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autoregressive model

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Testing for strict stationarity in a random coefficient autoregressive model

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Abstract: We propose a procedure to decide between the null hypothesis of (strict) stationarity and the alternative of non-stationarity, in the context of a Random Coefficient AutoRegression (RCAR). The procedure is based on randomising a diagnostic which diverges to positive infinity under the null, and drifts to zero under the alternative. Thence, we propose a randomised test which can be used directly and - building on it - a decision rule to discern between the null and the alternative. The procedure can be applied under very general circumstances: albeit developed for an RCAR model, it can be used in the case of a standard AR(1) model, without requiring any modifications or prior knowledge. Also, the test works (again with no modification or prior knowledge being required) in the presence of infinite variance, and in general requires minimal assumptions on the existence of moments.

Keywords: Random Coefficient AutoRegression, Stationarity, Unit Root, Heavy Tails, Randomised Tests.

AMS 2000 subject classification: Primary 62F05; secondary 62M10.

1. Introduction

In this paper, we propose a procedure to decide in favour of, or against, the strict stationarity of a series generated by a Random Coefficient AutoRegressive (RCAR) model:

$$X_t = (\varphi + b_t) X_{t-1} + e_t, \quad 1 \leq t < \infty, \quad \text{where } X_0 \text{ is an initial value.} \quad (1.1)$$

Model (1.1) has been paid considerable attention by the literature, mainly due to its flexibility and analytical tractability - see [Nicholls and Quinn \(2012\)](#) and the references in [Aue and Horváth \(2011\)](#), and also the article by [Diaconis and Freedman \(1999\)](#) where several examples are discussed. Equation (1.1) has also become increasingly popular in econometrics. Indeed, it is immediate to see that (1.1) nests the AR(1) model as a special case, with the advantage that it can be viewed as a competitor for a model with an abrupt break in the autoregressive root (see, especially, a related paper by [Giraitis et al., 2014](#)). Further, a closely related specification is the so-called Double AutoRegressive (DAR) model $X_t = \varphi X_{t-1} + v_t$ with $v_t = \sqrt{a + bX_{t-1}^2} \epsilon_t$ and ϵ_t is an *i.i.d.* process; this model, in turn, nests the popular ARCH specification (see Theorem 1 in [Tsay, 1987](#)). Finally, (1.1) has been employed as a more general alternative to deterministic unit root processes, with $\varphi = 1$ and b_t not identically zero - this is known in the literature as the Stochastic Unit Root (STUR) process, and we refer to the contributions by [Granger and Swanson \(1997\)](#), [McCabe and Tremayne \(1995\)](#) and [Leybourne et al. \(1996\)](#), among others, for an overview.

Various aspects of the inference on (1.1) are well developed. The estimation of φ , in particular, has been studied in numerous contributions, for both the stationary case ([Aue et al., 2006](#)) and the non-stationary case ([Berkes et al., 2009](#)); [Aue and Horváth \(2011\)](#) suggest using the Quasi-Maximum Likelihood (QML) estimator, showing that the estimator of φ is always consistent and asymptotically normal, irrespective of the stationarity (or lack thereof) of X_t , as long as b_t is not equal zero almost surely. The same result has also been shown for several other estimators, like the Weighted Least Squares (WLS) estimator (see [Horváth and Trapani, 2016](#)) and the Empirical Likelihood (EL) estimator, with [Hill and Peng \(2014\)](#) and [Hill et al. \(2016\)](#) showing that standard normal inference holds for all possible cases.

Conversely, other parts of the inference on (1.1) are not fully established. In particular, only few results are available as far as testing for the stationarity/ergodicity of X_t is concerned. Most contributions focus on restricted versions of (1.1), e.g. testing whether X_t is a genuine unit root process versus the alternative of a STUR process - see [McCabe and Tremayne \(1995\)](#), [Leybourne et al. \(1996\)](#), [Distaso \(2008\)](#) and [Nagakura \(2009\)](#). Indeed, such approaches are not fully clear about X_t being stationary

or not, since a STUR process may well be strictly stationary (we refer to [Yoon, 2006](#) for an insightful discussion). The case of a stochastic unit root is of great interest, since it is a marked departure from the usual deterministic unit root case. Indeed, whilst the unit root hypothesis may often hold for several series, nonstationarity itself may not; in this respect, the literature has explored, especially in the context of financial data, the notion of “volatility induced stationarity”, where a series may have a unit root, but its variance grows to infinity so as to compensate the deviations from the mean (see [Nielsen and Rahbek, 2014](#) and [Kanaya, 2011](#)). There are several reasons why it would be useful to find out whether X_t is stationary and ergodic or not: estimating the variance of the error term e_t is possible only under stationarity; further, in order to recover standard normal inference when X_t is nonstationary through e.g. the QML estimator discussed in [Aue and Horváth \(2011\)](#), it is required that b_t is not equal to zero almost surely, otherwise the standard OLS estimator has a faster rate of convergence ([Wang and Yu, 2015](#)). Although there is a plethora of tests for a unit root in the AR(1) case, to the best of our knowledge there are no contributions which have a sufficient amount of generality to be applicable to the context of AR(1) and RCAR(1) models. In addition to this, even in the AR(1) case, testing procedures usually require the existence of at least the first two moments; tests for a unit root in presence of infinite variance have been studied ([Phillips, 1990](#)), but their implementation usually requires using a different limiting distribution (and, consequently, different critical values) - note however that, in a related contribution, [Cavaliere et al. \(2016\)](#) provide a solution by using the bootstrap, but their results do not cover the RCAR case and it is not immediately clear how to generalise the bootstrap to the RCAR model under general conditions (see [Fink and Kreiss, 2013](#)).

Such a lack of procedures to check for the (strict) stationarity of an RCAR model is undesirable, in the light of the empirical potential of this family of models. Indeed, the literature has developed a fair amount of tests for stationarity (either as the null, or, perhaps more frequently, the alternative hypothesis) in various contexts. There are, as is well-known, many contributions for the unit root problem in the case of a linear model (such as the AR(1), which as noted above is nested in the RCAR set-up), but there are also quite a few contributions for more general frameworks. For example, [Kapetanios \(2007\)](#), [Lima and Neri \(2013\)](#) and [Busetti and Harvey \(2010\)](#) propose tests for the null of stationarity by using quantile-based techniques, thus not needing to specify a model for the data. In addition, there are also several tests that deal with specific nonlinear models, such as TAR ([Tsay, 1997](#), [Caner and Hansen, 2001](#)), STAR ([Kapetanios et al., 2003](#)) and ESTAR ([Kılıç, 2011](#)). In particular, building on the notion of top Lyapunov exponent, [Guo et al. \(2016\)](#) propose a test for the null of strict

stationarity in the context of a DAR model; see also [Shintani and Linton \(2004\)](#), and [Ling \(2004\)](#) and [Francq and Zakoïan \(2012\)](#). However, all the contributions cited above have, in common, the assumption that some moments of the distribution of the data (typically, the variance) need to exist.

This paper extends the literature in at least three directions. Firstly, as mentioned above, contributions on testing for the stationarity of the RCAR model are rare; furthermore, the existing exceptions (e.g. [Zhao and Wang, 2012](#)) typically require moment restrictions such as having finite variance. Furthermore, our test, which is constructed for the null of (strict) stationarity in the RCAR framework, can also be applied to the case of an AR(1) model with no modifications and no need to test for genuine randomness in (1.1). Indeed, even though in the main part of the paper we consider the construction of a test for stationarity without having any deterministic in (1.1), as is typical in the case of infinite variance, we also consider extensions to include these: our test can be applied without any pre-treatment of the data in the presence of bounded deterministic (including constant and piecewise constant functions) and, subject to detrending, to the case of trends. We also propose, as a by-product, a test for the null of non-stationarity with the same level of generality as discussed above. Finally, our test can be applied in the presence of infinite variance and even infinite mean (indeed, all that is required is the existence of some moments, i.e. $E|e_0|^\epsilon < \infty$ and $E|b_0|^\epsilon < \infty$, for arbitrarily small ϵ); again, the test does not require any prior knowledge as to the existence of moments, and it can be applied directly with no modifications (even in this case, irrespective of having an RCAR or an AR(1) model). Having a test which can be readily applied in the presence of heavy tails with no need to estimate nuisance parameters is arguably useful in empirical applications: data with infinite variance (or even infinite mean) could occur in applications to finance and also to other disciplines - we refer to the textbook by [Embrechts et al. \(2013\)](#) for details and discussed examples. An important feature of several inferential procedures is that they rely on the estimation of the so-called “tail index”, which characterizes the largest existing moment of a random variable: this parameter is however notoriously difficult to estimate (see [Embrechts et al., 2013](#)). Note also that, since we allow for $E|X_t|^2 = \infty$, our focus is on strict, as opposed to weak, stationarity.

