \exp\{\tfrac{1}{2} b^2 \sigma_u^{-4} \hat{u}' D_\eta (H_{\lfloor \eta T \rfloor}^{-1} + b^2\sigma_u^{-2} D_\eta' M_\tau D_\eta)^{-1} D_\eta' \hat{u}\}.$$

Taking a first order Taylor series expansion of $LR(\tau)$ in the locality of $b^2 = 0$, we find that the stochastic component of $LR(\tau)$, up to a constant of proportionality, is given by

$$S(\tau) = \sum_{\substack{\lfloor \eta T \rfloor \in \Lambda_T, \\ \lfloor \eta T \rfloor \neq \lfloor \tau T \rfloor}} \hat{u}' D_\eta H_{\lfloor \eta T \rfloor} D'_\eta \hat{u}. \tag{7}$$

This represents the locally best invariant test with respect to b^2 that maximizes weighted average power, for given $H_{\lfloor \eta T \rfloor}$.

We specify $H_{\lfloor \eta T \rfloor}$ separately under the models (3) and (4), and, as in EM, we construct the elements of $H_{\lfloor \eta T \rfloor}$ using particular scalings of $\lfloor \tau T \rfloor$ and $(T - \lfloor \tau T \rfloor)$ such that the resulting $S(\tau)$ tests have asymptotic distributions under H_0 that do not depend on τ. This choice is justified by the convenience of allowing the same asymptotic critical value to apply to each of the sequence of individual tests over $\lfloor \tau T \rfloor \in \Lambda_T$. Given these choices for $H_{\lfloor \eta T \rfloor}$, explicit forms for (7) can be derived under both (3) with $\rho = 0$ and (4) with $\rho = 1$, as detailed in the following lemma.

Lemma 1

(a) Under DGP (3) ($\rho = 0$), when

$$H_{\lfloor \eta T \rfloor} = \begin{cases} \begin{bmatrix} \lfloor \tau T \rfloor^{-2} & 0 \\ 0 & \lfloor \tau T \rfloor^{-4} \end{bmatrix} & \text{if } \lfloor \eta T \rfloor < \lfloor \tau T \rfloor \\ \begin{bmatrix} (T - \lfloor \tau T \rfloor)^{-2} & 0 \\ 0 & (T - \lfloor \tau T \rfloor)^{-4} \end{bmatrix} & \text{if } \lfloor \eta T \rfloor > \lfloor \tau T \rfloor \end{cases} \tag{8}$$

it follows from (7) that, for testing H_0 against H_1, the locally best invariant test with respect to b^2 is given by

$$\begin{aligned} S_0(\tau) \;=\; & \lfloor \tau T \rfloor^{-2} \sum_{t=2}^{\lfloor \tau T \rfloor - 1} \left(\sum_{s=1}^{t} \hat{u}_s \right)^2 + \lfloor \tau T \rfloor^{-4} \sum_{t=2}^{\lfloor \tau T \rfloor - 1} \left(\sum_{s=1}^{t} (s-t) \hat{u}_s \right)^2 \\ & + (T - \lfloor \tau T \rfloor)^{-2} \sum_{t=\lfloor \tau T \rfloor + 1}^{T-2} \left(\sum_{s=\lfloor \tau T \rfloor + 1}^{t} \hat{u}_s \right)^2 + (T - \lfloor \tau T \rfloor)^{-4} \sum_{t=\lfloor \tau T \rfloor + 1}^{T-2} \left(\sum_{s=\lfloor \tau T \rfloor + 1}^{t} (s-t) \hat{u}_s \right)^2 \end{aligned} \tag{9}$$

where $\{\hat{u}_t\}_{t=1}^{T}$ denote the residuals from OLS estimation of (3) when $\lfloor \tau_0 T \rfloor$ is replaced by $\lfloor \tau T \rfloor$.

(b) Under DGP (4) ($\rho = 1$), when

$$H_{\lfloor \eta T \rfloor} = \begin{cases} \begin{bmatrix} \lfloor \tau T \rfloor^{-1} & 0 \\ 0 & \lfloor \tau T \rfloor^{-2} \end{bmatrix} & \text{if } \lfloor \eta T \rfloor < \lfloor \tau T \rfloor \\ \begin{bmatrix} (T - \lfloor \tau T \rfloor)^{-1} & 0 \\ 0 & (T - \lfloor \tau T \rfloor)^{-2} \end{bmatrix} & \text{if } \lfloor \eta T \rfloor > \lfloor \tau T \rfloor \end{cases} \tag{10}$$

it follows from (7) that, for testing H_0 against H_1, the locally best invariant test with respect to b^2 is given by

$$S_1(\tau) = \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor -1} \hat{u}_{t+1}^2 + \lfloor \tau T \rfloor^{-2} \sum_{t=2}^{\lfloor \tau T \rfloor -1} \left(\sum_{s=2}^{t} \hat{u}_s \right)^2 + (T - \lfloor \tau T \rfloor)^{-1} \sum_{t=\lfloor \tau T \rfloor +1}^{T-2} \hat{u}_{t+1}^2 + (T - \lfloor \tau T \rfloor)^{-2} \sum_{t=\lfloor \tau T \rfloor +1}^{T-2} \left(\sum_{s=\lfloor \tau T \rfloor +1}^{t} \hat{u}_s \right)^2 \quad (11)$$

where $\{\hat{u}_t\}_{t=2}^{T}$ denote the residuals from OLS estimation of (4) when $\lfloor \tau_0 T \rfloor$ is replaced by $\lfloor \tau T \rfloor$.

3.1 Large sample properties of the test procedures

Now we have the structures of the tests in place, we can derive their large sample properties under more general assumptions regarding ρ and u_t. Here we make one of the two following assumptions:

Assumption I(0) Let $|\rho| < 1$, $u_t = C(L)\zeta_t$, $C(L) = \sum_{i=0}^{\infty} C_i L^i$, $C_0 = 1$, with $C(z) \neq 0$ for all $|z| \leq 1$ and $\sum_{i=0}^{\infty} i|C_i| < \infty$, and where ζ_t is an IID sequence with mean zero, variance σ^2 and finite fourth moment.

Under Assumption I(0) we define the long-run variance of u_t as $\omega_u^2 = \lim_{T\to\infty} T^{-1} E(\sum_{t=1}^{T} u_t)^2 = \sigma^2 C(1)^2$. Note that the long-run variance of ε_t is then given by $\omega_\varepsilon^2 = \omega_u^2/(1-\rho)^2$.

Assumption I(1) Let $\rho = 1$ with u_t defined as in Assumption I(0).

Under Assumption I(1) we also define the short-run variance of u_t as $\sigma_u^2 = E(u_t^2)$. The theorem below gives the null limiting distributions of the efficient tests $S_0(\tau)$ and $S_1(\tau)$ under Assumptions I(0) and I(1), respectively.

Theorem 1

(a) Under $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ and Assumption I(0),

$$\omega_\varepsilon^{-2} S_0(\tau) \Rightarrow \int_0^1 B_2(r)^2 dr + \int_0^1 K(r)^2 dr + \int_0^1 B_2'(r)^2 dr + \int_0^1 K'(r)^2 dr \equiv \mathcal{L}_0.$$

(b) Under $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ and Assumption I(1),

$$\omega_u^{-2} \{S_1(\tau) - 2\sigma_u^2\} \Rightarrow \int_0^1 B_1(r)^2 dr + \int_0^1 B_1'(r)^2 dr \equiv \mathcal{L}_1$$

where

$$\begin{aligned} B_1(r) &= B(r) - rB(1), \\ B_2(r) &= B(r) - rB(1) + 6r(1-r)\left\{ \frac{1}{2}B(1) - \int_0^1 B(s)ds \right\}, \\ K(r) &= -r^2(1-r)B(1) - \int_0^r B(s)ds + r^2(3-2r)\int_0^1 B(s)ds \end{aligned}$$

with $B(r)$ a standard Brownian motion process, and where $B_1'(r)$, $B_2'(r)$and $K'(r)$ take the same forms as $B_1(r)$, $B_2(r)$ and $K(r)$, respectively, but with $B(r)$ replaced by $B'(r)$, with $B'(r)$ a standard Brownian motion process independent of $B(r)$. (Note that $B_1(r)$, $B_2(r)$ and $K(r)$ are tied down and $B_j(r)$ is a j'th level Brownian bridge.)

Remark 1 Note that, as desired, $\omega_\varepsilon^{-2}S_0(\tau)$ and $\omega_u^{-2}\{S_1(\tau) - 2\sigma_u^2\}$ have nuisance-parameter free distributions that do not depend on τ. This property arises from the specific functions for $H_{\lfloor \eta T \rfloor}$ adopted, justifying the $H_{\lfloor \eta T \rfloor}$ choices made in Lemma 1. Note also that the $\mathcal{L}_1$ distribution coincides with the null limit distribution of the test proposed by EM in the case of a single regressor that is subject to a break.

Remark 2 The result in Theorem 1 (b) is obtained because both the first and third terms of $S_1(\tau)$ in (11) converge in probability to σ_u^2. These components of $S_1(\tau)$ are associated with testing on the one-time dummy variable in (4), and it can easily be shown that these terms also converge in probability to σ_u^2 under the alternative H_1 when only a level break occurs under Assumption I(1), i.e. when an outlier of magnitude δ_1 is present in the I(0) first differences of the series. As such, $S_1(\tau)$ does not have asymptotic power for identifying the date of a break in level in I(1) data. This is to be expected given that an unscaled level break is asymptotically irrelevant in an I(1) series. However, retaining these terms in the statistic (11), along with a judicious choice of σ_u^2 estimator (discussed below), can yield finite sample performance benefits, hence we do not omit these terms from the $S_1(\tau)$ statistic.

Remark 3 A theoretical alternative to our approach would be to attempt to endow the first and third terms of $S_1(\tau)$ with a null limit distribution rather than a probability limit. However, this would require a rescaled and centered variant of the form $\lfloor \tau T \rfloor^{-1/2} \sum_{t=2}^{\lfloor \tau T \rfloor -1} (\hat{u}_{t+1}^2 - \sigma_u^2)$ for the first component (and similarly for the third component). This introduces two complications; first, σ_u^2 is unknown and ultimately needs replacing with an estimator, which we generically denote $\tilde{\sigma}_u^2$. Since $\tilde{\sigma}_u^2$ is at best $O_p(T^{-1/2})$-consistent for σ_u^2, it follows that the asymptotic distribution of $\lfloor \tau T \rfloor^{-1/2} \sum_{t=2}^{\lfloor \tau T \rfloor -1} (\hat{u}_{t+1}^2 - \tilde{\sigma}_u^2)$ will be different to that of $\lfloor \tau T \rfloor^{-1/2} \sum_{t=2}^{\lfloor \tau T \rfloor -1} (\hat{u}_{t+1}^2 - \sigma_u^2)$. Secondly, even if σ_u^2 is known, $\lfloor \tau T \rfloor^{-1/2} \sum_{t=2}^{\lfloor \tau T \rfloor -1} (\hat{u}_{t+1}^2 - \sigma_u^2)$ implicitly involves the partial sum process $T^{-1/2} \sum_{t=1}^{\lfloor rT \rfloor} (u_t^2 - \sigma_u^2)$, while the third term of $S_1(\tau)$ involves the partial sum process $T^{-1/2} \sum_{t=1}^{\lfloor rT \rfloor} u_t$; the joint limit distribution of these two partial sum processes depends on the third moment of u_t, which is also unknown. As a result, we adopt the more analytically tractable specification outlined in Lemma 1 (b).

Table 1 gives simulated (upper tail) α-level critical values for the limit distributions $\mathcal{L}_0$ and $\mathcal{L}_1$. These were obtained by direct simulation of the limiting distributions given in Theorem 1, approximating the Brownian motion processes using $NIID(0,1)$ random variates, and with the integrals approximated by normalized sums of 2000 steps. The simulations were programmed in Gauss 9.0 using 50,000 Monte Carlo replications. If these critical values are applied to each of the sequence of tests $\omega_\varepsilon^{-2} S_0(\tau)$ under Assumption I(0), and $\omega_u^{-2}\{S_1(\tau) - 2\sigma_u^2\}$ under Assumption I(1), the corresponding confidence set based on inverting these tests will have asymptotically correct coverage of $(1-\alpha)$, regardless of the magnitude of the break in level and/or trend.

We next consider the behaviour of $S_0(\tau)$ and $S_1(\tau)$ under H_0 when an incorrect assumption regarding the value of ρ is made.

Theorem 2

(a) Under $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ and Assumption I(1),

$$\omega_\varepsilon^{-2} S_0(\tau) = O_p(T^2).$$

(b) Under $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ and Assumption I(0),

$$\omega_u^{-2}\{S_1(\tau) - 2\sigma_u^2\} = \omega_u^{-2} 2\{E(\Delta\varepsilon_t)^2 - \sigma_u^2\} + O_p(T^{-1/2}).$$

Theorem 1 (a) shows that a (nominal) $(1-\alpha)$-level confidence set based on $\omega_\varepsilon^{-2} S_0(\tau)$ will be asymptotically empty (i.e. zero coverage) as all the test statistics diverge to $+\infty$ and thereby exceed the α-level critical value in the limit. Theorem 1 (b) shows that $\omega_u^{-2}\{S_1(\tau) - 2\sigma_u^2\}$ converges in probability to a constant that takes the value $\omega_u^{-2} 2\{E(\Delta\varepsilon_t)^2 - \sigma_u^2\}$. If this constant exceeds the α-level critical value, then the confidence set based on $\omega_u^{-2}\{S_1(\tau) - 2\sigma_u^2\}$ will also be asymptotically empty (zero coverage); if it is less than the α-level critical value, then the confidence set based on $\omega_u^{-2}\{S_1(\tau) - 2\sigma_u^2\}$ will be asymptotically full (i.e. coverage of unity). Which of these two cases pertains will depend on the values of ω_u^2, $E(\Delta\varepsilon_t)^2$ and σ_u^2. Trivially, a sufficient condition for the latter case is $E(\Delta\varepsilon_t)^2 \leq \sigma_u^2$, since then $\omega_u^{-2}\{S_1(\tau) - 2\sigma_u^2\}$ assumes a negative probability limit, which can never exceed the (positive) asymptotic critical value. Clearly then, an incorrect assumption regarding the order of integration of ε_t negates the validity of confidence sets based on inverting sequences of these efficient tests, an issue we revisit in section 4.

The tests considered so far are clearly infeasible since they depend on the unknown parameters ω_ε^2, or ω_u^2 and σ_u^2. In the next section we examine some feasible versions of the tests and reassess the content of Theorems 1 and 2 in the context of these.