From a technical point of view, we construct a scale-invariant statistic which diverges to positive infinity under the null of stationarity, and drifts to zero at a polynomial rate under the alternative hypothesis, in all possible circumstances. In the context of a non-linear model like the RCAR (also with the possibility of having infinite variance or even infinite mean) it is not easy to construct a statistic which converges to a unique and “easy” limiting distribution: hence, we do not derive any

distributional limit for our test statistic, only rates of convergence/divergence. Given that our proposed statistic does not have a usable limiting distribution, we propose to randomise it, in a similar spirit to [Corradi and Swanson \(2006\)](#) and [Bandi and Corradi \(2014\)](#). The test can then be employed as is; however, given that it is a randomised test whose outcome depends on the auxiliary randomisation, we complement our methodology by also proposing a strong decision rule which is independent of the randomisation, thereby giving the same outcome to different users.

The paper is organised as follows. The main assumptions, the test statistic, and the asymptotics, are all in [Section 2](#). In [Section 3](#) we report extensions to: including deterministics, testing for the null of non-stationarity, and developing a family of related statistics. Monte Carlo evidence is reported in [Section 4](#), where we also carry out an empirical application to illustrate the use of our procedure and in particular of the proposed strong decision rule. [Section 5](#) concludes. Technical results and all proofs are in [Appendix](#).

NOTATION We use “ \rightarrow ” to denote the ordinary limit; “a.s.” stands for almost sure (convergence); C_0, C_1, \dots denote positive and finite constants that do not depend on the sample size (unless otherwise stated), and whose value may change from line to line; $I_A(x)$ is the indicator function of a set A ; strictly positive, arbitrarily small constants are denoted as ϵ - again, the value of ϵ may change from line to line. Finally, since all results in the paper hold almost surely, orders of magnitude for an a.s. convergent sequence (say s_T) are denoted as $O(T^\varsigma)$ and $o(T^\varsigma)$ when, for some $\epsilon > 0$ and $\tilde{T} < \infty$, $P \left[|T^{-\varsigma} s_T| < \epsilon \text{ for all } T \geq \tilde{T} \right] = 1$ and $T^{-\varsigma} s_T \rightarrow 0$ a.s., respectively.

2. Testing for strict stationarity

This section contains all the relevant theory. In [Section 2.1](#) we spell out the necessary and sufficient conditions required for strict stationarity, and discuss under which circumstances the second moment of X_t is finite. Assumptions are in [Section 2.2](#), and the test statistic is reported in [Section 2.3](#).

2.1. Classification and hypothesis testing framework

Recall [\(1.1\)](#)

$$X_t = (\varphi + b_t) X_{t-1} + e_t.$$

It is well known that, under minimal assumptions such as the existence of logarithmic moments for e_0 and $\varphi + b_0$, three separate regimes can hold for the solutions of [\(1.1\)](#), depending on the value taken

by $E \ln |\varphi + b_0|$:

(i) If $-\infty \leq E \ln |\varphi + b_0| < 0$, then X_t converges exponentially fast (for all initial values X_0) to

$$\bar{X}_t = \sum_{s=-\infty}^t e_s \prod_{z=s+1}^t (\varphi + b_z). \quad (2.1)$$

Note that, when $E b_0^2 > 0$, $E \ln |\varphi + b_0|$ can be negative even when $\varphi = 1$: thus, the STUR process can converge to a strictly stationary solution, although in such a case \bar{X}_t has an infinite second moment (see [Hwang and Basawa, 2005](#)). More generally, the variance of \bar{X}_t needs not be finite under strict stationarity; a necessary and sufficient condition for this is $E b_0^2 + \varphi^2 < 1$ ([Quinn, 1982](#)).

(ii) If $E \ln |\varphi + b_0| > 0$, then X_t is nonstationary and it exhibits an explosive behaviour. This case has also been studied in depth by the literature: [Berkes et al. \(2009\)](#) show that $|X_t| \rightarrow \infty$ exponentially fast.

(iii) In the boundary case $E \ln |\varphi + b_0| = 0$, X_t is nonstationary (see also the comments after [Assumption 2](#)). Even in this case $|X_t|$ diverges, but at a slower rate than exponential. This case has been paid comparatively less attention in the literature.

Clearly, this classification also holds for the basic AR(1) model, i.e. for the case $b_0 = 0$ a.s.

On the grounds of the classification above, we propose a procedure to decide between

$$\begin{cases} H_0 : & X_t \text{ is strictly stationary} \\ H_A : & X_t \text{ is nonstationary} \end{cases} \quad (2.2)$$

2.2. Assumptions

We now introduce and discuss the main assumptions. The first assumption must be satisfied by X_t irrespective of the regime it belongs to, and it can be compared to the assumptions in [Aue et al. \(2006\)](#).

Assumption 1. *It holds that: (i) $\{b_t, -\infty < t < \infty\}$ and $\{e_t, -\infty < t < \infty\}$ are independent sequences; (ii) $\{b_t, -\infty < t < \infty\}$ are independent and identically distributed random variables; (iii) $\{e_t, -\infty < t < \infty\}$ are independent and identically distributed random variables; (iv) b_0 and e_0 are symmetric random variables; (v) $E |b_0|^\nu < \infty$ and $E |e_0|^\nu < \infty$ for some $\nu > 0$; (vi) X_0 is independent of $\{e_t, b_t, t \geq 1\}$ with $E |X_0|^\nu < \infty$.*

[Assumption 1](#) contains minimal requirements as far as the existence of moments is concerned, and, in this respect, it is very general. Note that by part (iv) of the assumption, we require that, when

the mean of e_t and b_t exists, this is zero. As far as the *i.i.d.* requirement for e_t and b_t is concerned, it is typical in this literature (see [Aue et al., 2006](#)), and it is imposed only in order for the main arguments in the proofs not to be overshadowed by technical details. Indeed, relaxing the assumption of independence is possible: the conditions for stationarity mentioned above hold as long as $\{e_t, b_t\}$ is strictly stationary and ergodic (see Theorem 4.1 in [Douc et al., 2014](#)) as well as having logarithmic moments. Also, the technical arguments used in the paper (essentially, the ergodic theorem, the SLLN, and the almost sure Invariance Principle) can all be extended to the case of weakly dependent data. A major advantage of our approach, in this case, is that our test statistic is based only on rates and therefore, even in the presence of dependence, our test statistic would not require any modifications such as, for example, the estimation of long run variance matrices.

Stationary units must also satisfy the following assumption.

Assumption 2. *If $E \ln |\varphi + b_0| < 0$, it holds that (i) $P(|\bar{X}_0| = 0) < 1$; (ii) $P(|e_0| = 0) < 1$.*

Assumption 2 is also quite standard in the literature. Part (i) is, in essence, a non degeneracy requirement; as far as part (ii) is concerned, its most immediate consequence is that, under the other assumptions in this paper, the condition $E \ln |\varphi + b_0| < 0$ is necessary and sufficient for strict stationarity - see [Aue et al. \(2006\)](#).

When X_t is non-stationary, we need the following assumptions in addition to Assumption 1.

Assumption 3. *If $E \ln |\varphi + b_0| \geq 0$, it holds that: (i) e_0 has bounded density; (ii) when $P(b_0 = 0) < 1$, $E |\ln |\varphi + b_0||^k < \infty$ for some $k > 2$; (iii) $EX_0^2 < \infty$.*

Assumption 4. *When $E \ln |\varphi + b_0| = 0$ with $b_0 = 0$ a.s., it holds that either (i) $E|X_0|^2 < \infty$ and $E|e_0|^{\nu'} < \infty$ for some $\nu' > 2$; or (ii) (a) $\{e_t, -\infty < t < \infty\}$ are symmetric random variables with common distribution $F(x)$ such that*

$$1 - F(x) = C_0 x^{-\gamma} + \varsigma(x) x^{-\gamma}, \quad x \geq x_0,$$

with $C_0 > 0$, $\gamma \in (0, 2]$ and $\varsigma(x) \rightarrow 0$ as $x \rightarrow \infty$, with $\varsigma(x) x^{-\gamma}$ decreasing for all $x \geq x_0$; and (b) $E|X_0|^\gamma < \infty$.