3.2 Feasible test procedures and their large sample properties

To make the tests feasible, we require suitable estimators of ω_ε^2 for $S_0(\tau)$ and ω_u^2 and σ_u^2 for $S_1(\tau)$. To estimate the long-run variances ω_ε^2 and ω_u^2 we consider both non-parametric and parametric approaches. In the non-parametric case, we employ the Bartlett kernel-based estimators

$$\hat{\omega}_{i,NP}^2(\tau) = \hat{\gamma}_{i,0}(\tau) + 2\sum_{l=1}^{\ell_{NP}} h(l, \ell_{NP})\hat{\gamma}_{i,l}(\tau), \quad \hat{\gamma}_{i,l}(\tau) = T^{-1}\sum_{t=l+1}^{T} \hat{u}_t \hat{u}_{t-l}$$

for $i = \{\varepsilon, u\}$, where the $\hat{u}_t$ are the residuals obtained from OLS estimation of regression (3) when $i = \varepsilon$ and (4) when $i = u$.[1] Here, $h(l, \ell_{NP}) = 1 - l/(\ell_{NP} + 1)$, with a lag truncation parameter ℓ_{NP} that is assumed to satisfy the standard condition that, as $T \to \infty$, $1/\ell_{NP} + \ell_{NP}^3/T \to 0$.

[1]For economy of notation we do not discriminate between the different numbers of $\hat{u}_t$ available in the two cases.

In the parametric case, we employ Berk-type autoregressive spectral density estimators which can be written as

$$\hat{\omega}^2_{i,P}(\tau) = \frac{s_i^2}{\hat{\pi}_i^2}$$

where $\hat{\pi}_i$ is obtained from the fitted OLS regression

$$\Delta\hat{u}_t = \hat{\pi}\hat{u}_{t-1} + \sum_{l=1}^{\ell_P} \hat{\psi}_j \Delta\hat{u}_{t-l} + \hat{e}_t, \quad t = \ell_P + 1, ..., T$$

and $s_i^2 = T^{-1}\sum_{t=\ell_P+1}^{T} \hat{e}_t^2$. Again, the $\hat{u}_t$ are obtained from (3) if $i = \varepsilon$, and from (4) if $i = u$. Also, ℓ_P is assumed to have the same properties as ℓ_{NP} above.

It is also natural to consider estimating σ_u^2 with $\hat{\sigma}_u^2(\tau) = \hat{\gamma}_{u,0}(\tau)$ using the $\hat{u}_t$ from (4). The following lemma gives the large sample behaviour of the various estimators.

Lemma 2

(a) Under $H_0 : \lfloor\tau_0 T\rfloor = \lfloor\tau T\rfloor$ and Assumption I(0),

$$\begin{aligned}
\hat{\omega}^2_{\varepsilon,NP}(\tau),\ \hat{\omega}^2_{\varepsilon,P}(\tau) &\xrightarrow{p} \omega^2_\varepsilon, \\
\hat{\omega}^2_{u,NP}(\tau) &= O_p(\ell_{NP}^{-1}), \\
\hat{\omega}^2_{u,P}(\tau) &= O_p(\ell_P^{-2}), \\
\hat{\sigma}^2_u(\tau) &= E(\Delta\varepsilon_t)^2 + O_p(T^{-1/2}).
\end{aligned}$$

(b) Under $H_0 : \lfloor\tau_0 T\rfloor = \lfloor\tau T\rfloor$ and Assumption I(1),

$$\begin{aligned}
\hat{\omega}^2_{\varepsilon,NP}(\tau) &= O_p(\ell_{NP}T), \\
\hat{\omega}^2_{\varepsilon,P}(\tau) &= O_p(T^2), \\
\hat{\omega}^2_{u,NP}(\tau),\ \hat{\omega}^2_{u,P}(\tau) &\xrightarrow{p} \omega^2_u, \\
\hat{\sigma}^2_u(\tau) &\xrightarrow{p} \sigma^2_u.
\end{aligned}$$

The results for $\hat{\omega}^2_{\varepsilon,NP}(\tau)$, $\hat{\omega}^2_{u,NP}(\tau)$ and $\hat{\sigma}^2_u(\tau)$ arise from a simple adaptation of results shown in Harvey *et al.* (2009); those for $\hat{\omega}^2_{\varepsilon,P}(\tau)$ and $\hat{\omega}^2_{u,P}(\tau)$ arise similarly from Harvey *et al.* (2010).

We can now define feasible versions of the statistics as

$$\begin{aligned}
\hat{S}^\tau_{0,j}(\tau) &= \hat{\omega}^{-2}_{\varepsilon,j}(\tau) S_0(\tau), \\
\hat{S}^\tau_{1,j}(\tau) &= \hat{\omega}^{-2}_{u,j}(\tau)\{S_1(\tau) - 2\hat{\sigma}^2_u(\tau)\}
\end{aligned}$$

for $j = \{NP, P\}$. Based on Theorem 1 and Lemma 2, we then have the following corollary.

Corollary 1

(a) Under $H_0 : \lfloor\tau_0 T\rfloor = \lfloor\tau T\rfloor$ and Assumption I(0),

$$\hat{S}^\tau_{0,NP}(\tau),\ \hat{S}^\tau_{0,P}(\tau) \Rightarrow \mathcal{L}_0.$$

(b) Under $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ and Assumption I(1),

$$\hat{S}^{\tau}_{1,NP}(\tau),\ \hat{S}^{\tau}_{1,P}(\tau) \Rightarrow \mathcal{L}_1.$$

These results simply show that when a correct order of integration is assumed (and therefore the appropriate limit critical values are employed), confidence sets based on the feasible tests will continue to provide asymptotically correct coverage. From Theorem 2 and Lemma 2 we have the following corollary.

Corollary 2

(a) Under $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ and Assumption I(1),

$$\begin{aligned} \hat{S}^{\tau}_{0,NP}(\tau) &= O_p(\ell_{NP}^{-1} T) \xrightarrow{p} \infty, \\ \hat{S}^{\tau}_{0,P}(\tau) &= O_p(1). \end{aligned}$$

(b) Under $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ and Assumption I(0),

$$\begin{aligned} \hat{S}^{\tau}_{1,NP}(\tau) &= O_p(\ell_{NP} T^{-1/2}) \xrightarrow{p} 0, \\ \hat{S}^{\tau}_{1,P}(\tau) &= O_p(\ell_P^2 T^{-1/2}) \begin{cases} \xrightarrow{p} 0 & \ell_P = o(T^{1/4}) \\ = O_p(1) & \ell_P = O(T^{1/4}) \\ \xrightarrow{p} \infty & \ell_P^{-1} = o(T^{1/4}) \end{cases}. \end{aligned}$$

Corollary 2 (a) shows that a (nominal) $(1-\alpha)$-level confidence set based on $\hat{S}^{\tau}_{0,NP}(\tau)$ will be asymptotically empty, thereby paralleling the behaviour of its infeasible counterpart. However, the behaviour of a confidence set based on $\hat{S}^{\tau}_{0,P}(\tau)$ is uncertain since it is an $O_p(1)$ variate (whose behaviour will actually depend on ω_u^2). It is, however, almost certain to be the case that this confidence set will have incorrect coverage asymptotically. From Corollary 2 (b), a confidence set based on $\hat{S}^{\tau}_{1,NP}(\tau)$ will be asymptotically full. All possibilities - unit, incorrect (dependent on ω_ε^2) or zero coverage - can arise with $\hat{S}^{\tau}_{1,P}(\tau)$, contingent on how ℓ_P is chosen. The results of Corollary 2 therefore reinforce the importance of assuming a correct order of integration, since use of an incorrect assumption results in a procedure with asymptotic coverage different from $(1-\alpha)$.

We should be aware that the properties of $\hat{\omega}^2_{i,j}(\tau)$ and $\hat{\sigma}^2_u(\tau)$ shown in Lemma 2 - particularly their consistency properties, will *not* hold in general under $H_1 : \lfloor \tau_0 T \rfloor \neq \lfloor \tau T \rfloor$ (the exception being when a level break alone occurs under Assumption I(1)). In view of this, we might entertain employing alternate estimators of $\hat{\omega}^2_{i,j}(\tau)$ and $\hat{\sigma}^2_u(\tau)$ based on some estimator of τ_0. Below we will consider the break fraction estimator derived in Harvey and Leybourne (2013), therein referred to as $\hat{\tau}_{D_m}$. This estimator is the value of τ that yields the minimum sum of squared residuals from an OLS regression of $\mathbf{y}_{\bar{\rho}} = [y_1, y_2 - \bar{\rho} y_1, ..., y_T - \bar{\rho} y_{T-1}]'$ on $\mathbf{Z}_{\bar{\rho},\tau} = [\mathbf{z}_1, \mathbf{z}_2 - \bar{\rho}\mathbf{z}_1, ..., \mathbf{z}_T - \bar{\rho}\mathbf{z}_{T-1}]'$ where $\mathbf{z}_t = [1, t, 1(t > \lfloor \tau T \rfloor), (t - \lfloor \tau T \rfloor) 1(t > \lfloor \tau T \rfloor)]'$ across $\lfloor \tau T \rfloor \in \Lambda_T$ and across $\bar{\rho} \in D_m$. In what follows we set $\Lambda_T = \{\lfloor 0.01T \rfloor, ..., \lfloor 0.99T \rfloor\}$ and, following Harvey and Leybourne (2013), we set $D_m = \{0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.975, 1\}$. It can be shown that $\hat{\omega}^2_{i,j}(\hat{\tau}_{D_m})$ and $\hat{\sigma}^2_u(\hat{\tau}_{D_m})$ have the same asymptotic properties as those for $\hat{\omega}^2_{i,j}(\tau)$ and $\hat{\sigma}^2_u(\tau)$ shown in Lemma 2, and also that these properties will continue to hold under H_1. This gives rise to the potential for power improvements under H_1, and therefore potentially narrower confidence sets. In what follows we therefore also consider versions of the $\hat{S}^{\tau}_{i,j}(\tau)$ procedures where $\hat{\omega}^2_{i,j}(\tau)$ and $\hat{\sigma}^2_u(\tau)$ are replaced with $\hat{\omega}^2_{i,j}(\hat{\tau}_{D_m})$ and $\hat{\sigma}^2_u(\hat{\tau}_{D_m})$, respectively, i.e.

$$\begin{aligned}\hat{S}^{\hat{\tau}}_{0,j}(\tau) &= \hat{\omega}^{-2}_{\varepsilon,j}(\hat{\tau}_{D_m})S_0(\tau),\\ \hat{S}^{\hat{\tau}}_{1,j}(\tau) &= \hat{\omega}^{-2}_{u,j}(\hat{\tau}_{D_m})\{S_1(\tau)-2\hat{\sigma}^2_u(\hat{\tau}_{D_m})\}.\end{aligned}$$

4 Selecting between I(0)- and I(1)-based confidence sets

Given the foregoing discussions, it should be clear that we want to base confidence set construction on the $\hat{S}^k_{0,j}(\tau)$ ($j=\{NP,P\}$, $k=\{\tau,\hat{\tau}\}$) suite of test statistics under Assumption I(0) and the $\hat{S}^k_{1,j}(\tau)$ statistics under Assumption I(1). One way or another, in practice this has to involve deciding whether a given data set is more compatible with Assumption I(0) or Assumption I(1) and then applying $\hat{S}^k_{0,j}(\tau)$ or $\hat{S}^k_{1,j}(\tau)$ as appropriate. The most direct way of doing this is to apply a unit root test in the role of a pre-test. To this end, we employ the infimum GLS-detrended Dickey-Fuller test of Perron and Rodríguez (2003) and Harvey *et al.* (2013). In the current context, this statistic is calculated as

$$MDF=\inf_{\lfloor\tau T\rfloor\in\Lambda^*_T} DF^{GLS}_{\bar{c}}(\tau)$$

where $\Lambda^*_T=[\lfloor\tau_l T\rfloor,\lfloor\tau_U T\rfloor]$ with τ_l and τ_U representing trimming parameters. Here $DF^{GLS}_{\bar{c}}(\tau)$ denotes the standard t-ratio associated with $\tilde{\pi}$ in the fitted ADF-type regression

$$\Delta\tilde{u}_t=\tilde{\pi}\tilde{u}_{t-1}+\sum_{j=1}^{\ell_{DF}}\tilde{\psi}_j\Delta\tilde{u}_{t-j}+\tilde{e}_t,\quad t=k+2,...,T,$$

with ℓ_{DF} having the same properties as ℓ_{NP} above, and

$$\tilde{u}_t=y_t-\tilde{\beta}_1-\tilde{\beta}_2 t-\tilde{\delta}_1 1(t>\lfloor\tau T\rfloor)-\tilde{\delta}_2(t-\lfloor\tau T\rfloor)1(t>\lfloor\tau T\rfloor)$$

where $[\tilde{\beta}_1,\tilde{\beta}_2,\tilde{\delta}_1,\tilde{\delta}_2]'$ is obtained from a local GLS regression of $\mathbf{y}_{\bar{\rho}}$ on $\mathbf{Z}_{\bar{\rho},\tau}$ with $\bar{\rho}=1+\bar{c}/T$.

The limiting distribution of the *MDF* statistic under the null hypothesis of Assumption I(1) when $\delta_1=\delta_2=0$ is given by the expression in equation (11) of Perron and Rodríguez (2003) on setting $c=0$. Let cv_α denote an asymptotic α-level (left-tail) critical value from this distribution. Our pre-test-based decision rule is then to select $\hat{S}^k_{0,j}(\tau)$ if $MDF<cv_\alpha$ and select $\hat{S}^k_{1,j}(\tau)$ if $MDF\geq cv_\alpha$. Under Assumption I(0), *MDF* diverges to $-\infty$ at the rate $O_p(T^{1/2})$ so that $\hat{S}^k_{0,j}(\tau)$ is selected with probability one in the limit; this occurs regardless of whether δ_1 and δ_2 are zero or non-zero. Under Assumption I(1), $\hat{S}^k_{1,j}(\tau)$ is selected with limit probability $1-\alpha$ when $\delta_2=0$, irrespective of the magnitude of δ_1. When $\delta_2\neq 0$ (and again irrespective of δ_1), the asymptotic size of *MDF* is only slightly below α, so that $\hat{S}^k_{1,j}(\tau)$ is selected with limit probability a little above $1-\alpha$. In order to ensure that $\hat{S}^k_{1,j}(\tau)$ is selected with limit probability one under Assumption I(1), whilst also selecting

$\hat{S}^k_{0,j}(\tau)$ with probability one in the limit under Assumption I(0), the *MDF* pre-test can be conducted at a significance level that shrinks with the sample size, by replacing cv_α with $cv_{\alpha,T}$, where $cv_{\alpha,T}$ satisfies $cv_{\alpha,T} \to -\infty$ and $cv_{\alpha,T} = o(T^{1/2})$, i.e. a critical value that diverges to $-\infty$ at a rate slower than $T^{1/2}$.

In what follows, we denote our pre-test-based tests of $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$ as follows:

$$\hat{S}^k_{pre,j}(\tau) = \begin{cases} \hat{S}^k_{0,j}(\tau) & \text{if } MDF < cv_{\alpha,T} \\ \hat{S}^k_{1,j}(\tau) & \text{if } MDF \geq cv_{\alpha,T} \end{cases}, \quad j = \{NP, P\},\ k = \{\tau, \hat{\tau}\}.$$

In the limit, it follows that under $H_0 : \lfloor \tau_0 T \rfloor = \lfloor \tau T \rfloor$,

$$\hat{S}^k_{pre,j}(\tau) \Rightarrow \begin{cases} \mathcal{L}_0 & \text{under Assumption I(0)} \\ \mathcal{L}_1 & \text{under Assumption I(1)} \end{cases}, \quad j = \{NP, P\},\ k = \{\tau, \hat{\tau}\}$$

and so comparison of $\hat{S}^k_{pre,j}(\tau)$ with critical values from $\mathcal{L}_0$ if $MDF < cv_{\alpha,T}$ or from $\mathcal{L}_1$ if $MDF < cv_{\alpha,T}$, will lead to correctly sized tests asymptotically. Inference based on the inversion of sequences of such tests offers the possibility of reliable confidence set construction without the need to make an *a priori* (and possibly incorrect) assumption regarding the order of integration. Given the uncertainty surrounding the unit root properties of typical economic and financial series, particularly those that are subject to a break in level/trend, such an approach has obvious appeal.