Assumption 3 is relatively common in this literature (see e.g. [Berkes et al., 2009](#)). The main part of the assumption is part (ii), which poses a moment restriction on $\ln |\varphi + b_0|$: asymptotics is based on this quantity rather than on $|\varphi + b_0|$; note that this is not, therefore, a requirement on the existence of

e.g. the second moment of b_0 . Assumption 4 deals with the standard unit root case, where $\varphi = 1$ and $b_0 = 0$ a.s.; all other cases are covered by Assumption 3. Part (ii) of the assumption, in particular, allows for infinite variance, and indeed only requires minimal moment existence conditions - all that is needed is that the tail index, γ , be strictly positive, so that the variance, and even the first absolute moment, need not be finite. Note that we do not need to estimate γ at any stage of the proposed testing procedure. The requirements on the tail behaviour of the distribution function are rather standard in the literature (Berkes et al., 1986; Berkes and Dehling, 1989). Some of the technical results derived under this assumption are of general interest, such as the anti-concentration bound in Lemma 8.

2.3. Detecting strict stationarity

We start by discussing the rationale underpinning the construction of the test statistic. In the light of the comments above, $|X_t| \rightarrow \infty$ or not according as X_t is non-stationary or stationary: this holds under quite general circumstances, e.g. whether $b_t = 0$ or not, or whether $E|X_t|^2 < \infty$ or not. Thus, it could be possible to exploit this fact to decide between H_0 and H_A . Heuristically, a possible indicator could be based on

$$E(u_t^2 | I_{t-1}) = Ee_0^2 + X_{t-1}^2 Eb_0^2, \quad (2.3)$$

where $u_t = X_t - \varphi X_{t-1}$ and I_{t-1} is the information set available up to $t-1$ - (2.3) represents the (conditional) variance of the “error term” u_t . Testing for stationarity or the lack thereof based on the growth rate of variances has already been considered (see e.g. Cai and Shintani, 2006, Bandi and Corradi, 2014, Corradi, 1999). Albeit natural, this approach suffers from several drawbacks: Eb_0^2 and/or Ee_0^2 may not exist (thus limiting the applicability of the test); or Eb_0^2 could be zero, which would prevent the test from being applied to genuine $AR(1)$ specifications; finally, approaches based on (2.3) may require estimates of Eb_0^2 and Ee_0^2 , with the latter not being always consistent (see Horváth and Trapani, 2017). In order to overcome all these difficulties, one could instead think of using the transformation

$$Y_t = \frac{a}{a + X_t^2},$$

where $0 < a < \infty$ is chosen so as to ensure scale invariance. Heuristically, since $a > 0$, Y_t should not be equal to 0 when X_t is stationary; conversely, since $a < \infty$, when X_t is nonstationary, Y_t should drift to 0. The variable Y_t is not affected by X_t having infinite variance or infinite mean, since all moments of Y_t exist by construction. Also, in the definition of Y_t there is no dependence on Eb_0^2 : thus, Y_t uses the full force of X_t when this diverges, even in the presence of a genuine $AR(1)$ specification

for which $b_t = 0$. Finally, upon choosing a in a suitable way, no estimation of Eb_0^2 or Ee_0^2 is required, making the problem much more tractable. We build on these considerations in order to propose a test for the strict stationarity of X_t , based on the following scale-invariant transformation

$$D_T = \frac{1}{T-p} \sum_{t=p+1}^T \frac{v_p}{v_p + X_t^2}, \quad (2.4)$$

where $v_p = p^{-1} \sum_{t=1}^p X_t^2$ can be viewed as a sample second moment (or, if using demeaned data, a sample variance). We point out that, as shown in the remainder of the paper, v_p can still be used even when the second moment of X_t is not finite. Based on the heuristic considerations above, D_T should converge to a strictly positive number under stationarity, and to zero otherwise. As a final, ancillary comment, we note that a related quantity to D_T is used in the context of the estimation of φ - see [Janečková and Prášková \(2004\)](#).

The computation of v_p should satisfy the following assumption:

Assumption 5. *It holds that $p = p(T)$ with: (i) $\lim_{T \rightarrow \infty} p(T) = \infty$; (ii) $\limsup_{T \rightarrow \infty} \frac{p(T)}{\ln \ln T} = C_0 < \infty$.*

The rates of convergence are summarised in the following theorem.

Theorem 1. *Under Assumptions 1-5, it holds that*

$$D_T = C_0 + o(1) \text{ when } E \ln |\varphi + b_0| < 0, \quad (2.5)$$

$$D_T = O(T^{-\epsilon}) \text{ when } E \ln |\varphi + b_0| \geq 0, \quad (2.6)$$

where $0 < C_0 < \infty$ and $\epsilon > 0$.

The constant $C_0 \leq 1$ in (2.5) is explicitly calculated in the appendix, where it is shown that its value differs according as $EX_0^2 < \infty$ or $= \infty$ - in the latter case, $C_0 = 1$. However, for the purpose of the implementation of the test, it suffices to have $C_0 > 0$. The result in (2.6) means that, under nonstationarity, D_T drifts to zero. We do not know anything about the value of ϵ in general; however, on account of (2.6), D_T converges a.s. to zero at a polynomial rate.

Based on Theorem 1, we can propose a test for H_0 . Consider a sequence $\psi(T)$ such that, as $T \rightarrow \infty$

$$\psi(T) \rightarrow \infty \quad \text{and} \quad T^{-\epsilon} \psi(T) \rightarrow 0. \quad (2.7)$$

Then, by virtue of (2.5) and (2.6), we can assume that

$$\lim_{T \rightarrow \infty} \psi(T) D_T = \infty \text{ under } H_0, \quad (2.8)$$

$$\lim_{T \rightarrow \infty} \psi(T) D_T = 0 \text{ under } H_A. \quad (2.9)$$

Thus, $\psi(T) D_T$ diverges to positive infinity under the null, whereas it drifts to zero under the alternative. Equations (2.8)-(2.9) suggest that $\psi(T) D_T$ can be a suitable diagnostic to discriminate between $E \ln |\varphi + b_0| < 0$ and $E \ln |\varphi + b_0| \geq 0$.

Let now $g(\cdot)$ be a continuous, monotonically increasing function such that $g(0) = 0$ and $\lim_{x \rightarrow \infty} g(x) = \infty$, and define

$$l_T = g(\psi(T) D_T);$$

based on (2.8) and (2.9), it holds that

$$P\{\omega : \lim_{T \rightarrow \infty} l_T = \infty\} = 1 \quad \text{under } H_0, \quad (2.10)$$

$$P\{\omega : \lim_{T \rightarrow \infty} l_T = 0\} = 1 \quad \text{under } H_A, \quad (2.11)$$

and consequently we can assume that $\lim_{T \rightarrow \infty} l_T = \infty$ under H_0 , and $= 0$ under H_A . Note that, on account of (2.7), $\psi(T)$ may not be allowed to diverge too fast; $\psi(T) = (\ln T)^\beta$, for some $\beta > 0$, is a possible choice. However, depending on the choice of the function $g(\cdot)$, the sequence l_T can be made to diverge arbitrarily fast.

Our test is based on a randomised version of l_T . We propose a ‘‘classical’’ randomisation scheme, which has been employed in the literature - we refer to [Corradi and Swanson \(2006\)](#) and [Bandi and Corradi \(2014\)](#) *inter alia*. Of course, other schemes are also possible.

Step 1 Generate an *i.i.d.* sequence $\{\xi_j\}$, $1 \leq j \leq R$, with common distribution $G(\cdot)$.

Step 2 Generate the Bernoulli sequence $\zeta_j = I(l_T^{1/2} \xi_j \leq u)$, with u extracted from a distribution $F(u)$.

Step 3 Compute

$$\vartheta_{R,T}(u) = \frac{[G(0)(1 - G(0))]^{-1/2}}{R^{1/2}} \sum_{j=1}^R (\zeta_j - G(0)). \quad (2.12)$$

Step 4 Define

$$\Theta_{R,T} = \int_{-\infty}^{+\infty} |\vartheta_{R,T}(u)|^2 dF(u). \quad (2.13)$$

We need the following regularity conditions on $G(\cdot)$ and $F(\cdot)$:

Assumption 6. *It holds that: (i) $G(\cdot)$ has bounded density and $G(0) \neq 0$ or 1; (ii) $\int_{-\infty}^{+\infty} u^2 dF(u) < \infty$.*

We are now ready to present the main results. Let P^* denote the conditional probability with respect of $\{e_t, b_t, -\infty < t < \infty\}$; we use the notation “ $\xrightarrow{D^*}$ ” and “ $\xrightarrow{P^*}$ ” to define, respectively, conditional convergence in distribution and in probability according to P^* .