Thus far we have considered the cases $|\rho| < 1$ and $\rho = 1$ to evaluate the behaviour of the different procedures under stationary and unit root assumptions. It is also important to assess the behaviour of $\hat{S}^k_{pre,j}(\tau)$ under a local-to-unity specification for ρ. Adopting the usual Pitman drift specification $\rho = 1 + cT^{-1}$, $c \leq 0$, *MDF* is an $O_p(1)$ variate, and hence, due to the fact that $cv_{\alpha,T} \to -\infty$, $\hat{S}^k_{pre,j}(\tau) = \hat{S}^k_{1,j}(\tau)$ in the limit. It can then be easily shown (along the lines of the proof of Theorem 1) that, for $c \leq 0$ under H_0,

$$\hat{S}^k_{pre,j}(\tau) \Rightarrow \mathcal{L}^c_1(\tau), \quad j = \{NP, P\},\ k = \{\tau, \hat{\tau}\}$$

where

$$\mathcal{L}^c_1(\tau) = \tau^{-2} \int_0^\tau \left\{ B_c(r) - \frac{r}{\tau} B_c(\tau) \right\}^2 dr + (1-\tau)^{-2} \int_\tau^1 \left\{ B_c(r) - B_c(\tau) - \frac{r-\tau}{1-\tau} (B_c(1) - B_c(\tau)) \right\}^2 dr \tag{12}$$

with $B_c(r) = \int_0^r e^{(r-s)c} dB(s)$. Note that on setting $c = 0$ we obtain $\mathcal{L}^0_1(\tau) \overset{d}{=} \mathcal{L}_1\ \forall \tau$. Table 2 reports asymptotic coverage rates for nominal 0.90-, 0.95- and 0.99-level confidence sets constructed from the $\hat{S}^k_{pre,j}(\tau)$ tests, using critical values from Table 1 (which are appropriate for $c = 0$). The coverage rates were obtained by direct simulation of (12) in the same manner as the simulations for Table 1, and results are reported for $c = \{0, -5, -10, -20, -30, -40, -50\}$ and $\tau_0 = \{0.1, 0.2, ..., 0.9\}$, noting that the $\mathcal{L}^c_1(\tau_0)$ distribution depends on τ_0 unless $c = 0$. It is clear from the results that in the local-to-unity setting, confidence sets based on the $\hat{S}^k_{pre,j}(\tau)$ tests do not suffer from any under-coverage across c or τ_0; indeed, over-coverage is observed, increasing in $-c$ for a given τ_0. This arises from the individual $\hat{S}^k_{pre,j}(\tau)$ tests being under-sized for local-to-unity generating processes given that critical values appropriate for a pure unit root are being applied, and translates to conservative confidence sets that asymptotically include the true break date with a probability at least as great as the nominal coverage rate. This reassuring property indicates that asymptotic under-coverage is not a feature of our proposed pre-test-based confidence sets for any value of ρ, be it unity, local-to-unity, or strictly less than one.

Finally, an alternative feasible approach to constructing a confidence set with correct asymptotic coverage under both Assumption I(0) and Assumption I(1) (and with over-coverage under a local-to-unity specification) is to consider taking a union of an I(0)-based confidence set and an I(1)-based confidence set. Given the results of Corollary 2, it is evident that asymptotically correct coverage, i.e. a coverage rate of $(1-\alpha)$ in both the I(0) and I(1) cases, would be obtained only from a union of the confidence sets corresponding to $\hat{S}^k_{0,NP}(\tau)$ and $\hat{S}^k_{1,P}(\tau)$, with the latter requiring we set $\ell_P^{-1} = o(T^{1/4})$. All other unions would lead to either asymptotically full coverage (i.e. a coverage rate of one), or a coverage rate that depends on nuisance parameters (ω_u^2 or ω_ε^2). We investigated the finite sample properties of such a union, and while the coverage rates were found to be comparable to those of the best of the pre-test procedures, the union confidence set lengths were generally greater than those afforded by the best pre-test approach (in some cases substantially so), hence we do not pursue the union further here.

In the next section we evaluate the finite sample properties of our pre-test-based approaches in comparison with those that are based on a maintained assumption regarding the integration properties of the data, both in terms of coverage and length.

5 Finite sample performance

In this section we examine the finite sample performance of confidence sets based on the $\hat{S}^k_{0,j}(\tau)$, $\hat{S}^k_{1,j}(\tau)$ and $\hat{S}^k_{pre,j}(\tau)$ tests ($j = \{NP, P\}$, $k = \{\tau, \hat{\tau}\}$). We simulate the DGP (1)-(2) with $\beta_1 = \beta_2 = 0$ (without loss of generality) and a range of break magnitudes, δ_1 and δ_2, and timings, τ_0, for the sample sizes $T = 150$ and $T = 300$. We consider $\rho \in \{0.00, 0.50, 0.80, 0.90, 0.95, 1.00\}$ to encompass both I(1) and a range of I(0) DGPs, and set $u_t \sim NIID(0,1)$. The $\hat{S}^k_{0,j}(\tau)$ and $\hat{S}^k_{1,j}(\tau)$ tests are applied at the nominal 0.05-level using the asymptotic critical values provided in Table 1, with $\ell_{NP} = \ell_{\max} = \lfloor 12(T/100)^{1/4} \rfloor$ and ℓ_P selected via the Bayesian information criterion with maximum value $\ell_{\max}$. For the $\hat{S}^k_{pre,j}(\tau)$ tests, we select between $\hat{S}^k_{0,j}(\tau)$ and $\hat{S}^k_{1,j}(\tau)$ on the basis of *MDF* conducted at the 0.05-level with $\bar{c} = -17.6$ (following Harvey *et al.* (2013)), $\tau_l = 1 - \tau_U = 0.01$,[2] and where ℓ_{DF} is selected according to the MAIC procedure of Ng and Perron (2001), as modified by Perron and Qu (2007), with maximum lag order $\ell_{\max}$. All simulations were conducted using 10,000 Monte Carlo replications, and in the tables we report results for confidence set coverage (the proportion of replications for which the true break date is contained in the confidence set) and confidence set length (in each replication, length is

[2] From simulation of the asymptotic null distribution of *MDF* in this case, we find that $cv_{0.05} = -3.88$. For simplcity, we conduct *MDF* at the nominal 0.05-level for both $T = 150$ and $T = 300$, rather than shrinking the significance level with increasing sample size.

calculated as the number of dates included in the confidence set as a proportion of the sample size; we then report the average length over Monte Carlo replications).

Table 3 reports results for $\tau_0 = 0.3$, $\delta_1 = 5$ and $\delta_2 = 0.5$, such that both a level and trend break occur before the sample mid-point. Consider first the behaviour of the confidence sets based on $\hat{S}^k_{0,j}(\tau)$ ($j = \{NP, P\}$, $k = \{\tau, \hat{\tau}\}$). When $\rho = 0$, we find that (approximately) correct coverage is achieved for the two $\hat{S}^k_{0,P}(\tau)$ sets, whereas the two $\hat{S}^k_{0,NP}(\tau)$ sets display correct coverage only for $T = 300$, with under-coverage apparent for $T = 150$. When $\rho = 1$, the $\hat{S}^k_{0,NP}(\tau)$ sets deliver substantial under-coverage, increasingly so in the larger sample size, as our asymptotic results in Corollary 2 suggest. In contrast, the $\hat{S}^k_{0,P}(\tau)$ sets (the tests for which were found to be $O_p(1)$), display over-coverage for both sample sizes, which is clearly less of a concern. For $\rho = 0.5$, the coverage rates for the $\hat{S}^k_{0,j}(\tau)$ sets are seen to be broadly similar to those for $\rho = 0$, then as ρ increases towards one, coverage moves closer to those observed in the $\rho = 1$ case, as we might expect in finite samples.

Turning now to the $\hat{S}^k_{1,j}(\tau)$ sets, all are seen to provide (approximately) correct coverage when $\rho = 1$, in line with our theoretical results; indeed, coverage never deviates from 0.95 by more than 0.01 across both sample sizes. At the other extreme, when $\rho = 0$ we find that all the $\hat{S}^k_{1,j}(\tau)$ sets show under-coverage for both $T = 150$ and $T = 300$ (which is somewhat surprising in the case of the two $\hat{S}^k_{1,NP}(\tau)$ sets, since the tests converge in probability to zero under Assumption I(0), although unreported simulations confirm that coverage does start to increase for larger samples); under-coverage is also seen in some cases when $\rho = 0.5$, while for the larger values of $\rho < 1$, coverage is closer to the correct coverage seen when $\rho = 1$ (in fact some over-coverage is displayed in these cases).

For our proposed pre-test-based procedures $\hat{S}^k_{pre,j}(\tau)$, we see that in each case, coverage is very close to the corresponding $\hat{S}^k_{1,j}(\tau)$ coverage for $\rho = 1$ and ρ values close to 1, but then for small values of ρ assumes the more accurate coverage rates of the corresponding $\hat{S}^k_{0,j}(\tau)$ sets. Of course, the coverage of any given $\hat{S}^k_{pre,j}(\tau)$ set is limited by the coverage performance of the corresponding underlying $\hat{S}^k_{0,j}(\tau)$ and $\hat{S}^k_{1,j}(\tau)$ sets, thus for the two $\hat{S}^k_{pre,NP}(\tau)$ sets, under-coverage is still manifest for some settings, due to the under-coverage inherent in the $\hat{S}^k_{0,NP}(\tau)$ sets. However, the $\hat{S}^k_{pre,P}(\tau)$ sets show good finite sample coverage rates across the range of settings considered in the table, in particular avoiding problems of under-coverage.

When considering our results for the length of the confidence sets implied by the different tests, as we would expect, length generally decreases (since test power generally increases) as T increases and as ρ decreases. Comparing the different procedures, the most striking feature is that any given $\hat{S}^{\hat{\tau}}_{i,j}(\tau)$ or $\hat{S}^{\hat{\tau}}_{pre,j}(\tau)$ set (where the short and long run variance estimators used in the tests are based on $\hat{\tau}_{D_m}$) substantially outperforms the corresponding $\hat{S}^{\tau}_{i,j}(\tau)$ or $\hat{S}^{\tau}_{pre,j}(\tau)$ set (where the estimators in the tests are evaluated at each τ). This is entirely to be expected, since under the alternative hypothesis, use of a consistent estimator of the true break fraction allows consistent estimation of σ^2_u and ω^2_u under Assumption I(1) and consistent estimation of ω^2_ε under Assumption I(0). In contrast, the estimators $\hat{\sigma}^2_u(\tau)$, $\hat{\omega}^2_u(\tau)$ and $\hat{\omega}^2_\varepsilon(\tau)$ are not consistent when $\tau \neq \tau_0$, and are likely to over-state the values of the true parameters, thereby reducing the values of the test statistics and increasing the confidence set length. Of the better performing $\hat{S}^{\hat{\tau}}_{i,j}(\tau)$ and $\hat{S}^{\hat{\tau}}_{pre,j}(\tau)$ sets, $\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$, $\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$,

$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$ and $\hat{S}^{\hat{\tau}}_{1,P}(\tau)$ were found to suffer from problems of under-coverage, making them unreliable on that measure. Overall, then, it is clear that the two procedures that can be deemed in some sense satisfactory, on both coverage and length grounds, are $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$ and $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$. Of these two procedures, $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ suffers from less over-coverage, and also has arguably the best length properties across the range of ρ values considered; specifically, $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ and $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$ have similar length for $\rho = 0$, 0.5, 0.8 and 0.95, and while $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ has somewhat greater length than $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$ for $\rho = 0.9$, it offers a more marked improvement in length when $\rho = 1$, as we would expect given the ability of $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ to select the better-performing $\hat{S}^{\hat{\tau}}_{1,P}(\tau)$ set in this scenario. It is also reassuring to see that for values of ρ less than but close to one, the preferred $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ procedure has decent length properties. For these large values of $\rho < 1$, the local-to-unity asymptotic results are potentially relevant, and it is clear that despite the $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ procedure being conservative in such cases (displaying over-coverage), the procedure retains an ability to achieve a reasonably short length, demonstrating that while the underlying tests may be under-sized for local-to-unity processes, they still have power to reject for incorrect break dates.

Table 4 reports results for the same settings as Table 3, except with a larger magnitude level and trend break, with $\delta_1 = 10$ and $\delta_2 = 1$. As regards coverage, much the same comments apply as for Table 3.[3] As we would expect, the lengths of the confidence sets are generally smaller in this case of larger, more detectable, breaks. Once more, we find that $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ is the best performing procedure overall; indeed, compared to the only other procedure with reliable coverage and decent length, $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$, we see that $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ now displays equal or shorter length across all values of ρ, with decreases in length of up to 0.28 seen.

Table 5 reports results for the case of $\delta_1 = 10$ and $\delta_2 = 0$ so that only a level break occurs. Consider first the results for $\rho = 1$. From Remark 1, it follows that here the $\hat{S}^{k}_{1,j}(\tau)$ tests have zero asymptotic power to identify the date of the level break; this can be seen in the table as the lengths of all the $\hat{S}^{k}_{1,j}(\tau)$ sets increase between $T = 150$ and $T = 300$. What we observe, however, is that, for a given T, the sets based on $\hat{S}^{\hat{\tau}}_{1,j}(\tau)$ are very much shorter than those based on $\hat{S}^{\tau}_{1,j}(\tau)$.[4] Taking the results across the different values of ρ together, we again find $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ to be the best procedure when considering both coverage and length, with the gains in length over $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$ when $\rho = 1$ now even more marked than was observed in Tables 3 and 4.

In Table 6 we have $\delta_1 = 0$ and $\delta_2 = 1$ so that only a trend break is present. Here we find the

[3]Note that the coverage rates for the $\hat{S}^{\tau}_{i,j}(\tau)$ sets are numerically identical across different δ_1 and δ_2 settings, since they are invariant to these parameters by construction under H_0.

[4]This arises because there is an upward bias in $\hat{\sigma}^2_u(\tau)$ relative to $\hat{\sigma}^2_u(\hat{\tau}_{D_m})$ resulting from the former being based on residuals from a regression containing a mis-specified break component whenever $\tau \neq \tau_0$, while the latter uses an estimator of τ_0 which, albeit not consistent, can nonetheless perform reasonably in finite samples. This relative upward bias translates to lower values of $\hat{S}^{\tau}_{1,j}(\tau)$ compared to $\hat{S}^{\hat{\tau}}_{1,j}(\tau)$, negatively affecting the power of the former and the length of the corresponding confidence set. Indeed, the lengths of the $\hat{S}^{\tau}_{1,j}(\tau)$ sets are close to the nominal coverage rates, and similar to what would be obtained if the first and third terms of $S_1(\tau)$ (and consequently the $2\hat{\sigma}^2_u(\tau)$ centering) were simply omitted from the statistic, unlike $\hat{S}^{\hat{\tau}}_{1,j}(\tau)$ where inclusion of the first and third terms of $S_1(\tau)$ (together with the $2\hat{\sigma}^2_u(\hat{\tau}_{D_m})$ centering) contribute substantially to shortening the confidence set length.

pattern of results mimic those of Table 4, albeit with lengths tending to be somewhat greater due to the lack of contribution of a level break. What is clear from all these results is that $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ is the preferred test for construction of confidence sets.