Theorem 2. *Under Assumptions 1-6, as $\min(R, T) \rightarrow \infty$ with (2.7) and*

$$\frac{R^{1/2}}{g(\psi(T))} \rightarrow 0, \quad (2.14)$$

it holds that

$$\Theta_{R,T} \xrightarrow{D^*} \chi_1^2 \quad \text{under } H_0, \quad (2.15)$$

for almost all realisations of $\{b_t, e_t, -\infty < t < \infty\}$.

Under Assumptions 1 and 3-5, as $\min(R, T) \rightarrow \infty$ with (2.7), it holds that

$$\frac{G(0)}{1-G(0)} \frac{1}{R} \Theta_{R,T} \xrightarrow{P^*} \frac{\int_{-\infty}^{+\infty} [I_{[0,\infty)}(u) - G(0)]^2 dF(u)}{G(0)[1-G(0)]} \quad \text{under } H_A, \quad (2.16)$$

for almost all realisations of $\{b_t, e_t, -\infty < t < \infty\}$.

Theorem 2 provides the limiting behaviour of the test statistic $\Theta_{R,T}$ under the null and under the alternative. The results are derived conditional on the sample, and they hold for all possible realisations, apart from a set of measure zero. Considering (2.16), note that the drift term is maximised for $G(0) = F(0) = \frac{1}{2}$, which entails that, under H_A , $R^{-1}\Theta_{R,T} \xrightarrow{P^*} 1$.

The theorem illustrates how the choice of R impacts on the behaviour of $\Theta_{R,T}$. On the one hand, one should choose R as large as possible, in order to maximise the rate of divergence of $\Theta_{R,T}$ under the alternative - this is evident from (2.16). On the other hand, the test statistic has a non-centrality parameter which grows with R , under the null: this is illustrated by (2.14). Consequently, R should not be too large, in order to avoid size distortion.

Theorem 2 implies that

$$\lim_{\min(T,R) \rightarrow \infty} P^*\{\Theta_{R,T} \geq c_\alpha\} = \alpha \quad \text{under } H_0, \quad (2.17)$$

$$\lim_{\min(T,R) \rightarrow \infty} P^*\{\Theta_{R,T} \geq c_\alpha\} = 1 \quad \text{under } H_A, \quad (2.18)$$

for almost all realisations of $\{b_t, e_t, -\infty < t < \infty\}$, where c_α is defined as $P\{N(0, 1) \geq c_\alpha\} = \alpha$, $\alpha \in (0, 1)$.

2.3.1. Deciding between H_0 and H_A

Testing using $\Theta_{R,T}$ is, in essence, based on checking a rate of convergence, and it is therefore very similar to the idea in [Bandi and Corradi \(2014\)](#) - we also refer to a contribution by [Kanaya \(2011\)](#) for a discussion. In principle (although, possibly, with some interpretational difficulties related to having a null hypothesis spelt out in terms of a rate of divergence), the test can be employed as it is, and its main properties (size and power) are reported in (2.17) and (2.18). Based on the latter equation, the test rejects the null with probability one when false, thus being consistent. Conversely, the meaning of (2.17) is non-standard. The randomness in $\Theta_{R,T}$ is added by the researcher, and indeed it is the only randomness present in the statistic: such randomness does not vanish asymptotically. Thus, different researchers using the same dataset will obtain different p -values; indeed, if an infinite number of researchers apply the test to the same data (and the null holds), the resulting p -values will be uniformly distributed on $[0, 1]$.

This is a well-known feature of randomised tests, and it may be viewed as an undesirable issue, which may explain their relative infrequent use - see the discussion and the solutions proposed in the contribution by [Geyer and Meeden \(2005\)](#). We build on the notion of “randomised confidence function”, proposed by [Song \(2016\)](#), in order to propose a strong rule to decide between H_0 and H_A , whose outcome is the same for all researchers using the same dataset. In order to remove the randomness from the test statistic, each researcher, instead of computing $\Theta_{R,T}$ just once, will compute the test statistic S times using, at each iteration s , an independent sequence $\{\xi_j^{(s)}\}$ for $1 \leq j \leq R$ and $1 \leq s \leq S$, then defining

$$Q(\alpha) = S^{-1} \sum_{s=1}^S I \left[\Theta_{R,T}^{(s)} \leq c_\alpha \right]; \quad (2.19)$$

in our context, $Q(\alpha)$ is the randomised confidence function proposed by [Song \(2016\)](#), computed under the null. Intuitively, based on [Theorem 2](#), $Q(\alpha)$ should converge to $1 - \alpha$ under H_0 , and to 0 under H_A . This dichotomous behaviour is not subject to the randomness added by the researcher (which is washed away as $S \rightarrow \infty$), and it could be employed to construct a decision rule based on the Law of the Iterated Logarithm; in particular, we propose to decide in favour of H_0 if

$$Q(\alpha) \geq D_{\alpha,S}, \quad (2.20)$$

with

$$D_{\alpha,S} = (1 - \alpha) - \sqrt{\alpha(1 - \alpha)} \sqrt{\frac{2 \ln \ln S}{S}}. \quad (2.21)$$

It holds that

Corollary 1. *Under the same assumptions as Theorem 2, as $\min(T, R, S) \rightarrow \infty$ with (2.14), it holds that*

$$P^* \left(\lim_{\min(T, R, S) \rightarrow \infty} Q(\alpha) \geq D_{\alpha, S} \right) = 1 \quad \text{under } H_0, \quad (2.22)$$

and

$$P^* \left(\lim_{\min(T, R, S) \rightarrow \infty} Q(\alpha) = 0 \right) = 1 \quad \text{under } H_A, \quad (2.23)$$

for almost all realisations of $\{b_t, e_t, -\infty < t < \infty\}$ and for every $\alpha > 0$.

The rule in (2.20) is similar to the rule proposed by Corradi (1999), who proposes a bound to discern between $I(0)$ and $I(1)$ - related contributions, where criteria are proposed in the context of choosing between $I(0)$ and $I(1)$, have also been developed by Stock (1994), Phillips and Ploberger (1994) and Phillips and Ploberger (1996). A typical advantage of this approach is that, at least asymptotically, it yields a probability zero of having both a Type I error and a Type II error, as (2.22) and (2.23) show. In addition to this, as far as our context is concerned, we note that, based on Corollary 1, under H_0 each researcher will make the same decision (“accept”), with no discrepancies among researchers and probability one of being correct, and similarly under H_A .

3. Discussion and extensions

In this section, we discuss possible extensions and generalisations of the basic test statistic. We consider three possible generalisations: (a) the case of deterministic components being present in (1.1); (b) the construction of a test for the null of non-stationarity; and finally, (c) the construction of different, but related test statistics. All results are shown in appendix when necessary; the proofs of some results however follow readily from the existing proofs, and thus, when possible, we omit the details.

3.1. Employing the test in the presence of deterministic components

So far, we have assumed no deterministic components in (1.1). This is common in this literature, when heavy tails are considered, and we also refer to the comments in Cavaliere et al. (2016). In this section, we show that it is possible to consider an extension to incorporate deterministic components in (1.1). We assume that the observed data - say X_t^* - are generated as

$$X_t^* = d_t + X_t, \quad (3.1)$$

where X_t is defined as in (1.1). There are two types of deterministic processes d_t . In the case of square integrable d_t , viz.

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T |d_t|^2 < \infty, \quad (3.2)$$

we show that the test can be used with no modifications. Condition (3.2) includes several possible cases: d_t can be constant; it can be piecewise constant, thus allowing for shifts in the mean; or it could be a weighted average of sines and cosines, which could be useful to model seasonalities and, in general, smooth, bounded processes (see [Enders and Lee, 2012](#)). In all these cases, and indeed whenever (3.2) is satisfied, the test can be applied directly, with no modifications or prior knowledge of the nature of d_t .

It is also possible to apply the test in the presence of trends, and in general when (3.2) does not hold. In this case, it is necessary to detrend the data first, by estimating d_t via, say, \hat{d}_t . The test statistic D_T is modified as

$$D_T^* = \frac{1}{T-p} \sum_{t=p+1}^T \frac{v_p}{v_p + Y_t^2},$$

where

$$Y_t = \begin{cases} X_t^* & \text{according as (3.2) holds true} \\ X_t^* - \hat{d}_t & \text{(3.2) does not hold} \end{cases}.$$

We formalise our discussion in the following assumption.