Tables 7 and 8 report results for the same settings as Tables 3 and 4, respectively (i.e. cases where both a level and trend break occur), but with the breaks occurring at the sample mid-point, i.e. $\tau_0 = 0.5$, rather than $\tau_0 = 0.3$. Comparing the coverage results across $\tau_0 = 0.3$ and $\tau_0 = 0.5$, while the under-coverage associated with the $\hat{S}^k_{0,NP}(\tau)$ sets for $\rho = 1$ is exaggerated for a mid-point break, the most noticeable feature is that the under-coverage seen for the $\hat{S}^k_{1,j}(\tau)$ sets for the smaller values of ρ is here replaced by *over*-coverage. This ensues partly because when $\tau = \tau_0 = 0.5$, it can easily be shown that the difference between the sum of the first and third components of $S_1(\tau)$ in (11) and $2\hat{\sigma}^2_u(\tau)$ (or $2\hat{\sigma}^2_u(\hat{\tau}_{D_m})$) is $o_p(T^{-1/2})$, as opposed to when $\tau = \tau_0 \neq 0.5$ where this difference is only $O_p(T^{-1/2})$ and tends to be positive. Other things equal, therefore, when $\tau = \tau_0 = 0.5$ the chance of the $\hat{S}^k_{1,j}(\tau)$ test rejecting in finite samples is reduced relative to when $\tau = \tau_0 \neq 0.5$. However, despite $\hat{S}^k_{1,j}(\tau)$ performing relatively well for these mid-point breaks, one could not rely on this approach to deliver reliable confidence sets in general, given the absence of knowledge regarding τ_0 and the possibility of under-coverage for non-central breaks. Taking the results of Tables 7 and 8 as a whole, it is still the case that $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ performs very well.

Unreported results for the case of $\tau_0 = 0.7$ also confirm that $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ is the best performing procedure overall. Therefore, our recommendation would clearly be for the $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ procedure, given its reliable finite sample coverage and good performance in terms of confidence set length.

5.1 Confidence sets based on trimming

An issue that may be relevant in finite samples is that when τ is close to zero the first two components of $S_0(\tau)$ in (9) and $S_1(\tau)$ in (11) are based on only a few of the $\hat{u}_t$ residuals; similarly, when τ is close to one the same is true of the last two components of $S_0(\tau)$ and $S_1(\tau)$. Therefore, it is possible that for values of τ near the $(0, 1)$ extremities, the finite sample behaviour of the tests may differ markedly from the behaviour of the same tests evaluated at less extreme values of τ. In our above simulations, coverages were calculated for $\tau = \tau_0 = 0.3$ and 0.5 - values well away from the extremities, so no such problems should arise there. That said, there is clearly a potential for values of $\hat{S}^k_{i,j}(\tau)$ calculated near the extremities of τ to adversely influence the *lengths* of the resulting confidence sets (these being potentially non-contiguous). To investigate this, we recalculated the lengths of the sets based on $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$, $\hat{S}^{\hat{\tau}}_{1,P}(\tau)$ and our preferred test $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ only for $\lfloor \tau T \rfloor \in \Lambda'_T = \{\lfloor 0.1T \rfloor, ..., \lfloor 0.9T \rfloor\}$, which can be thought of as a 10% trimming, akin to the assumption that no break can occur in the first and last 10% of the observed data, an assertion frequently made in the associated structural change literature.

The results are shown in Table 9. The first block of results in Table 9 is for $\tau_0 = 0.3$, $\delta_1 = 5$ and $\delta_2 = 0.5$ and is to be compared with the corresponding results in Table 3. For $T = 150$, we observe length reductions of up to about 0.13.[5] This implies that, in some cases, a significant proportion of non-rejections of H_0 are incorrectly occurring for tests being evaluated at the extreme values of τ, since τ_0 itself is not close to these extremes. When $T = 300$, the length reduction is up to about 0.07 so that, for this specification, trimming is less effective with the larger sample size, implying that the untrimmed confidence sets contain relatively few anomalous extreme dates. The second block of results is for $\tau_0 = 0.3$, $\delta_1 = 10$ and $\delta_2 = 1$, i.e. where the break magnitudes are doubled. Comparing with Table 4 we find that, for both $T = 150$ and $T = 300$, there appears to be very little (if any) reduction in length arising from trimming, again implying, for this specification, few spurious rejections of H_0 occur for tests evaluated at extreme values of τ. In the third block of Table 9 where $\tau_0 = 0.3$, $\delta_1 = 10$ and $\delta_2 = 0$, we see, on comparing with Table 5, that trimming is again effective, and more so for $T = 300$ than for $T = 150$. For the remaining specifications in Table 9 (the lower blocks), comparison with Tables 6-8 shows generally only very modest shortenings arising from trimming. Overall, however, we conclude that trimming can be of possible benefit in improving the length of confidence sets, potentially removing spurious dates from the set that have arisen purely due to the sampling variability involved in the tests when evaluated near the extremes.

6 Empirical illustrations

As empirical illustrations of our confidence set procedures for dating a break in level and/or trend, we apply them to two US macroeconomic series. These are the nominal money supply M2 (seasonally adjusted, measured in logarithms) and the effective federal funds rate, using monthly data over the period 1959:1-2012:12 ($T = 648$). The data were obtained from the FRED database of the Federal Reserve Bank of St Louis. We construct 0.95-level confidence sets employing the three procedures $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$, $\hat{S}^{\hat{\tau}}_{1,P}(\tau)$ and $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ (note the confidence set for $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ is either that for $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$ or $\hat{S}^{\hat{\tau}}_{1,P}(\tau)$, depending on the outcome of *MDF*), using the same settings as were applied in the Monte Carlo simulations above.

Results for the M2 series are shown in Figure 1, where the confidence sets are represented by the shaded regions, while the series overlays the sets. Figure 1 (a) reports the confidence set for $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$ which is contiguous here with a length of 0.51 (330 observations) covering the interval 1971:4-1998:9. In Figure 1 (b), we see that the confidence set for $\hat{S}^{\hat{\tau}}_{1,P}(\tau)$ is much shorter, with length 0.33 (213 observations), but is not contiguous. In particular, the set is comprised of an almost contiguous subset of dates covering the interval 1978:6-1994:2 (the dates 1986:10-1987:2 inclusive are exceptions to this), plus a number of dates towards the extremes of the sample, the latter lying within $0.03T$ of the sample's beginning and end. If we view the end-point behaviour as spurious and apply a trimming rule of at least 3%, cf. section 5.1, we effectively ignore the non-rejections associated with these very early and very late dates. The resulting confidence set then contains the almost contiguous subset of dates alone, with the length of the set reducing to 0.28. Visual inspection of the plot of the M2 series confirms that a break in this date range is plausible. The confidence set selected by our pre-test

[5] Note that the maximum possible reduction in length with 10% trimming is 0.20.

procedure $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ is that of $\hat{S}^{\hat{\tau}}_{1,P}(\tau)$, and hence the shorter and more plausible of the two, reinforcing the case for using such an approach in practice.

Figure 2 gives the results for the federal funds rate series. Here $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$ yields a contiguous confidence set with length 0.28 (181 observations) covering the interval 1972:12-1987:12, which again appears consistent with the visual plot of the data. The confidence set associated with $\hat{S}^{\hat{\tau}}_{1,P}(\tau)$ has length 0.98, which is rather meaningless as a confidence set for a break since it includes nearly all observations in the sample. Our pre-test procedure $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ selects the confidence set $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$, which is without any doubt the more plausible of the two. These examples taken together highlight the potential shortcomings of simply constructing confidence sets based on $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$ or $\hat{S}^{\hat{\tau}}_{1,P}(\tau)$ alone, while simultaneously demonstrating the benefits of the $\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$ approach.

7 Conclusions

In this paper we have proposed methods for constructing confidence sets for the timing of a break in level and/or trend that have asymptotically correct coverage regardless of the order of integration (and are asymptotically conservative in the case of local-to-unity processes). Our approach follows the work of EM, and is based on inverting a sequence of tests for the break location, evaluated across the full spectrum of possible break dates. We propose two locally best invariant tests upon which the confidence sets can be based, each of which corresponds to a particular order of integration (i.e. I(0) or I(1) data generating processes). Under their respective assumptions, these confidence sets provide correct asymptotic coverage regardless of the magnitude of the break in level/trend, and also display good finite sample properties in terms of both coverage and length. When the tests are applied under an incorrect assumption regarding the order of integration, they perform relatively poorly, however. Consequently, we propose use of a pre-test procedure to select between the I(0)- and I(1)-based confidence sets. Monte Carlo evidence shows that our recommended pre-test based procedure works well across both I(0) and I(1) environments, offering practitioners a reliable and robust approach to constructing confidence sets without the need to make an *a priori* assumption concerning the data's integration order. Application to two US macroeconomic series provides further evidence as to the efficacy of these procedures.

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Appendix

Proof of Lemma 1

(a) To show (9), note that

$$D'_\eta \hat{u} = \begin{bmatrix} \sum_{t=\lfloor \eta T \rfloor+1}^{T} \hat{u}_t \\ \sum_{t=\lfloor \eta T \rfloor+1}^{T} (t - \lfloor \eta T \rfloor)\hat{u}_t \end{bmatrix}. \tag{13}$$

Also, since we have the orthogonality condition $D'_\tau \hat{u} = 0$,

$$\begin{bmatrix} \sum_{t=\lfloor \tau T \rfloor+1}^{T} \hat{u}_t \\ \sum_{t=\lfloor \tau T \rfloor+1}^{T} (t - \lfloor \tau T \rfloor)\hat{u}_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and from the orthogonality condition $D'\hat{u} = 0$,

$$\begin{bmatrix} \sum_{t=1}^{T} \hat{u}_t \\ \sum_{t=1}^{T} t\hat{u}_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So, for $\lfloor \eta T \rfloor < \lfloor \tau T \rfloor$, (13) can be written as

$$D'_\eta \hat{u} = \begin{bmatrix} -\sum_{t=1}^{\lfloor \eta T \rfloor} \hat{u}_t \\ -\sum_{t=1}^{\lfloor \eta T \rfloor} (t - \lfloor \eta T \rfloor)\hat{u}_t \end{bmatrix}.$$

For $\lfloor \eta T \rfloor > \lfloor \tau T \rfloor$,

$$\begin{aligned} \sum_{t=\lfloor \eta T \rfloor+1}^{T} \hat{u}_t &= \sum_{t=\lfloor \tau T \rfloor+1}^{T} \hat{u}_t - \sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \eta T \rfloor} \hat{u}_t \\ &= -\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \eta T \rfloor} \hat{u}_t \end{aligned}$$

$$\begin{aligned} \sum_{t=\lfloor \eta T \rfloor+1}^{T} (t - \lfloor \eta T \rfloor)\hat{u}_t &= \sum_{t=\lfloor \tau T \rfloor+1}^{T} (t - \lfloor \eta T \rfloor)\hat{u}_t - \sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \eta T \rfloor} (t - \lfloor \eta T \rfloor)\hat{u}_t \\ &= \sum_{t=\lfloor \tau T \rfloor+1}^{T} (t - \lfloor \tau T \rfloor)\hat{u}_t + (\lfloor \tau T \rfloor - \lfloor \eta T \rfloor) \sum_{t=\lfloor \tau T \rfloor+1}^{T} \hat{u}_t - \sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \eta T \rfloor} (t - \lfloor \eta T \rfloor)\hat{u}_t \\ &= -\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \eta T \rfloor} (t - \lfloor \eta T \rfloor)\hat{u}_t \end{aligned}$$

such that (13) can be written as

$$D'_\eta \hat{u} = \left[\begin{array}{c} -\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \eta T \rfloor} \hat{u}_t \\ -\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \eta T \rfloor} (t-\lfloor \eta T \rfloor)\hat{u}_t \end{array} \right].$$

Using (8), it follows that

$$\begin{aligned} S(\tau) &= \sum_{\lfloor \eta T \rfloor=2}^{\lfloor \tau T \rfloor-1} \left\{ \lfloor \tau T \rfloor^{-2} \left(\sum_{t=1}^{\lfloor \eta T \rfloor} \hat{u}_t \right)^2 + \lfloor \tau T \rfloor^{-4} \left(\sum_{t=1}^{\lfloor \eta T \rfloor} (t-\lfloor \eta T \rfloor)\hat{u}_t \right)^2 \right\} \\ &+ \sum_{\lfloor \eta T \rfloor=\lfloor \tau T \rfloor+1}^{T-2} \left\{ (T-\lfloor \tau T \rfloor)^{-2} \left(\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \eta T \rfloor} \hat{u}_t \right)^2 + (T-\lfloor \tau T \rfloor)^{-4} \left(\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \eta T \rfloor} (t-\lfloor \eta T \rfloor)\hat{u}_t \right)^2 \right\} \\ &= \lfloor \tau T \rfloor^{-2} \sum_{t=2}^{\lfloor \tau T \rfloor-1} \left(\sum_{s=1}^{t} \hat{u}_s \right)^2 + \lfloor \tau T \rfloor^{-4} \sum_{t=2}^{\lfloor \tau T \rfloor-1} \left(\sum_{s=1}^{t} (s-t)\hat{u}_s \right)^2 \\ &+ (T-\lfloor \tau T \rfloor)^{-2} \sum_{t=\lfloor \tau T \rfloor+1}^{T-2} \left(\sum_{s=\lfloor \tau T \rfloor+1}^{t} \hat{u}_s \right)^2 + (T-\lfloor \tau T \rfloor)^{-4} \sum_{t=\lfloor \tau T \rfloor+1}^{T-2} \left(\sum_{s=\lfloor \tau T \rfloor+1}^{t} (s-t)\hat{u}_s \right)^2 \\ &= S_0(\tau). \end{aligned}$$

(b) To show (11), paralleling the proof of Lemma 1(a), we find

$$\begin{aligned} D'_\eta \hat{u} &= \left[\begin{array}{c} \hat{u}_{\lfloor \eta T \rfloor+1} \\ \sum_{t=\lfloor \eta T \rfloor+1}^{T} \hat{u}_t \end{array} \right] \\ &= \begin{cases} \left[\begin{array}{c} \hat{u}_{\lfloor \eta T \rfloor+1} \\ -\sum_{t=2}^{\lfloor \eta T \rfloor} \hat{u}_t \end{array} \right] & \text{for } \lfloor \eta T \rfloor < \lfloor \tau T \rfloor \\ \left[\begin{array}{c} \hat{u}_{\lfloor \eta T \rfloor+1} \\ -\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \eta T \rfloor} \hat{u}_t \end{array} \right] & \text{for } \lfloor \eta T \rfloor > \lfloor \tau T \rfloor \end{cases}. \end{aligned}$$

Then, using (10),

$$\begin{aligned} S(\tau) &= \sum_{\lfloor \eta T \rfloor=2}^{\lfloor \tau T \rfloor-1} \left\{ \lfloor \tau T \rfloor^{-1} \hat{u}_{\lfloor \eta T \rfloor+1}^2 + \lfloor \tau T \rfloor^{-2} \left(\sum_{t=2}^{\lfloor \eta T \rfloor} \hat{u}_t \right)^2 \right\} \\ &+ \sum_{\lfloor \eta T \rfloor=\lfloor \tau T \rfloor+1}^{T-2} \left\{ (T-\lfloor \tau T \rfloor)^{-1} \hat{u}_{\lfloor \eta T \rfloor+1}^2 + (T-\lfloor \tau T \rfloor)^{-2} \left(\sum_{t=\lfloor \tau T \rfloor+1}^{\lfloor \eta T \rfloor} \hat{u}_t \right)^2 \right\} \\ &= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor-1} \hat{u}_{t+1}^2 + \lfloor \tau T \rfloor^{-2} \sum_{t=2}^{\lfloor \tau T \rfloor-1} \left(\sum_{s=2}^{t} \hat{u}_s \right)^2 \\ &+ (T-\lfloor \tau T \rfloor)^{-1} \sum_{t=\lfloor \tau T \rfloor+1}^{T-2} \hat{u}_{t+1}^2 + (T-\lfloor \tau T \rfloor)^{-2} \sum_{t=\lfloor \tau T \rfloor+1}^{T-2} \left(\sum_{s=\lfloor \tau T \rfloor+1}^{t} \hat{u}_s \right)^2 \\ &= S_1(\tau). \end{aligned}$$

Proof of Theorem 1

In what follows we may set $\beta_1 = \beta_2 = 0$ and $\delta_1 = \delta_2 = 0$ without loss of generality.

(a) Let $W(r) = \omega_\varepsilon B(r)$. In view of $S_0(\tau)$, the limits we require are those of (i) $T^{-1/2}\sum_{t=1}^{\lfloor rT\rfloor}\hat{u}_t$ for $t \leq \lfloor \tau T\rfloor$, (ii) $T^{-1/2}\sum_{t=\lfloor \tau T\rfloor+1}^{\lfloor rT\rfloor}\hat{u}_t$ for $t > \lfloor \tau T\rfloor$ and (iii) $T^{-3/2}\sum_{t=1}^{\lfloor rT\rfloor}(t-\lfloor rT\rfloor)\hat{u}_t$ for $t \leq \lfloor \tau T\rfloor$, (iv) $T^{-3/2}\sum_{t=\lfloor \tau T\rfloor+1}^{\lfloor rT\rfloor}(t-\lfloor rT\rfloor)\hat{u}_t$ for $t > \lfloor \tau T\rfloor$. To show (i) write

$$
\begin{aligned}
T^{-1/2}\sum_{t=1}^{\lfloor rT\rfloor}\hat{u}_t &= T^{-1/2}\sum_{t=1}^{\lfloor rT\rfloor}\left(u_t - \lfloor \tau T\rfloor^{-1}\sum_{s=1}^{\lfloor \tau T\rfloor}u_s\right) \\
&\quad - \left(\frac{T^{-3/2}\sum_{s=1}^{\lfloor \tau T\rfloor}u_s\left(s-\lfloor \tau T\rfloor^{-1}\sum_{j=1}^{\lfloor \tau T\rfloor}j\right)}{T^{-3}\sum_{s=1}^{\lfloor \tau T\rfloor}\left(s-\lfloor \tau T\rfloor^{-1}\sum_{j=1}^{\lfloor \tau T\rfloor}j\right)^2}\right)T^{-2}\sum_{t=1}^{\lfloor rT\rfloor}\left(t-\lfloor \tau T\rfloor^{-1}\sum_{s=1}^{\lfloor \tau T\rfloor}s\right) \\
&= T^{-1/2}\sum_{t=1}^{\lfloor rT\rfloor}u_t - \lfloor rT\rfloor\lfloor \tau T\rfloor^{-1}T^{-1/2}\sum_{s=1}^{\lfloor \tau T\rfloor}u_s \\
&\quad - \left(\frac{T^{-3/2}\sum_{s=1}^{\lfloor \tau T\rfloor}su_s - \lfloor \tau T\rfloor^{-1}T^{-1}\sum_{j=1}^{\lfloor \tau T\rfloor}jT^{-1/2}\sum_{s=1}^{\lfloor \tau T\rfloor}u_s}{T^{-3}\sum_{s=1}^{\lfloor \tau T\rfloor}\left(s-\lfloor \tau T\rfloor^{-1}\sum_{j=1}^{\lfloor \tau T\rfloor}j\right)^2}\right) \\
&\quad \times\left(T^{-2}\sum_{t=1}^{\lfloor rT\rfloor}t - \lfloor rT\rfloor\lfloor \tau T\rfloor^{-1}T^{-2}\sum_{s=1}^{\lfloor \tau T\rfloor}s\right) \\
&\Rightarrow W(r) - \frac{r}{\tau}W(\tau) - \left(\frac{\int_0^\tau sdW(s) - \frac{\tau^2}{2\tau}W(\tau)}{\frac{\tau^3}{12}}\right)\left(\frac{r^2}{2}-\frac{r\tau^2}{2\tau}\right) \\
&= W(r) - \frac{r}{\tau}W(\tau) + \frac{6r(\tau-r)}{\tau^3}\left(\frac{\tau}{2}W(\tau) - \int_0^\tau W(s)ds\right)
\end{aligned}
$$

and for (ii),

$$
\begin{aligned}
T^{-1/2}\sum_{t=\lfloor \tau T\rfloor+1}^{\lfloor rT\rfloor}\hat{u}_t &= T^{-1/2}\sum_{t=\lfloor \tau T\rfloor+1}^{\lfloor rT\rfloor}\left(u_t - (T-\lfloor \tau T\rfloor)^{-1}\sum_{s=\lfloor \tau T\rfloor+1}^{T}u_s\right) \\
&\quad - \left(\frac{T^{-3/2}\sum_{s=\lfloor \tau T\rfloor+1}^{T}u_s\left(s-(T-\lfloor \tau T\rfloor)^{-1}\sum_{j=\lfloor \tau T\rfloor+1}^{T}j\right)}{T^{-3}\sum_{s=\lfloor \tau T\rfloor+1}^{T}\left(s-(T-\lfloor \tau T\rfloor)^{-1}\sum_{j=\lfloor \tau T\rfloor+1}^{T}j\right)^2}\right) \\
&\quad \times T^{-2}\sum_{t=\lfloor \tau T\rfloor+1}^{\lfloor rT\rfloor}\left(t-(T-\lfloor \tau T\rfloor)^{-1}\sum_{s=\lfloor \tau T\rfloor+1}^{T}s\right)
\end{aligned}
$$

$$
\begin{aligned}
&= T^{-1/2}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor} u_t - (\lfloor rT\rfloor-\lfloor\tau T\rfloor)(T-\lfloor\tau T\rfloor)^{-1}T^{-1/2}\sum_{s=\lfloor\tau T\rfloor+1}^{T} u_s \\
&\quad -\left(\frac{T^{-3/2}\sum_{s=\lfloor\tau T\rfloor+1}^{T} su_s-(T-\lfloor\tau T\rfloor)^{-1}T^{-1}\sum_{j=\lfloor\tau T\rfloor+1}^{T} jT^{-1/2}\sum_{s=\lfloor\tau T\rfloor+1}^{T} u_s}{T^{-3}\sum_{s=\lfloor\tau T\rfloor+1}^{T}\left(s-(T-\lfloor\tau T\rfloor)^{-1}\sum_{j=\lfloor\tau T\rfloor+1}^{T} j\right)^2}\right) \\
&\quad \times\left(T^{-2}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor} t-(\lfloor rT\rfloor-\lfloor\tau T\rfloor)(T-\lfloor\tau T\rfloor)^{-1}T^{-2}\sum_{s=\lfloor\tau T\rfloor+1}^{T} s\right) \\
&\Rightarrow W(r)-W(\tau)-\frac{r-\tau}{1-\tau}(W(1)-W(\tau)) \\
&\quad -\left(\frac{\int_\tau^1 sdW(s)-\frac{1-\tau^2}{2(1-\tau)}(W(1)-W(\tau))}{\frac{(1-\tau)^3}{12}}\right)\left(\frac{r^2-\tau^2}{2}-\frac{(r-\tau)(1-\tau^2)}{2(1-\tau)}\right) \\
&= W(r)-W(\tau)-\frac{r-\tau}{1-\tau}(W(1)-W(\tau)) \\
&\quad +\frac{6(r-\tau)(1-r)}{(1-\tau)^3}\left(\frac{1-\tau}{2}(W(1)-W(\tau))-\int_\tau^1(W(s)-W(\tau))ds\right).
\end{aligned}
$$

For (iii) write

$$
T^{-3/2}\sum_{t=1}^{\lfloor rT\rfloor}(t-\lfloor rT\rfloor)\hat{u}_t = T^{-3/2}\sum_{t=1}^{\lfloor rT\rfloor} t\hat{u}_t - rT^{-1/2}\sum_{t=1}^{\lfloor rT\rfloor}\hat{u}_t
$$

where, for the first right hand side term

$$
\begin{aligned}
T^{-3/2}\sum_{t=1}^{\lfloor rT\rfloor} t\hat{u}_t &= T^{-3/2}\sum_{t=1}^{\lfloor rT\rfloor} t\left(u_t-\lfloor\tau T\rfloor^{-1}\sum_{s=1}^{\lfloor\tau T\rfloor}u_s\right) \\
&\quad -\left(\frac{T^{-3/2}\sum_{s=1}^{\lfloor\tau T\rfloor}u_s\left(s-\lfloor\tau T\rfloor^{-1}\sum_{j=1}^{\lfloor\tau T\rfloor}j\right)}{T^{-3}\sum_{s=1}^{\lfloor\tau T\rfloor}\left(s-\lfloor\tau T\rfloor^{-1}\sum_{j=1}^{\lfloor\tau T\rfloor}j\right)^2}\right)T^{-3}\sum_{t=1}^{\lfloor rT\rfloor}t\left(t-\lfloor\tau T\rfloor^{-1}\sum_{s=1}^{\lfloor\tau T\rfloor}s\right) \\
&= T^{-3/2}\sum_{t=1}^{\lfloor rT\rfloor}tu_t-\lfloor\tau T\rfloor^{-1}T^{-1}\sum_{t=1}^{\lfloor rT\rfloor}t\left(T^{-1/2}\sum_{s=1}^{\lfloor\tau T\rfloor}u_s\right) \\
&\quad -\left(\frac{T^{-3/2}\sum_{s=1}^{\lfloor\tau T\rfloor}su_s-\lfloor\tau T\rfloor^{-1}T^{-1}\sum_{j=1}^{\lfloor\tau T\rfloor}jT^{-1/2}\sum_{s=1}^{\lfloor\tau T\rfloor}u_s}{T^{-3}\sum_{s=1}^{\lfloor\tau T\rfloor}\left(s-\lfloor\tau T\rfloor^{-1}\sum_{j=1}^{\lfloor\tau T\rfloor}j\right)^2}\right) \\
&\quad \times\left(T^{-3}\sum_{t=1}^{\lfloor rT\rfloor}t^2-\lfloor\tau T\rfloor^{-1}T^{-1}\sum_{t=1}^{\lfloor rT\rfloor}t\left(T^{-2}\sum_{s=1}^{\lfloor\tau T\rfloor}s\right)\right) \\
&\Rightarrow \int_0^r sdW(s)-\frac{r^2}{2\tau}W(\tau)-\left(\frac{\int_0^\tau sdW(s)-\frac{\tau^2}{2\tau}W(\tau)}{\frac{\tau^3}{12}}\right)\left(\frac{r^3}{3}-\frac{r^2\tau^2}{4\tau}\right) \\
&= rW(r)-\int_0^r W(s)ds-\frac{r^2}{2\tau}W(\tau)-\frac{r^3-3r^2(\tau-r)}{\tau^3}\left(\frac{\tau}{2}W(\tau)-\int_0^\tau W(s)ds\right)
\end{aligned}
$$

so that

$$
\begin{aligned}
T^{-3/2}\sum_{t=1}^{\lfloor rT\rfloor}(t-\lfloor rT\rfloor)\hat{u}_t \;\Rightarrow\;& rW(r)-\int_0^r W(s)ds-\frac{r^2}{2\tau}W(\tau)-\frac{r^3-3r^2(\tau-r)}{\tau^3}\left(\frac{\tau}{2}W(\tau)-\int_0^\tau W(s)ds\right)\\
&-r\left\{W(r)-\frac{r}{\tau}W(\tau)+\frac{6r(\tau-r)}{\tau^3}\left(\frac{\tau}{2}W(\tau)-\int_0^\tau W(s)ds\right)\right\}\\
=\;& -\frac{r^2(\tau-r)}{\tau^2}W(\tau)-\int_0^r W(s)ds+\frac{r^2(3\tau-2r)}{\tau^3}\int_0^\tau W(s)ds.
\end{aligned}
$$