Assumption 7. *It holds that (i) either (a) (3.2) holds; or (b) it holds that $E \left| \hat{d}_t - d_t \right|^2 = O(T^{-\epsilon_1})$ for some $\epsilon_1 > 0$; (ii) when $E \ln |\varphi + b_0| < 0$, $P(|\bar{X}_0| = c) < 1$ for all $c \in R$.*

Part (i)(b) of the assumption is very similar, in spirit, to Assumption 5 in [Kapetanios \(2007\)](#) and it essentially requires that \hat{d}_t be a consistent estimator of the (trend) function d_t . Although several choices are possible, we refer to the contribution by [Peng and Yao \(2004\)](#) on the estimation of trend functions in the presence of heavy tails, where MSE consistency is still ensured; note that we do not need ϵ_1 to be any special value, as long as the MSE drifts to zero at a polynomial rate. Part (ii) strengthens Assumption 2, and it is again a non-degeneracy condition.

It holds that

Theorem 3. *Under Assumptions 1-5 and 7, equations (2.5) and (2.6) hold.*

Theorem 3 entails that D_T^* can be used in the same way as D_T , obtaining the same results for the corresponding test (under the same assumptions), and it can be generalised as we do with D_T in the next sections.

3.2. Testing for the null of non-stationarity

In the spirit of the KPSS test (see Kwiatkowski et al., 1992; see also the contribution by Giraitis et al., 2006), and of other contributions in the context of nonlinear models (see e.g. Kapetanios, 2007), all the results developed so far are based on the hypothesis testing framework set out in (2.2), where H_0 : X_t is strictly stationary.

However, a test for

$$\begin{cases} H_0^* : & X_t \text{ is nonstationary} \\ H_A^* : & X_t \text{ is strictly stationary} \end{cases} \quad (3.3)$$

can be readily derived from the theory developed above.

Our testing approach requires having a test statistic which diverges under the null, whilst being bounded under the alternative. On account of (2.8) and (2.9), one could use

$$l_T^* = g\left(\frac{1}{\psi(T)D_T}\right), \quad (3.4)$$

where $\psi(T)$ is defined in (2.7); by continuity, it follows that

$$\begin{aligned} P\{\omega : \lim_{T \rightarrow \infty} l_T^* = \infty\} &= 1 \quad \text{under } H_0^*, \\ P\{\omega : \lim_{T \rightarrow \infty} l_T^* = 0\} &= 1 \quad \text{under } H_A^*, \end{aligned}$$

from which a test, based on a randomised version of l_T^* , can be constructed. Using the same algorithm as proposed above, we would obtain a test statistic denoted by $\Theta_{R,T}^*$, whose asymptotics is in the following theorem, reported without proof.

Theorem 4. *We assume that Assumptions 1-6 are satisfied. Then, under H_0^* , as $\min(T, R) \rightarrow \infty$ with*

$$\frac{R^{1/2}}{g\left(\frac{T^\epsilon}{\psi(T)}\right)} \rightarrow 0, \quad (3.5)$$

for every $\epsilon > 0$, it holds that $\Theta_{R,T}^* \xrightarrow{D^*} N(0, 1)$ for almost all realisations of $\{b_t, e_t, -\infty < t < \infty\}$.

Under H_A^* , it holds that as $\min(T, R) \rightarrow \infty$

$$\frac{G(0)}{1 - G(0)} \frac{1}{R} \Theta_{R,T}^* \xrightarrow{P^*} \frac{\int_{-\infty}^{+\infty} [I_{[0, \infty)}(u) - G(0)]^2 dF(u)}{G(0)[1 - G(0)]},$$

for almost all realisations of $\{b_t, e_t, -\infty < t < \infty\}$.

As for $\Theta_{R,T}$, (3.5) provides a selection rule for R . If, as suggested in the next section, one were to choose $g(x) = \exp(x) - 1$ - and $\psi(T) = (\ln T)^\beta$, as recommended above - then setting $R = T$ would satisfy (3.5). Finally, note that instead of X_t the same arguments could be applied to Y_t .

3.3. Modifications of the test statistic

Heuristically, our test statistic is based on studying the variance of $u_t = e_t + b_t X_{t-1}$ in

$$X_t = \varphi X_{t-1} + u_t.$$

The main intuition is that under stationarity $E(u_t^2 | I_{t-1})$ should be bounded as t elapses, whereas it grows as $t \rightarrow \infty$ when X_t is nonstationary. Owing to the reasons discussed at the beginning of Section 2.3, the testing procedure uses a modified statistic, rather than $E(u_t^2 | I_{t-1})$ directly.

In a similar vein, one use different moments of the error term u_t , say $E(|u_t|^\zeta | I_{t-1})$ for some $\zeta > 0$: it can be expected that the stationarity/nonstationarity of X_t will still entail the convergence/divergence of $E(|u_t|^\zeta | I_{t-1})$. From a technical point of view, our arguments and our proofs differ depending on whether $E|X_t|^2 = \infty$ or not, and therefore it can be envisaged that, if relying upon $E(|u_t|^\zeta | I_{t-1})$, proofs will differ according as $E|X_t|^\zeta = \infty$ or not - but apart from this, the final results will be the same.

To formalise the discussion, consider

$$D_T(\zeta) = \frac{1}{T-p} \sum_{t=p+1}^T \frac{v_p(\zeta)}{v_p(\zeta) + |X_t|^\zeta}, \quad (3.6)$$

where $v_p(\zeta) = p^{-1} \sum_{t=1}^p |X_t|^\zeta$. The following result is reported without proof.

Theorem 5. *Under Assumptions 1-5, equations (2.5) and (2.6) hold.*

Theorem 5 entails that $D_T(\zeta)$ has the same properties as D_T , and it can therefore be used, and generalised, in the same way. Indeed, upon making sure that the artificial samples used in the various randomisations are generated independently, it would even be possible to run a meta-test by trying several values of ζ and combining the outcomes e.g. according to Fisher's method. As before, the same ideas could be applied to Y_t . Although this extension is theoretically possible, unreported simulations show that $\zeta = 2$ affords the best results; choosing $\zeta < 2$ makes tests very conservative (e.g. when $\zeta = 1$ the power is cut by two thirds), whereas $\zeta > 2$ results in the opposite problem.

4. Simulations and empirical illustration

This section contains two separate contributions. In Section 4.1 we report some evidence from synthetic data on the empirical rejection frequencies of our test in order to assess size and power; we analyse the performance of decision rules based on $D_{\alpha, \zeta}$; and we discuss possible guidelines for the implementation

of the test statistic. In Section 4.2, we illustrate our approach, and in particular the use of $D_{\alpha,S}$, through an application to several US macro aggregates.

4.1. Monte Carlo evidence

The design of the reported experiments is as follows.

We use (1.1) as a DGP; b_t is generated as *i.i.d.* $N(0, \sigma_b^2)$, with $\sigma_b^2 \in \{0, 0.1, 0.25\}$ in order to consider the genuine AR case as well as cases with random coefficients. According to the theory, it would also be possible to consider a heavy tailed distribution for b_t ; we noticed through few trials that doing this does not change results in a decisive way. We report results for three different specifications of e_t : *i.i.d.* $N(0, \sigma_e^2)$, *i.i.d.* t_2 and *i.i.d.* t_1 , where t_k denotes a Student t distribution with k degrees of freedom, so as to consider the cases of infinite variance and infinite mean. In the Gaussian case, we have used $\sigma_e^2 = 1$; we note however that the test is completely insensitive to the value of σ_e^2 , which suggests that the use of v_p is very effective at ensuring scale invariance.

We have used $\varphi \in \{0, 0.5, 0.75, 0.95, 1, 1.05\}$; larger values of φ , for the nonstationary cases, could also be considered but in those cases - as can be expected - the test has unit power even for very small samples.

The various combinations $\{\varphi, \sigma_b^2\}$ deserve attention. All cases where $\varphi \leq 0.95$ entail that X_t is stationary: the corresponding empirical rejection frequencies represent the size of the test. Also, upon computing the value of $E \ln |\varphi + b_0|$, it can be noted that the two cases $\{\varphi, \sigma_b^2\} = \{1, 0.1\}$ and $\{\varphi, \sigma_b^2\} = \{1, 0.25\}$ correspond to a stationary STUR; even in these cases the empirical rejection frequencies represent the size, and it should be noted that, when X_t is a (stationary) STUR process, it has infinite variance irrespective of the distributions of b_t and e_t . Finally, again upon computing $E \ln |\varphi + b_0|$, it turns out that the case $\{\varphi, \sigma_b^2\} = \{1.05, 0.25\}$ is also an instance of X_t being stationary, again with infinite variance.