Finally, for (iv),

$$
T^{-3/2}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor}(t-\lfloor rT\rfloor)\hat{u}_t=T^{-3/2}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor}t\hat{u}_t-rT^{-1/2}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor}\hat{u}_t
$$

and for the first right hand side term

$$
\begin{aligned}
T^{-3/2}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor}t\hat{u}_t =\;& T^{-3/2}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor}t\left(u_t-(T-\lfloor\tau T\rfloor)^{-1}\sum_{s=\lfloor\tau T\rfloor+1}^{T}u_s\right)\\
&-\left(\frac{T^{-3/2}\sum_{s=\lfloor\tau T\rfloor+1}^{T}u_s\left(s-(T-\lfloor\tau T\rfloor)^{-1}\sum_{j=\lfloor\tau T\rfloor+1}^{T}j\right)}{T^{-3}\sum_{s=\lfloor\tau T\rfloor+1}^{T}\left(s-(T-\lfloor\tau T\rfloor)^{-1}\sum_{j=\lfloor\tau T\rfloor+1}^{T}j\right)^2}\right)\\
&\times T^{-3}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor}t\left(t-(T-\lfloor\tau T\rfloor)^{-1}\sum_{s=\lfloor\tau T\rfloor+1}^{T}s\right)\\
=\;& T^{-3/2}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor}tu_t-(T-\lfloor\tau T\rfloor)^{-1}T^{-1}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor}t\left(T^{-1/2}\sum_{s=\lfloor\tau T\rfloor+1}^{T}u_s\right)\\
&-\left(\frac{T^{-3/2}\sum_{s=\lfloor\tau T\rfloor+1}^{T}su_s-(T-\lfloor\tau T\rfloor)^{-1}T^{-1}\sum_{j=\lfloor\tau T\rfloor+1}^{T}jT^{-1/2}\sum_{s=\lfloor\tau T\rfloor+1}^{T}u_s}{T^{-3}\sum_{s=\lfloor\tau T\rfloor+1}^{T}\left(s-(T-\lfloor\tau T\rfloor)^{-1}\sum_{j=\lfloor\tau T\rfloor+1}^{T}j\right)^2}\right)\\
&\times\left(T^{-3}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor}t^2-(T-\lfloor\tau T\rfloor)^{-1}T^{-1}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor}t\left(T^{-2}\sum_{s=\lfloor\tau T\rfloor+1}^{T}s\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\;& \int_\tau^r sdW(s)-\frac{r^2-\tau^2}{2(1-\tau)}(W(1)-W(\tau))\\
&-\left(\frac{\int_\tau^1 sdW(s)-\frac{1-\tau^2}{2(1-\tau)}(W(1)-W(\tau))}{\frac{(1-\tau)^3}{12}}\right)\left(\frac{r^3-\tau^3}{3}-\frac{(r^2-\tau^2)(1-\tau^2)}{4(1-\tau)}\right)\\
=\;& rW(r)-\int_0^r W(s)ds-\left(\tau W(\tau)-\int_0^\tau W(s)ds\right)-\frac{r^2-\tau^2}{2(1-\tau)}(W(1)-W(\tau))\\
&+(r-\tau)\frac{3r+3\tau-r\tau-\tau^2-4r^2}{(1-\tau)^3}\left\{W(1)-\int_0^1 W(s)ds\right.\\
&\left.-\left(\tau W(\tau)-\int_0^\tau W(s)ds\right)-\frac{1-\tau^2}{2(1-\tau)}(W(1)-W(\tau))\right\}
\end{aligned}
$$

so

$$
\begin{aligned}
T^{-3/2}\sum_{t=\lfloor\tau T\rfloor+1}^{\lfloor rT\rfloor}(t-\lfloor rT\rfloor)\hat{u}_t \;\Rightarrow\;& rW(r)-\int_0^r W(s)ds-\left(\tau W(\tau)-\int_0^\tau W(s)ds\right)-\frac{r^2-\tau^2}{2(1-\tau)}(W(1)-W(\tau))\\
&+(r-\tau)\frac{3r+3\tau-r\tau-\tau^2-4r^2}{(1-\tau)^3}\left\{W(1)-\int_0^1 W(s)ds\right.\\
&\left.-\left(\tau W(\tau)-\int_0^\tau W(s)ds\right)-\frac{1-\tau^2}{2(1-\tau)}(W(1)-W(\tau))\right\}\\
&-r\left\{W(r)-W(\tau)-\frac{r-\tau}{1-\tau}(W(1)-W(\tau))+\frac{6(r-\tau)(1-r)}{(1-\tau)^3}\right.\\
&\left.\times\left(\frac{1-\tau}{2}(W(1)-W(\tau))-\int_\tau^1(W(s)-W(\tau))ds\right)\right\}\\
=\;& -\frac{(r-\tau)^2(1-r)}{(1-\tau)^2}(W(1)-W(\tau))-\int_\tau^r(W(s)-W(\tau))ds\\
&+\frac{(r-\tau)^2(3(1-\tau)-2(r-\tau))}{(1-\tau)^3}\int_\tau^1(W(s)-W(\tau))ds.
\end{aligned}
$$

Taking each term in $S_0(\tau)$ separately

$$
\begin{aligned}
\lfloor\tau T\rfloor^{-2}\sum_{t=2}^{\lfloor\tau T\rfloor-1}\left(\sum_{s=1}^{t}\hat{u}_s\right)^2 \;\Rightarrow\;& \tau^{-2}\int_0^\tau\left\{W(r)-\frac{r}{\tau}W(\tau)+\frac{6r(\tau-r)}{\tau^3}\left(\frac{\tau}{2}W(\tau)-\int_0^\tau W(s)ds\right)\right\}^2dr\\
=\;& \int_0^1\left\{\tau^{-1/2}W(\tau r^*)-r^*\tau^{-1/2}W(\tau)\right.\\
&\left.+\tau^{-1/2}\frac{6r(\tau-r)}{\tau^3}\tau^{3/2}\left(\frac{1}{2}\tau^{-1/2}W(\tau)-\tau^{-3/2}\int_0^\tau W(s)ds\right)\right\}^2dr^*\\
=\;& \int_0^1\left\{W_\tau(r^*)-r^*W_\tau(1)+6r^*(1-r^*)\left(\frac{1}{2}W_\tau(1)-\int_0^1 W_\tau(s^*)ds^*\right)\right\}^2dr^*
\end{aligned}
$$

using $r^*=r\tau^{-1}$ and $W_\tau(r^*)=\tau^{-1/2}W(\tau r^*)$. This has the same distribution as

$$
\int_0^1\left\{W(r)-rW(1)+6r(1-r)\left(\frac{1}{2}W(1)-\int_0^1 W(s)ds\right)\right\}^2dr=\omega_\varepsilon^2\int_0^1 B_2(r)^2dr
$$

where $B_2(r)$ denotes a second level Brownian bridge. Next,

$$
\begin{aligned}
\lfloor\tau T\rfloor^{-4}\sum_{t=2}^{\lfloor\tau T\rfloor-1}\left(\sum_{s=1}^{t}(s-t)\hat{u}_s\right)^2 \;\Rightarrow\;& \tau^{-4}\int_0^\tau\left\{-\frac{r^2(\tau-r)}{\tau^2}W(\tau)-\int_0^r W(s)ds+\frac{r^2(3\tau-2r)}{\tau^3}\int_0^\tau W(s)ds\right\}^2dr\\
=\;& \tau^{-4}\int_0^1\left\{-\frac{\tau^2r^{*2}(\tau-\tau r^*)}{\tau^2}W(\tau)-\int_0^{r^*}W(\tau s^*)\tau ds^*\right.\\
&\left.+\frac{\tau^2r^{*2}(3\tau-2\tau r^*)}{\tau^3}\int_0^1 W(\tau s^*)\tau ds^*\right\}^2\tau dr^*\\
=\;& \int_0^1\left\{-r^{*2}(1-r^*)W_\tau(1)-\int_0^{r^*}W_\tau(s^*)ds^*\right.\\
&\left.+r^{*2}(3-2r^*)\int_0^1 W_\tau(s^*)ds^*\right\}^2dr^*
\end{aligned}
$$

which has the same distribution as

$$\int_0^1 \left\{ -r^2(1-r)W(1) - \int_0^r W(s)ds + r^2(3-2r)\int_0^1 W(s)ds \right\}^2 dr = \omega_\varepsilon^2 \int_0^1 K(r)^2 dr.$$

In a similar way, it can also be shown that

$$\begin{aligned}
(T-\lfloor \tau T \rfloor)^{-2} \sum_{t=\lfloor \tau T \rfloor+1}^{T-2} \left(\sum_{s=\lfloor \tau T \rfloor+1}^{t} \hat{u}_s \right)^2 &\Rightarrow \omega_\varepsilon^2 \int_0^1 B_2'(r)^2 dr \\
(T-\lfloor \tau T \rfloor)^{-4} \sum_{t=\lfloor \tau T \rfloor+1}^{T-2} \left(\sum_{s=\lfloor \tau T \rfloor+1}^{t} (s-t)\hat{u}_s \right)^2 &\Rightarrow \omega_\varepsilon^2 \int_0^1 K'(r)^2 dr
\end{aligned}$$

where $B_2'(r)$ and $K'(r)$ take the same forms as $B_2(r)$ and $K(r)$, respectively, but where the implied $B(r)$ and $B'(r)$ Brownian motion processes are independent. Hence,

$$S_0(\tau) \Rightarrow \omega_\varepsilon^2 \left\{ \int_0^1 B_2(r)^2 dr + \int_0^1 K(r)^2 dr + \int_0^1 B_2'(r)^2 dr + \int_0^1 K'(r)^2 dr \right\}.$$

(b) Let $W(r) = \omega_u B(r)$. Note that $\hat{u}_{\lfloor \tau T \rfloor+1} = 0$. For $t \leq \lfloor \tau T \rfloor$,

$$\begin{aligned}
T^{-1/2} \sum_{t=1}^{\lfloor rT \rfloor} \hat{u}_t &= T^{-1/2} \sum_{t=1}^{\lfloor rT \rfloor} \left(u_t - \lfloor \tau T \rfloor^{-1} \sum_{s=1}^{\lfloor \tau T \rfloor} u_s \right) \\
&= T^{-1/2} \sum_{t=1}^{\lfloor rT \rfloor} u_t - \lfloor rT \rfloor \lfloor \tau T \rfloor^{-1} T^{-1/2} \sum_{s=1}^{\lfloor \tau T \rfloor} u_s \\
&\Rightarrow W(r) - \frac{r}{\tau} W(\tau)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\lfloor \tau T \rfloor^{-2} \sum_{t=2}^{\lfloor \tau T \rfloor -1} \left(\sum_{s=2}^{t} \hat{u}_s \right)^2 &\Rightarrow \tau^{-2} \int_0^\tau \left\{ W(r) - \frac{r}{\tau} W(\tau) \right\}^2 dr \\
&\overset{d}{=} \int_0^1 \{W(r) - rW(1)\}^2 dr \\
&= \omega_u^2 \int_0^1 B_1(r)^2 dr.
\end{aligned}$$

The following is obtained in an analogous way

$$\begin{aligned}
(T-\lfloor \tau T \rfloor)^{-2} \sum_{t=\lfloor \tau T \rfloor+1}^{T-2} \left(\sum_{s=\lfloor \tau T \rfloor+1}^{t} \hat{u}_s \right)^2 &\Rightarrow (1-\tau)^{-2} \int_\tau^1 \left\{ W(r) - W(\tau) - \frac{r-\tau}{1-\tau}(W(1) - W(\tau)) \right\}^2 dr \\
&\overset{d}{=} \omega_u^2 \int_0^1 B_1'(r)^2 dr
\end{aligned}$$

noting that $\hat{u}_{\lfloor \tau T \rfloor+1} = 0$.

Finally, it is easily shown that

$$\begin{aligned}
\lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor -1} \hat{u}_{t+1}^2 &= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor -1} u_t^2 + O_p(T^{-1/2}) \\
&= \sigma_u^2 + O_p(T^{-1/2}), \\
(T - \lfloor \tau T \rfloor)^{-1} \sum_{t=\lfloor \tau T \rfloor +1}^{T-2} \hat{u}_{t+1}^2 &= (T - \lfloor \tau T \rfloor)^{-1} \sum_{t=\lfloor \tau T \rfloor +1}^{T-2} u_t^2 + O_p(T^{-1/2}) \\
&= \sigma_u^2 + O_p(T^{-1/2}).
\end{aligned}$$

So,

$$S_1(\tau) \Rightarrow \omega_u^2 \left\{ \int_0^1 B_1(r)^2 dr + \int_0^1 B_1'(r)^2 dr \right\} + 2\sigma_u^2.$$

Proof of Theorem 2

(a) For the second term of $S_0(\tau)$ consider

$$\begin{aligned}
\sum_{t=1}^{\lfloor rT \rfloor} t\hat{u}_t &= \sum_{t=1}^{\lfloor rT \rfloor} t \left(u_t - \lfloor \tau T \rfloor^{-1} \sum_{s=1}^{\lfloor \tau T \rfloor} u_s \right) \\
&\quad - \left(\frac{\sum_{s=1}^{\lfloor \tau T \rfloor} u_s \left(s - \lfloor \tau T \rfloor^{-1} \sum_{j=1}^{\lfloor \tau T \rfloor} j \right)}{T^{-3} \sum_{s=1}^{\lfloor \tau T \rfloor} \left(s - \lfloor \tau T \rfloor^{-1} \sum_{j=1}^{\lfloor \tau T \rfloor} j \right)^2} \right) T^{-3} \sum_{t=1}^{\lfloor rT \rfloor} t \left(t - \lfloor \tau T \rfloor^{-1} \sum_{s=1}^{\lfloor \tau T \rfloor} s \right) \\
&= O_p(T^{5/2})
\end{aligned}$$

Hence

$$\begin{aligned}
\lfloor \tau T \rfloor^{-4} \sum_{t=2}^{\lfloor \tau T \rfloor -1} \left(\sum_{s=1}^{t} (s-t)\hat{u}_s \right)^2 &= O_p(T^5) \lfloor \tau T \rfloor^{-3} O_p(1) \\
&= O_p(T^2)
\end{aligned}$$

Similarly, the fourth term is also $O_p(T^2)$. The first and third terms are also easily shown to be $O_p(T^2)$. The result for $\omega_\varepsilon^{-2} S_0(\tau)$ follows directly.

(b) For the first term of $S_1(\tau)$ we can show that

$$\begin{aligned}
\lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor -1} \hat{u}_{t+1}^2 &= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor -1} (\Delta\varepsilon_t)^2 + O_p(T^{-1/2}) \\
&= E\{(\Delta\varepsilon_t)^2\} + O_p(T^{-1/2})
\end{aligned}$$

The third term can be shown to behave likewise. For the second term of $S_1(\tau)$ consider

$$\begin{aligned}
\sum_{t=1}^{\lfloor rT \rfloor} \hat{u}_t &= \sum_{t=1}^{\lfloor rT \rfloor} \left(\Delta\varepsilon_t - \lfloor \tau T \rfloor^{-1} \sum_{s=1}^{\lfloor \tau T \rfloor} \Delta\varepsilon_s \right) \\
&= (\varepsilon_{\lfloor rT \rfloor} - \varepsilon_1) - \lfloor \tau T \rfloor^{-1} \lfloor rT \rfloor (\varepsilon_{\lfloor \tau T \rfloor} - \varepsilon_1) \\
&= O_p(1).
\end{aligned}$$

Hence,

$$\begin{aligned} \lfloor \tau T \rfloor^{-2} \sum_{t=2}^{\lfloor \tau T \rfloor -1} \left(\sum_{s=2}^{t} \hat{u}_s \right)^2 &= \lfloor \tau T \rfloor^{-1} O_p(1) \\ &= O_p(T^{-1}). \end{aligned}$$

In a similar way, the fourth term is also shown to be $O_p(T^{-1})$. So,

$$S_1(\tau) = 2E\{(\Delta \varepsilon_t)^2\} + O_p(T^{-1/2})$$

and the result for $\omega_u^{-2}\{S_1(\tau) - 2\sigma_u^2\}$ follows directly.

Table 1. Asymptotic α-level critical values from the $\mathcal{L}_0$ and $\mathcal{L}_1$ distributions.

	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
$\mathcal{L}_0$	0.220	0.257	0.349
$\mathcal{L}_1$	0.607	0.749	1.063

Table 2. Asymptotic coverage of nominal $(1-\alpha)$-level $\hat{S}^k_{pre,j}(\tau)$, $j = \{NP, P\}$, $k = \{\tau, \hat{\tau}\}$ confidence sets: $\rho = 1 + cT^{-1}$.