Thus, the nonstationary cases considered in our experiment are a pure explosive case corresponding to $\{\varphi, \sigma_b^2\} = \{1.05, 0\}$, and the pure unit root case $\{\varphi, \sigma_b^2\} = \{1, 0\}$. In the latter case, clearly $E \ln |\varphi + b_0| = 0$ and therefore X_t is on the cusp between explosive and strictly stationary behaviour - note that we are considering, by virtue of the several possible distributions of e_t , also cases of random walk with infinite variance and mean, in a similar spirit to Cavaliere et al. (2016). However, in our case the null is stationarity, not unit root, and therefore the empirical rejection frequencies represent the power of our test. Finally, we point out that the case $\{\varphi, \sigma_b^2\} = \{1.05, 0.1\}$ is of particular interest

because $E \ln |\varphi + b_0| = 3.3 \times 10^{-3}$ - that is, it is positive (and, therefore, X_t is nonstationary) but very small. Finally, in a separate experiment (the outcomes are in Tables 2 and 4), we have considered several combinations of $\{\varphi, \sigma_b^2\}$ for which $E \ln |\varphi + b_0| = 0$, so as to evaluate the behaviour of the test in those cases.

We now turn to describing the specifications of the test; as a general note, their impact vanishes for large samples. Our reported experiments are based on the following choices, which delivered the best results and are thus recommended as guidelines to the applied user. We choose:

$$\psi(T) = (\ln T)^{5/4}, \quad (4.1)$$

$$g(x) = \exp(\exp(x) - 1) - 1; \quad (4.2)$$

the choices in (4.1) and (4.2) are designed in order to ensure that $g(\psi(T) D_T) \asymp \exp(T)$ under H_0 , and that $g(\psi(T) D_T)$ drifts to zero as $T \rightarrow \infty$ under H_A . Thus, the double exponential in (4.2) serves the purpose of “divaricating” as much as possible the case where $\psi(T) D_T$ diverges from the case where it does not; other choices would also be possible, but (4.1) and (4.2) work well in all cases considered. Based on (2.14), we set $R = T$. Finally, by Assumption 5, in the computation of v_p (which we carry out with demeaned data), we need to choose $p = C_0 \ln \ln T$ for some C_0 , which implies that p does not vary too much as T increases. We have used $C_0 = 2$, rounding $C_0 \ln \ln T$ to the nearest, largest integer; varying p around this number does not affect the results anyway. Finally, we have implemented the decision rule based on $D_{\alpha, S}$ using $S = 1000$; we note that increasing this number results in better outcomes, at the (obvious) cost of a higher computational burden. Under each scenario, we compute the percentage of times that the decision rule is in favour of H_0 , using this as a measure of performance.

We generate $\{\xi_j\}_{j=1}^R$ as *i.i.d.* $N(0, 1)$, and u is $\{-\sqrt{2}, \sqrt{2}\}$ with equal probability. Finally, the sample sizes have been chosen as $T \in \{250, 500, 1000, 2000\}$; the first 1000 observations have been discarded to avoid dependence on initial conditions. The number of replications is set equal to 2000; when evaluating the size, this entails that empirical rejection frequencies have a confidence interval of $[0.04, 0.06]$.

[Insert Tables 1 and 2 somewhere here]

Table 1 contains the empirical rejection frequencies for the test for $H_0 : X_t$ is strictly stationary, using the combinations of $\{\varphi, \sigma_b^2\}$ indicated above and, in brackets, the percentage of times that the decision rule based on $D_{\alpha,S}$ leads to accepting H_0 . As can be noted, the test has the correct size for almost all cases considered - exceptions are cases on the boundary (such as $\{\varphi, \sigma_b^2\} = \{1, 0.1\}$, or $\{\varphi, \sigma_b^2\} = \{1.05, 0.25\}$), but even in these cases the size becomes correct as T increases. The distribution of the error term e_t does not affect, in general, the values of the empirical rejection frequencies, with few exceptions. As far as power is concerned, in the pure unit root case - viz. when $\{\varphi, \sigma_b^2\} = \{1, 0\}$ - the test exhibits good power, which is found to be higher than 50% whenever $T \geq 500$, despite not being designed explicitly for that specific alternative hypothesis; even in this case, the results are broadly similar for different distributions of e_t . As a conclusion, the test seems to work well in discerning between a genuinely unit root process, and a (stationary) STUR process. The test is also powerful in the purely explosive case $\{\varphi, \sigma_b^2\} = \{1.05, 0\}$, and has some power also versus the “boundary” case $\{\varphi, \sigma_b^2\} = \{1.05, 0.1\}$; in this case, the power is affected by the distribution of the error term for small T , and it declines as the tails of the distribution of e_t become heavier, but this seems to vanish as T increases. Similar considerations hold for $D_{\alpha,S}$; note that, when data have heavy tails, in the case $\{\varphi, \sigma_b^2\} = \{0.95, 0\}$ the procedure requires, in order to work sufficiently well, $T \geq 1000$.

As mentioned above, we have also considered a broader set of cases where X_t is nonstationary, in which $E \ln |\varphi + b_0| = 0$. The power of our test versus these alternatives is in Table 2; the test has very good power in all cases considered, the only possible exception being the case with normally distributed errors and $T = 250$, but even in that case the power picks up for larger T . Note the major increase in power when the error term e_t has a Student t distribution; this does not seem to be sensitive to the degrees of freedom of the distribution. Note also the excellent performance of $D_{\alpha,S}$ for large T .

We have also considered, based on the discussion in Section 3.2, testing for $H_0 : X_t$ is nonstationary. For brevity, we did not experiment with $D_{\alpha,S}$; otherwise, the design of the simulations, and the specification of the test statistic, are carried out exactly as in the previous case. As suggested in Section 3.2, we use

$$l_T^* = g \left(\frac{1}{\psi(T) D_T} \right),$$

in the construction of the test.

[Insert Tables 3 and 4 somewhere here]

The test has the correct size in all cases considered. The power versus stationarity is strong when X_t is “very stationary” - i.e. when $\varphi = 0$ or 0.5 , and it is anyway above 50% in the “less stationary” case of having $\varphi = 0.95$ when $T \geq 500$. Similarly, Table 4 shows that the test has the correct size even in the nonstationary, but boundary, case $E \ln |\varphi + b_0| = 0$. We note that, in all cases considered, the distribution of e_t does not seem to play a role on the final results.

4.2. Empirical illustration

The purpose of this section is primarily to illustrate the use of $Q(\alpha)$ and of the decision rule based on $D_{\alpha,S}$. We apply our procedure to several US macroeconomic aggregates (similarly to [Hill and Peng, 2014](#)). We consider the logs of: real GDP, M2 (as a measure of the aggregate money supply), CPI (and we also consider inflation, defined as the log-difference of CPI), and the Industrial Production index. We also apply our methodology to the (untransformed) rate of unemployment. Finally, we also consider the 3-months T-bill, inspired by the contribution by [Nielsen and Rahbek \(2014\)](#).

The decision rule (2.20) is applied in order to choose between

$$\begin{cases} H_0 : & X_t \text{ is strictly stationary} \\ H_A : & X_t \text{ is nonstationary} \end{cases}$$

As a further illustration, we apply the test to first-differenced data, in the cases where a series is found to be nonstationary. Finally, we also use (2.20) to decide between

$$\begin{cases} H_0 : & X_t \text{ is nonstationary} \\ H_A : & X_t \text{ is strictly stationary} \end{cases},$$

again considering data in levels and (if need be) in first difference. As far as the implementation is concerned, a note on deterministic is in order. We know from Section 3.1 that our test can, in general, be applied to non-zero mean data; the test can also be applied in the presence of trends (albeit only after detrending), which should make our procedure particularly suitable for macroeconomic aggregates. There has been much debate on the presence (or absence) of a linear trend in macroeconomic aggregates, and, in general, as to whether macroeconomic series are better characterised as having a linear trend or a unit root (the so-called “uncertain unit root”). Starting at least from the seminal paper by [Nelson and Plosser \(1982\)](#), various contributions have questioned whether such series ought to be modelled as having a unit root or a linear trend. GDP is a prime example of this debate, and it has been the subject of several studies: we refer to the classical paper by [Rudebusch \(1993\)](#), and also

to [Murray and Nelson \(2000\)](#) and the extensive literature review therein. Similarly, some studies seem to suggest that trends may be present in the CPI (see [Beechey and Österholm, 2008](#)), and that money aggregates also may have trends ([Brand, Gerdesmeier, and Roffia, Brand et al.](#)). Whilst the empirical exercise in this paper is not aimed at addressing the “uncertain unit root” debate in a comprehensive way, we have taken this literature into account by detrending all the series using the GLS detrending scheme proposed in [Elliot et al. \(1996\)](#).

[Insert Table 5 somewhere here]

In the computation of (2.20), we have used $Q(0.05)$, setting $S = 5000$. We have used the same specifications as described in the previous section, namely: $R = T$; $p = 5$; $\psi(T) = (\ln T)^{5/4}$; $g(x) = \exp(\exp(x) - 1) - 1$; and we compute v_p using demeaned data. In Table 6, we also report the estimated values of φ and σ_b^2 computed using the WLS estimator studied in [Horváth and Trapani \(2017\)](#). Based on (2.20), and on the fact that $\alpha = 0.05$ and $S = 5000$, the decision rule is based on not rejecting H_0 whenever

$$Q(0.05) \geq 0.9436, \tag{4.3}$$

rejecting otherwise. Results are reported in Table 6, where we have also reported, for illustration purposes, the outcomes of the unit root test by [Elliot et al. \(1996\)](#) and of the KPSS test (see [Kwiatkowski et al., 1992](#)), which we have carried out for those series which do not have a random autoregressive root.

[Insert Table 6 somewhere here]

As a preliminary comment, based on the test for no randomness ($H_0 : \sigma_b^2 = 0$) developed by [Horváth and Trapani \(2017\)](#), two series (Industrial Production and unemployment) are found to have a random autoregressive root, whereas the others do not. We have (very) heuristically checked these two series, by calculating the value of $E \ln |\varphi + b_0|$, using $\hat{\varphi}$ as face value and assuming that b_0 is Gaussian; in both cases, $E \ln |\varphi + b_0|$ turns out to be very close to zero. As far as testing for stationarity is concerned, results are quite clear-cut: all series are found to be nonstationary. This is perfectly in line with the findings from applying the other unit root tests reported in the table - although of course these can only be applied to the series with a deterministic autoregressive root. Interestingly, all series

become stationary after first differencing: this cannot be taken for granted, since some series have a random autoregressive coefficient and therefore there is no guarantee that first differencing may induce stationarity - see [Leybourne et al. \(1996\)](#). Exactly the same pattern of results is found when swapping the null and the alternative hypotheses.

Finally, we note that, as a robustness check, we have experimented with different specifications for our procedure - e.g. varying R from $T/2$ to $2T$; increasing p to $4 \ln \ln T$; and trying $S = 1000$ and $S = 10000$. In all these cases, we noted that results were entirely unchanged compared to the ones in Table 6, suggesting that the strong rule based on (2.20) is quite robust to different choices of user-defined parameters.

5. Conclusions

In this paper, we have developed a test for the null of strict stationarity applied to a RCAR(1) model. Our testing approach can be applied to a wide variety of situations, without requiring any modification or any prior knowledge. Chiefly, the test is still usable if the autoregressive root is not random, i.e. in the case of an AR(1) specification. Also, the test does not require (neither as assumptions, nor for the purpose of the actual implementation) the existence of the variance of X_t , or of virtually any moment. Finally, the test can be applied in the presence of deterministic: even in this case (with the exception of having trends), the implementation of the test does not require any prior analysis. To the best of our knowledge, no existing test has such a level of generality.

Technically, the test is based on the (almost sure) limiting behaviour of a statistic which either diverges to positive infinity or drifts to zero without having any randomness, we propose to use it as part of a randomisation procedure. Numerical evidence shows that the test performs very well, also showing very promising results in the cases where X_t is borderline between being stationary and non-stationary - that is, in cases where $E \ln |\varphi + b_0|$ is either positive or negative, but “small”.

6. Technical Lemmas

The first few lemmas are for the case $E \ln |\varphi + b_0| < 0$. The first lemma is an immediate consequence of Assumption 2, and we therefore report it without proof.

Lemma 1. *Under Assumption 2, if $E \ln |\varphi + b_0| < 0$ with $|\varphi| < 1$, it holds that $E |\bar{X}_0|^\delta > 0$, for all $\delta \geq 0$.*

Lemma 2. *Under Assumption 1, if $E \ln |\varphi + b_0| < 0$ it holds that*

$$\sum_{t=1}^T \left| |X_t|^\kappa - |\bar{X}_t|^\kappa \right| = O(1),$$

for any $\kappa > 0$.

Proof. Using Lipschitz and Hölder continuity, we have

$$\left| |X_t|^\kappa - |\bar{X}_t|^\kappa \right| \leq \begin{cases} C_0 \left(|\bar{X}_t|^{\kappa-1} + |X_t|^{\kappa-1} \right) |X_t - \bar{X}_t| & \text{according as } \kappa \geq 1 \\ C_0 |X_t - \bar{X}_t|^\kappa & \kappa < 1 \end{cases}.$$

Horváth and Trapani (2017) show that $|X_t - \bar{X}_t| = O(e^{-C_0 t})$ for some $C_0 > 0$. Also, by Lemma 2 in Aue et al. (2006), there exists a $\delta' > 0$ such that $E |\bar{X}_t|^{\delta'} < \infty$. Hence, by the Borel-Cantelli Lemma (see e.g. Chow and Teicher, 2012, Corollary 3 on p. 90), $|\bar{X}_t| = O\left(|t|^{1/\delta'} (\ln t)^{(1+\epsilon)/\delta'}\right)$. The desired result now follows immediately by putting everything together. \square

We now distinguish the cases $E\bar{X}_0^2 = \infty$ and $E\bar{X}_0^2 < \infty$. In the former case, the following lemmas are needed.

Lemma 3. *Under Assumptions 1 and 2, with $E \ln |\varphi + b_0| < 0$ and $\varphi^2 + Eb_0^2 \geq 1$, it holds that, as $T \rightarrow \infty$, $v_p \rightarrow \infty$ a.s.*

Proof. Let $\bar{v}_p = p^{-1} \sum_{t=1}^p \bar{X}_t^2$. By Lemma 2, it holds that $v_p - \bar{v}_p = O(p^{-1})$. Further, under the assumptions of the lemma, it holds that $E\bar{X}_0^2 = \infty$ (Quinn, 1982). Let now $\bar{v}_p(C_0) = p^{-1} \sum_{t=1}^p \min\{\bar{X}_t^2, C_0^2\}$, where $C_0 > 0$. By construction, $\bar{v}_p = \limsup_{C_0 \rightarrow \infty} \bar{v}_p(C_0) = \limsup_{C_0 \rightarrow \infty} E \min\{\bar{X}_0^2, C_0^2\}$ a.s., where the last passage follows from the ergodic theorem. Also, $\limsup_{C_0 \rightarrow \infty} E \min\{\bar{X}_0^2, C_0^2\} = \lim_{C_0 \rightarrow \infty} E \min\{\bar{X}_0^2, C_0^2\} = E\bar{X}_0^2$, by monotone convergence, which proves the lemma. \square

Lemma 4. *Under Assumptions 1 and 2, with $E \ln |\varphi + b_0| < 0$ and $\varphi^2 + Eb_0^2 \geq 1$, it holds that, as $T \rightarrow \infty$, $D_T \rightarrow 1$ a.s.*

Proof. We begin by showing that $v_p > 0$ a.s.; indeed, by the generalised mean inequality $v_p \geq (p^{-1} \sum_{t=1}^p X_t^\delta)^{2/\delta}$; using Lemma 2 and the ergodic theorem, we get that $v_p \geq (E\bar{X}_0^\delta)^{2/\delta}$, which is strictly positive by Lemma 1 for any $\delta \in (0, 2)$.