τ_0	$c = 0$	$c = -5$	$c = -10$	$c = -20$	$c = -30$	$c = -40$	$c = -50$
				(1 −α) = 0.90			
0.1	0.900	0.954	0.973	0.988	0.996	0.999	1.000
0.2	0.900	0.956	0.983	0.998	1.000	1.000	1.000
0.3	0.900	0.957	0.991	1.000	1.000	1.000	1.000
0.4	0.900	0.959	0.994	1.000	1.000	1.000	1.000
0.5	0.900	0.963	0.994	1.000	1.000	1.000	1.000
0.6	0.900	0.963	0.993	1.000	1.000	1.000	1.000
0.7	0.900	0.961	0.989	0.999	1.000	1.000	1.000
0.8	0.900	0.960	0.983	0.997	0.999	1.000	1.000
0.9	0.900	0.958	0.973	0.986	0.994	0.997	0.999
				(1 −α) = 0.95			
0.1	0.950	0.980	0.988	0.996	0.999	1.000	1.000
0.2	0.950	0.980	0.994	1.000	1.000	1.000	1.000
0.3	0.950	0.982	0.997	1.000	1.000	1.000	1.000
0.4	0.950	0.986	0.999	1.000	1.000	1.000	1.000
0.5	0.950	0.985	0.999	1.000	1.000	1.000	1.000
0.6	0.950	0.985	0.998	1.000	1.000	1.000	1.000
0.7	0.950	0.984	0.996	1.000	1.000	1.000	1.000
0.8	0.950	0.984	0.994	0.999	1.000	1.000	1.000
0.9	0.950	0.981	0.989	0.995	0.998	0.999	1.000
				(1 −α) = 0.99			
0.1	0.990	0.997	0.998	1.000	1.000	1.000	1.000
0.2	0.990	0.998	0.999	1.000	1.000	1.000	1.000
0.3	0.990	0.998	1.000	1.000	1.000	1.000	1.000
0.4	0.990	0.999	1.000	1.000	1.000	1.000	1.000
0.5	0.990	0.998	1.000	1.000	1.000	1.000	1.000
0.6	0.990	0.998	1.000	1.000	1.000	1.000	1.000
0.7	0.990	0.998	1.000	1.000	1.000	1.000	1.000
0.8	0.990	0.998	0.999	1.000	1.000	1.000	1.000
0.9	0.990	0.997	0.999	0.999	1.000	1.000	1.000

Table 3. Finite sample coverage and length of nominal 0.95-level confidence sets: $\tau_0 = 0.3$, $\delta_1 = 5$, $\delta_2 = 0.5$

Panel A. $T = 150$												
ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.852	0.757	0.850	0.947	0.739	0.946	0.851	0.756	0.849	0.946	0.737	0.945
0.50	0.861	0.843	0.861	0.969	0.943	0.971	0.860	0.843	0.860	0.969	0.942	0.971
0.80	0.854	0.950	0.881	0.992	0.996	0.993	0.853	0.950	0.880	0.990	0.996	0.992
0.90	0.805	0.970	0.933	0.996	0.998	0.996	0.804	0.969	0.931	0.995	0.997	0.995
0.95	0.720	0.969	0.953	0.996	0.989	0.989	0.719	0.968	0.951	0.994	0.989	0.987
1.00	0.600	0.955	0.944	0.996	0.958	0.958	0.600	0.952	0.940	0.990	0.957	0.956
						Length						
0.00	0.392	0.784	0.402	0.868	0.705	0.863	0.016	0.225	0.021	0.029	0.221	0.033
0.50	0.402	0.821	0.412	0.824	0.893	0.823	0.060	0.170	0.063	0.112	0.271	0.115
0.80	0.447	0.881	0.540	0.956	0.942	0.946	0.137	0.199	0.154	0.209	0.436	0.250
0.90	0.485	0.907	0.786	0.969	0.927	0.930	0.173	0.258	0.243	0.315	0.463	0.401
0.95	0.498	0.910	0.866	0.971	0.892	0.895	0.205	0.315	0.309	0.453	0.463	0.441
1.00	0.483	0.895	0.874	0.972	0.832	0.836	0.237	0.351	0.349	0.571	0.441	0.432

Panel B. $T = 300$												
ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.932	0.766	0.932	0.947	0.688	0.947	0.931	0.765	0.931	0.947	0.688	0.947
0.50	0.918	0.891	0.919	0.959	0.927	0.960	0.918	0.891	0.919	0.959	0.927	0.960
0.80	0.873	0.989	0.880	0.982	1.000	0.985	0.873	0.989	0.879	0.981	1.000	0.985
0.90	0.764	0.998	0.843	0.995	1.000	0.997	0.763	0.998	0.842	0.994	1.000	0.996
0.95	0.606	0.995	0.941	0.997	0.999	0.998	0.605	0.994	0.940	0.996	0.999	0.998
1.00	0.327	0.960	0.946	0.998	0.956	0.956	0.327	0.957	0.943	0.995	0.955	0.955
						Length						
0.00	0.146	0.372	0.146	0.917	0.289	0.916	0.020	0.126	0.020	0.030	0.112	0.030
0.50	0.152	0.362	0.153	0.867	0.410	0.864	0.056	0.084	0.056	0.075	0.119	0.076
0.80	0.164	0.463	0.170	0.943	0.608	0.935	0.091	0.116	0.091	0.122	0.283	0.125
0.90	0.171	0.616	0.286	0.980	0.627	0.886	0.103	0.166	0.122	0.169	0.313	0.193
0.95	0.166	0.701	0.593	0.983	0.612	0.691	0.101	0.213	0.196	0.235	0.319	0.284
1.00	0.152	0.688	0.670	0.980	0.566	0.580	0.087	0.292	0.288	0.441	0.336	0.329

Table 4. Finite sample coverage and length of nominal 0.95-level confidence sets: $\tau_0 = 0.3$, $\delta_1 = 10$, $\delta_2 = 1$

Panel A. $T = 150$												
ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.852	0.757	0.851	0.947	0.739	0.947	0.852	0.757	0.851	0.947	0.739	0.947
0.50	0.861	0.843	0.861	0.969	0.943	0.971	0.861	0.843	0.861	0.969	0.943	0.971
0.80	0.854	0.950	0.879	0.992	0.996	0.994	0.854	0.950	0.879	0.992	0.996	0.994
0.90	0.805	0.970	0.924	0.996	0.998	0.996	0.805	0.970	0.924	0.996	0.998	0.996
0.95	0.720	0.969	0.946	0.996	0.989	0.989	0.720	0.969	0.946	0.996	0.989	0.989
1.00	0.600	0.955	0.939	0.996	0.958	0.958	0.600	0.955	0.939	0.996	0.958	0.958
						Length						
0.00	0.392	0.748	0.399	0.910	0.501	0.901	0.007	0.013	0.007	0.010	0.013	0.010
0.50	0.393	0.761	0.401	0.950	0.582	0.941	0.015	0.006	0.015	0.030	0.010	0.030
0.80	0.404	0.781	0.482	0.969	0.631	0.890	0.067	0.007	0.053	0.126	0.018	0.101
0.90	0.414	0.797	0.663	0.973	0.640	0.744	0.098	0.008	0.038	0.180	0.020	0.064
0.95	0.413	0.804	0.742	0.973	0.639	0.685	0.106	0.011	0.026	0.233	0.020	0.041
1.00	0.400	0.798	0.765	0.972	0.621	0.648	0.102	0.017	0.024	0.316	0.022	0.033

Panel B. $T = 300$												
ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.932	0.766	0.932	0.947	0.688	0.947	0.932	0.766	0.932	0.947	0.688	0.947
0.50	0.918	0.891	0.918	0.959	0.927	0.959	0.918	0.891	0.918	0.959	0.927	0.959
0.80	0.873	0.989	0.877	0.982	1.000	0.984	0.873	0.989	0.877	0.982	1.000	0.984
0.90	0.764	0.998	0.822	0.995	1.000	0.997	0.764	0.998	0.822	0.995	1.000	0.997
0.95	0.606	0.995	0.910	0.997	0.999	0.998	0.606	0.995	0.910	0.997	0.999	0.998
1.00	0.327	0.960	0.939	0.998	0.956	0.956	0.327	0.960	0.939	0.998	0.956	0.956
						Length						
0.00	0.144	0.274	0.144	0.951	0.207	0.950	0.006	0.006	0.006	0.007	0.005	0.007
0.50	0.145	0.266	0.145	0.940	0.175	0.939	0.018	0.003	0.018	0.030	0.007	0.030
0.80	0.149	0.274	0.151	0.981	0.187	0.972	0.057	0.004	0.056	0.085	0.047	0.085
0.90	0.148	0.289	0.173	0.985	0.193	0.855	0.069	0.010	0.061	0.117	0.054	0.103
0.95	0.137	0.309	0.261	0.981	0.199	0.457	0.066	0.018	0.038	0.150	0.049	0.072
1.00	0.112	0.347	0.337	0.967	0.214	0.265	0.044	0.025	0.029	0.243	0.036	0.041

Table 5. Finite sample coverage and length of nominal 0.95-level confidence sets: $\tau_0 = 0.3$, $\delta_1 = 10$, $\delta_2 = 0$

Panel A. $T = 150$												
ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.852	0.757	0.846	0.947	0.739	0.944	0.852	0.757	0.846	0.947	0.739	0.944
0.50	0.861	0.843	0.862	0.969	0.943	0.973	0.861	0.843	0.862	0.969	0.943	0.973
0.80	0.854	0.950	0.902	0.992	0.996	0.996	0.854	0.950	0.902	0.992	0.996	0.996
0.90	0.805	0.970	0.946	0.996	0.998	0.997	0.805	0.970	0.946	0.996	0.998	0.997
0.95	0.720	0.969	0.951	0.996	0.989	0.989	0.720	0.969	0.951	0.996	0.989	0.989
1.00	0.600	0.955	0.936	0.996	0.958	0.958	0.600	0.955	0.936	0.996	0.958	0.958
						Length						
0.00	0.790	0.859	0.794	0.884	0.898	0.883	0.007	0.124	0.014	0.010	0.122	0.017
0.50	0.778	0.872	0.787	0.897	0.930	0.900	0.013	0.078	0.019	0.022	0.118	0.030
0.80	0.776	0.887	0.819	0.968	0.943	0.956	0.039	0.082	0.054	0.105	0.179	0.116
0.90	0.774	0.898	0.875	0.975	0.942	0.947	0.127	0.102	0.109	0.372	0.188	0.201
0.95	0.743	0.903	0.891	0.976	0.937	0.939	0.234	0.123	0.139	0.559	0.184	0.210
1.00	0.665	0.902	0.888	0.976	0.926	0.927	0.300	0.131	0.148	0.649	0.163	0.186

Panel B. $T = 300$												
ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.932	0.766	0.927	0.947	0.688	0.941	0.932	0.766	0.927	0.947	0.688	0.941
0.50	0.918	0.891	0.919	0.959	0.927	0.962	0.918	0.891	0.919	0.959	0.927	0.962
0.80	0.873	0.989	0.889	0.982	1.000	0.988	0.873	0.989	0.889	0.982	1.000	0.988
0.90	0.764	0.998	0.875	0.995	1.000	0.998	0.764	0.998	0.875	0.995	1.000	0.998
0.95	0.606	0.995	0.940	0.997	0.999	0.999	0.606	0.995	0.940	0.997	0.999	0.999
1.00	0.327	0.960	0.930	0.998	0.956	0.956	0.327	0.960	0.930	0.998	0.956	0.956
						Length						
0.00	0.248	0.865	0.285	0.893	0.886	0.892	0.006	0.168	0.017	0.008	0.153	0.017
0.50	0.262	0.895	0.274	0.815	0.949	0.817	0.012	0.112	0.014	0.016	0.156	0.018
0.80	0.332	0.924	0.372	0.942	0.981	0.945	0.027	0.131	0.034	0.048	0.524	0.075
0.90	0.435	0.947	0.619	0.983	0.982	0.982	0.076	0.206	0.130	0.274	0.580	0.346
0.95	0.468	0.957	0.865	0.988	0.978	0.979	0.189	0.294	0.290	0.589	0.544	0.520
1.00	0.368	0.945	0.916	0.988	0.952	0.952	0.267	0.329	0.335	0.753	0.410	0.418

Table 6. Finite sample coverage and length of nominal 0.95-level confidence sets: $\tau_0 = 0.3$, $\delta_1 = 0$, $\delta_2 = 1$

Panel A. $T = 150$

ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.852	0.757	0.846	0.947	0.739	0.943	0.833	0.712	0.825	0.942	0.694	0.936
0.50	0.861	0.843	0.862	0.969	0.943	0.973	0.847	0.782	0.843	0.964	0.915	0.966
0.80	0.854	0.950	0.890	0.992	0.996	0.996	0.843	0.907	0.871	0.988	0.994	0.993
0.90	0.805	0.970	0.944	0.996	0.998	0.997	0.794	0.936	0.918	0.993	0.993	0.992
0.95	0.720	0.969	0.959	0.996	0.989	0.989	0.708	0.928	0.921	0.989	0.977	0.976
1.00	0.600	0.955	0.950	0.996	0.958	0.958	0.602	0.899	0.895	0.981	0.933	0.932
						Length						
0.00	0.224	0.487	0.247	0.897	0.390	0.862	0.039	0.089	0.043	0.053	0.085	0.057
0.50	0.231	0.488	0.248	0.917	0.315	0.880	0.056	0.069	0.057	0.079	0.117	0.081
0.80	0.247	0.587	0.348	0.972	0.353	0.801	0.087	0.092	0.088	0.130	0.178	0.137
0.90	0.260	0.663	0.587	0.974	0.377	0.513	0.106	0.116	0.115	0.180	0.186	0.175
0.95	0.265	0.694	0.669	0.972	0.394	0.439	0.115	0.136	0.136	0.235	0.188	0.184
1.00	0.266	0.692	0.681	0.969	0.415	0.434	0.119	0.163	0.162	0.315	0.198	0.196

Panel B. $T = 300$

ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.932	0.766	0.925	0.947	0.688	0.937	0.926	0.738	0.917	0.945	0.654	0.934
0.50	0.918	0.891	0.921	0.959	0.927	0.962	0.914	0.854	0.915	0.958	0.906	0.959
0.80	0.873	0.989	0.889	0.982	1.000	0.989	0.872	0.979	0.887	0.983	0.999	0.990
0.90	0.764	0.998	0.867	0.995	1.000	0.998	0.764	0.994	0.866	0.994	1.000	0.997
0.95	0.606	0.995	0.958	0.997	0.999	0.999	0.603	0.992	0.954	0.998	0.999	0.998
1.00	0.327	0.960	0.952	0.998	0.956	0.955	0.327	0.942	0.934	0.995	0.946	0.946
						Length						
0.00	0.081	0.165	0.089	0.927	0.693	0.913	0.028	0.044	0.029	0.031	0.036	0.032
0.50	0.087	0.152	0.090	0.919	0.197	0.888	0.038	0.043	0.039	0.046	0.057	0.046
0.80	0.101	0.184	0.106	0.976	0.163	0.932	0.055	0.063	0.056	0.075	0.116	0.077
0.90	0.108	0.213	0.143	0.986	0.167	0.735	0.063	0.081	0.070	0.106	0.123	0.107
0.95	0.105	0.235	0.215	0.984	0.172	0.326	0.061	0.093	0.090	0.144	0.121	0.118
1.00	0.088	0.260	0.257	0.972	0.182	0.205	0.045	0.102	0.102	0.241	0.113	0.112

Table 7. Finite sample coverage and length of nominal 0.95-level confidence sets: $\tau_0 = 0.5$, $\delta_1 = 5$, $\delta_2 = 0.5$

Panel A. $T = 150$												
ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.892	1.000	0.896	0.942	0.999	0.949	0.891	0.998	0.895	0.942	0.997	0.948
0.50	0.882	1.000	0.886	0.963	1.000	0.968	0.881	1.000	0.885	0.962	0.999	0.967
0.80	0.832	1.000	0.883	0.991	1.000	0.995	0.830	0.999	0.881	0.990	1.000	0.994
0.90	0.722	0.999	0.938	0.996	1.000	0.999	0.720	0.998	0.938	0.994	1.000	0.997
0.95	0.578	0.995	0.963	0.995	0.996	0.995	0.577	0.993	0.961	0.990	0.996	0.994
1.00	0.436	0.980	0.960	0.990	0.965	0.964	0.435	0.975	0.955	0.983	0.964	0.962
						Length						
0.00	0.252	0.861	0.268	0.928	0.794	0.924	0.014	0.150	0.018	0.025	0.146	0.028
0.50	0.279	0.904	0.294	0.964	0.898	0.962	0.047	0.078	0.048	0.097	0.211	0.099
0.80	0.312	0.931	0.441	0.977	0.929	0.964	0.109	0.106	0.115	0.191	0.420	0.228
0.90	0.330	0.942	0.752	0.978	0.922	0.936	0.134	0.180	0.183	0.261	0.440	0.370
0.95	0.344	0.944	0.871	0.978	0.899	0.906	0.144	0.246	0.246	0.326	0.428	0.403
1.00	0.366	0.932	0.895	0.976	0.841	0.847	0.162	0.298	0.297	0.393	0.406	0.394
Panel B. $T = 300$												
ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.937	1.000	0.938	0.945	0.999	0.947	0.937	1.000	0.938	0.945	0.997	0.947
0.50	0.917	1.000	0.919	0.956	1.000	0.958	0.917	1.000	0.919	0.956	1.000	0.958
0.80	0.845	1.000	0.856	0.975	1.000	0.981	0.845	1.000	0.856	0.974	1.000	0.980
0.90	0.695	1.000	0.803	0.991	1.000	0.994	0.694	1.000	0.802	0.990	1.000	0.994
0.95	0.475	1.000	0.922	0.997	1.000	0.999	0.474	0.999	0.921	0.995	1.000	0.998
1.00	0.218	0.970	0.952	0.995	0.959	0.959	0.218	0.968	0.950	0.990	0.959	0.959
						Length						
0.00	0.114	0.481	0.117	0.959	0.305	0.955	0.016	0.043	0.016	0.023	0.032	0.023
0.50	0.129	0.421	0.131	0.954	0.415	0.951	0.049	0.024	0.049	0.069	0.063	0.069
0.80	0.144	0.504	0.153	0.985	0.537	0.973	0.082	0.070	0.082	0.117	0.256	0.119
0.90	0.142	0.609	0.258	0.988	0.547	0.881	0.086	0.131	0.103	0.161	0.284	0.180
0.95	0.124	0.673	0.560	0.989	0.538	0.643	0.077	0.179	0.165	0.215	0.282	0.252
1.00	0.098	0.667	0.648	0.988	0.516	0.533	0.063	0.252	0.249	0.314	0.295	0.289

Table 8. Finite sample coverage and length of nominal 0.95-level confidence sets: $\tau_0 = 0.5$, $\delta_1 = 10$, $\delta_2 = 1$

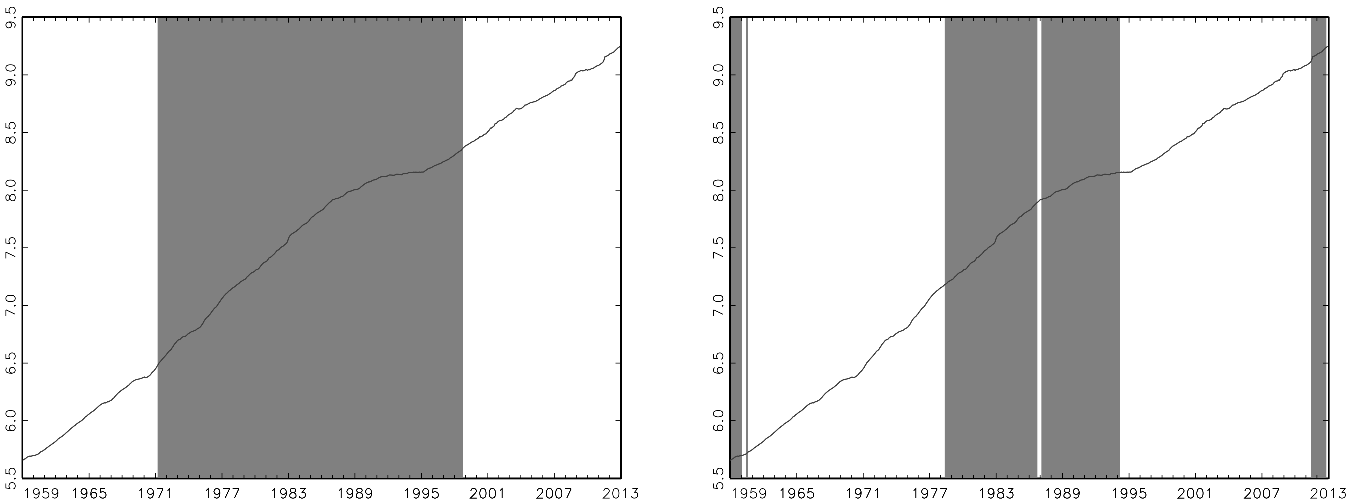
Panel A. $T = 150$												
ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.892	1.000	0.895	0.942	0.999	0.948	0.892	1.000	0.895	0.942	0.999	0.948
0.50	0.882	1.000	0.886	0.963	1.000	0.968	0.882	1.000	0.886	0.963	1.000	0.968
0.80	0.832	1.000	0.876	0.991	1.000	0.994	0.832	1.000	0.876	0.991	1.000	0.994
0.90	0.722	0.999	0.916	0.996	1.000	0.998	0.722	0.999	0.916	0.996	1.000	0.998
0.95	0.578	0.995	0.943	0.995	0.996	0.995	0.578	0.995	0.943	0.995	0.996	0.995
1.00	0.436	0.980	0.947	0.990	0.965	0.964	0.436	0.980	0.947	0.990	0.965	0.964
						Length						
0.00	0.226	0.940	0.239	0.954	0.580	0.948	0.007	0.007	0.007	0.010	0.007	0.010
0.50	0.248	0.962	0.262	0.971	0.636	0.964	0.013	0.007	0.013	0.025	0.007	0.024
0.80	0.274	0.967	0.398	0.975	0.684	0.918	0.047	0.007	0.040	0.108	0.007	0.088
0.90	0.277	0.967	0.698	0.976	0.690	0.801	0.071	0.007	0.034	0.160	0.007	0.059
0.95	0.270	0.963	0.831	0.976	0.683	0.739	0.078	0.007	0.022	0.193	0.007	0.034
1.00	0.262	0.950	0.875	0.976	0.663	0.699	0.079	0.007	0.017	0.223	0.007	0.023

Panel B. $T = 300$												
ρ	$\hat{S}^{\tau}_{0,NP}(\tau)$	$\hat{S}^{\tau}_{1,NP}(\tau)$	$\hat{S}^{\tau}_{pre,NP}(\tau)$	$\hat{S}^{\tau}_{0,P}(\tau)$	$\hat{S}^{\tau}_{1,P}(\tau)$	$\hat{S}^{\tau}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,NP}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
						Coverage						
0.00	0.937	1.000	0.937	0.945	0.999	0.945	0.937	1.000	0.937	0.945	0.999	0.945
0.50	0.917	1.000	0.918	0.956	1.000	0.957	0.917	1.000	0.918	0.956	1.000	0.957
0.80	0.845	1.000	0.852	0.975	1.000	0.979	0.845	1.000	0.852	0.975	1.000	0.979
0.90	0.695	1.000	0.773	0.991	1.000	0.994	0.695	1.000	0.773	0.991	1.000	0.994
0.95	0.475	1.000	0.878	0.997	1.000	0.999	0.475	1.000	0.878	0.997	1.000	0.999
1.00	0.218	0.970	0.938	0.995	0.959	0.959	0.218	0.970	0.938	0.995	0.959	0.959
						Length						
0.00	0.098	0.241	0.098	0.971	0.284	0.971	0.006	0.003	0.006	0.007	0.003	0.007
0.50	0.112	0.241	0.112	0.982	0.200	0.980	0.014	0.003	0.014	0.023	0.004	0.023
0.80	0.124	0.262	0.126	0.987	0.209	0.978	0.047	0.003	0.047	0.079	0.018	0.078
0.90	0.120	0.286	0.149	0.988	0.212	0.869	0.056	0.004	0.050	0.111	0.021	0.095
0.95	0.103	0.303	0.243	0.989	0.213	0.480	0.049	0.006	0.025	0.142	0.019	0.053
1.00	0.077	0.328	0.314	0.988	0.214	0.272	0.033	0.008	0.012	0.189	0.011	0.019

Table 9. Finite sample length of nominal 0.95-level confidence sets with 10% trimming

		$\tau_0 = 0.3,\ \delta_1 = 5,\ \delta_2 = 0.5$			$\tau_0 = 0.3,\ \delta_1 = 10,\ \delta_2 = 1$			$\tau_0 = 0.3,\ \delta_1 = 10,\ \delta_2 = 0$		
T	ρ	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
150	0.00	0.029	0.118	0.031	0.010	0.009	0.010	0.010	0.035	0.012
	0.50	0.112	0.166	0.113	0.030	0.010	0.030	0.022	0.034	0.023
	0.80	0.207	0.307	0.220	0.126	0.017	0.101	0.104	0.071	0.076
	0.90	0.291	0.330	0.301	0.179	0.018	0.063	0.342	0.076	0.108
	0.95	0.396	0.330	0.320	0.224	0.017	0.039	0.491	0.073	0.104
	1.00	0.489	0.319	0.315	0.288	0.015	0.027	0.560	0.061	0.087
300	0.00	0.030	0.065	0.030	0.007	0.005	0.007	0.008	0.056	0.011
	0.50	0.075	0.089	0.075	0.030	0.007	0.030	0.016	0.062	0.017
	0.80	0.122	0.221	0.123	0.085	0.047	0.085	0.048	0.380	0.066
	0.90	0.169	0.243	0.177	0.117	0.054	0.103	0.261	0.427	0.287
	0.95	0.233	0.247	0.228	0.150	0.049	0.072	0.515	0.390	0.387
	1.00	0.396	0.263	0.259	0.235	0.036	0.041	0.635	0.269	0.280

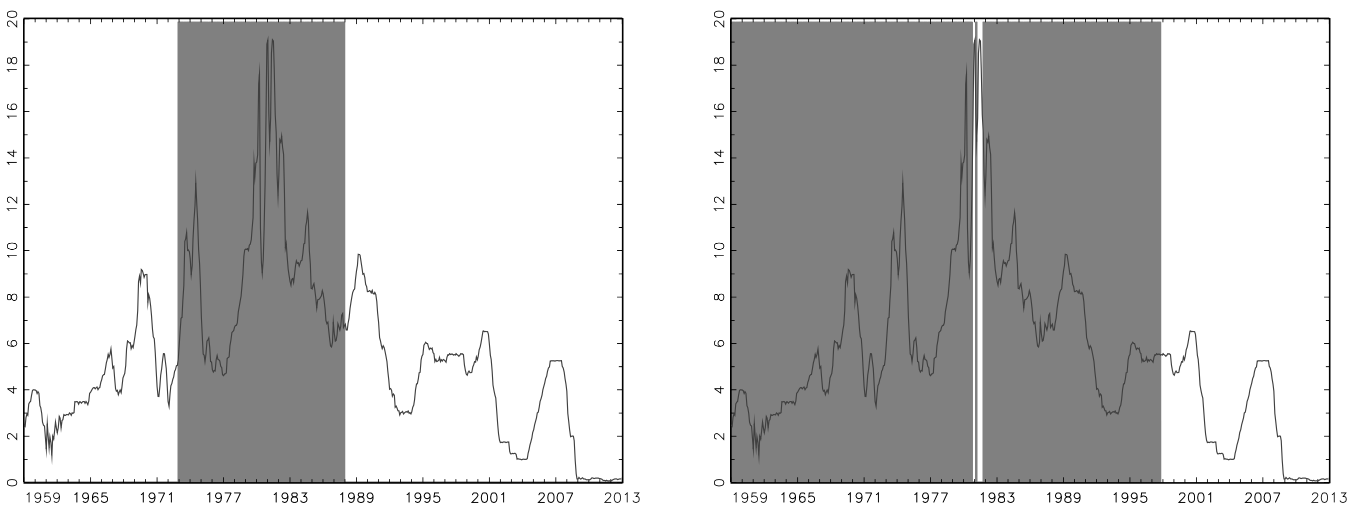
		$\tau_0 = 0.3,\ \delta_1 = 0,\ \delta_2 = 1$			$\tau_0 = 0.5,\ \delta_1 = 5,\ \delta_2 = 0.5$			$\tau_0 = 0.5,\ \delta_1 = 10,\ \delta_2 = 1$		
T	ρ	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{0,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{1,P}(\tau)$	$\hat{S}^{\hat{\tau}}_{pre,P}(\tau)$
150	0.00	0.053	0.078	0.056	0.025	0.072	0.026	0.010	0.007	0.010
	0.50	0.079	0.116	0.081	0.097	0.145	0.097	0.025	0.007	0.024
	0.80	0.130	0.176	0.136	0.191	0.333	0.210	0.108	0.007	0.088
	0.90	0.180	0.182	0.172	0.261	0.349	0.305	0.160	0.007	0.059
	0.95	0.233	0.181	0.177	0.323	0.338	0.323	0.193	0.007	0.034
	1.00	0.302	0.181	0.179	0.379	0.321	0.315	0.222	0.007	0.023
300	0.00	0.031	0.036	0.032	0.023	0.016	0.023	0.007	0.003	0.007
	0.50	0.046	0.057	0.046	0.069	0.060	0.069	0.023	0.004	0.023
	0.80	0.075	0.116	0.077	0.117	0.247	0.119	0.079	0.018	0.078
	0.90	0.106	0.123	0.107	0.161	0.273	0.178	0.111	0.021	0.095
	0.95	0.144	0.121	0.118	0.215	0.269	0.242	0.142	0.019	0.053
	1.00	0.236	0.113	0.112	0.311	0.266	0.261	0.189	0.011	0.019



(a) $\hat{S}^{\hat{\tau}}_{0,P}(\tau)$

(b) $\hat{S}^{\hat{\tau}}_{1,P}(\tau) = \hat{S}^{\hat{\tau}}_{pre,P}(\tau)$

Figure 1. US money supply M2 (1959:1-2012:12) and 0.95-level confidence sets for a break in level/trend



(a) $\hat{S}^{\hat{\tau}}_{0,P}(\tau) = \hat{S}^{\hat{\tau}}_{pre,P}(\tau)$

(b) $\hat{S}^{\hat{\tau}}_{1,P}(\tau)$

Figure 2. US effective federal funds rate (1959:1-2012:12) and 0.95-level confidence sets for a break in level/trend