Note now that

$$\sum_{t=p+1}^T \left| \frac{v_p}{v_p + X_t^2} - \frac{\bar{v}_p}{\bar{v}_p + \bar{X}_t^2} \right| \leq \sum_{t=p+1}^T \frac{|X_t^2 - \bar{X}_t^2|}{v_p + X_t^2} \leq C_0 \sum_{t=p+1}^T |X_t^2 - \bar{X}_t^2| = O(1); \quad (6.1)$$

the last passage follows from Lemma 2. Combining (6.1) with Lemma 3, this entails that for every C_0 , there is a p_0 such that for all $p \geq p_0$

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{\bar{X}_t^2}{v_p + \bar{X}_t^2} \leq \frac{1}{T-p} \sum_{t=p+1}^T \frac{\bar{X}_t^2}{C_0 + \bar{X}_t^2} \text{ a.s.}$$

By virtue of the ergodic theorem, this entails that

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{\bar{X}_t^2}{v_p + \bar{X}_t^2} \leq E \left(\frac{\bar{X}_0^2}{C_0 + \bar{X}_0^2} \right) \text{ a.s.}$$

Lemma 2 in Aue et al. (2006) entails that there exists a $\delta' > 0$ (with, clearly, $\delta' < 2$) such that $E |\bar{X}_0|^{\delta'} < \infty$, so that

$$E \left(\frac{\bar{X}_0^2}{C_0 + \bar{X}_0^2} \right) \leq E \left(\frac{\bar{X}_0^2}{C_0 + \bar{X}_0^2} \right)^{\delta'/2} \leq E \left(\frac{\bar{X}_0^2}{C_0} \right)^{\delta'/2} \leq C_1 C_0^{-\delta'/2},$$

where $C_1 < \infty$. Thus, for any $\epsilon > 0$ there exists a random p_0 such that for all $p \geq p_0$

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{X_t^2}{v_p + X_t^2} \leq \epsilon \text{ a.s.};$$

the desired result follows immediately. \square

Lemmas 3 and 4 allow to study the behaviour of D_T under the null when $E\bar{X}_0^2 = \infty$. When $E\bar{X}_0^2 < \infty$, the following result holds.

Lemma 5. *Under Assumptions 1 and 2, if $E \ln |\varphi + b_0| < 0$ with $Ee_0^2 < \infty$ and $\varphi^2 + Eb_0^2 < 1$, it holds that*

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{X_t^2}{v_p + X_t^2} = E \left(\frac{\bar{X}_0^2}{E\bar{X}_0^2 + \bar{X}_0^2} \right) + o(1).$$

Proof. Under the conditions of the lemma, it holds that $E\bar{X}_0^2 < \infty$ (see Lemma 3 in Aue et al., 2006).

We have

$$X_t^2 \frac{|E\bar{X}_0^2 - v_p|}{(E\bar{X}_0^2 + X_t^2)(v_p + X_t^2)} \leq \frac{|E\bar{X}_0^2 - v_p|}{E\bar{X}_0^2}.$$

By virtue of Lemma 1, we further have that $E\bar{X}_0^2 > 0$. By Lemma 2, we have $v_p - p^{-1} \sum_{t=1}^p \bar{X}_t^2 = O(p^{-1})$; also, the ergodic theorem entails that $p^{-1} \sum_{t=1}^p \bar{X}_t^2 - E\bar{X}_0^2 = o(1)$, so that putting everything together we have $E\bar{X}_0^2 - v_p = o(1)$. Hence

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{X_t^2}{v_p + X_t^2} - \frac{1}{T-p} \sum_{t=p+1}^T \frac{X_t^2}{E\bar{X}_0^2 + X_t^2} = o(1).$$

Note now that

$$\sum_{t=p+1}^T \left| \frac{X_t^2}{E\bar{X}_0^2 + X_t^2} - \frac{\bar{X}_t^2}{E\bar{X}_0^2 + \bar{X}_t^2} \right| \leq C_0 \sum_{t=p+1}^T |X_t - \bar{X}_t|, \quad (6.2)$$

for some $0 < C_0 < \infty$; thus, by Lemma 2

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{X_t^2}{E\bar{X}_0^2 + X_t^2} = \frac{1}{T-p} \sum_{t=p+1}^T \frac{\bar{X}_t^2}{E\bar{X}_0^2 + \bar{X}_t^2} + O(T^{-1}).$$

The final result obtains from the ergodic theorem. \square

We now report a series of lemmas to study the nonstationary case, $E \ln |\varphi + b_0| \geq 0$.

Lemma 6. *Under Assumptions 1, 3 and 4, it holds that*

$$\begin{aligned} O(t^{\max\{0, \nu-1\}} \exp(C_0 t)) & \quad E \ln |\varphi + b_0| \geq 0 \text{ with } P(b_0 = 0) < 1 \\ E |X_t|^\nu = O(t^{\max\{0, \nu-1\}} \exp(C_0 t)) & \quad \text{according as } E \ln |\varphi + b_0| > 0 \text{ with } P(b_0 = 0) = 1, \\ O(t^{1+\max\{0, \nu-1\}}) & \quad E \ln |\varphi + b_0| = 0 \text{ with } P(b_0 = 0) = 1 \end{aligned}$$

for some $0 < C_0 < \infty$; ν is defined in Assumption 1, and, when Assumption 4(ii) holds, it is chosen so that $\nu < \gamma$.

Proof. Consider the recursive solution

$$X_t = X_0 \prod_{s=1}^t (\varphi + b_s) + \sum_{s=1}^t e_s \prod_{z=s}^{t-1} (\varphi + b_{z+1}), \quad (6.3)$$

and note that

$$\begin{aligned} X_t^\nu & \leq C_0 \left[X_0^\nu \prod_{s=1}^t (\varphi + b_s)^\nu + \left| \sum_{s=1}^t e_s \prod_{z=s}^{t-1} (\varphi + b_{z+1}) \right|^\nu \right] \\ & \leq C_0 \left[X_0^\nu \prod_{s=1}^t (\varphi + b_s)^\nu + t^{\max\{0, \nu-1\}} \sum_{s=1}^t |e_s|^\nu \prod_{z=s}^{t-1} |\varphi + b_{z+1}|^\nu \right], \end{aligned}$$

where ν is defined in Assumption 1 and $C_0 = 1$ when $\nu \leq 1$ and $2^{\nu-1}$ when $\nu > 1$.

We begin by studying the case $P(b_0 = 0) < 1$. By Assumption 1, it holds that

$$\begin{aligned} & E |X_0|^\nu \prod_{s=1}^t E |\varphi + b_s|^\nu + t^{\max\{0, \nu-1\}} \sum_{s=1}^t E |e_s|^\nu \prod_{z=s}^{t-1} E |\varphi + b_{z+1}|^\nu \\ & \leq C_0 (E |\varphi + b_0|^\nu)^t \left(1 + t^{\max\{0, \nu-1\}} \sum_{s=1}^t (E |\varphi + b_0|^\nu)^{-s} \right). \end{aligned} \quad (6.4)$$

By Jensen's inequality, $\ln E |\varphi + b_0|^\nu \geq E \ln |\varphi + b_0|^\nu \geq 0$, so that $1 \leq E |\varphi + b_0|^\nu < \infty$. Thus, $(E |\varphi + b_0|^\nu)^t$ is at most of order $e^{C_0 t}$ for some $C_0 > 0$. When $P(b_0 = 0) = 1$ and $|\varphi| > 1$, the same result readily follows. Finally, when $|\varphi| = 1$ and $P(b_0 = 0) = 1$, (6.4) boils down to

$$E |X_0|^\nu \prod_{s=1}^t E |\varphi + b_s|^\nu + t^{\max\{0, \nu-1\}} \sum_{s=1}^t E |e_s|^\nu \prod_{z=s}^{t-1} E |\varphi + b_{z+1}|^\nu \leq C_0 \left(1 + t^{1+\max\{0, \nu-1\}} \right),$$

whence again the desired result follows immediately. \square

The next two lemmas contain some anti-concentration bounds for the case of nonstationary X_t .

Lemma 7. *Under Assumptions 1, 3 and 4(i), it holds that*

$$P(|X_t| \leq t^\alpha) \leq \begin{cases} C_0 t^\alpha \{ \exp(-t^{3/2(1+\nu)}) + t^{-(\nu-2)/2(1+\nu)} \} & E \ln |\varphi + b_0| > 0 \text{ with } P(b_0 = 0) < 1 \\ C_0 t^\alpha \exp(-t \ln |\varphi|) & E \ln |\varphi + b_0| > 0 \text{ with } P(b_0 = 0) = 1 \\ C_0 t^\alpha \{ \exp(-t^{3/2(1+\nu)}) + t^{-2} + t^{-(\nu-2)/2(1+\nu)} \} & \text{for } E \ln |\varphi + b_0| = 0 \text{ with } P(b_0 = 0) < 1 \\ C_0 t^{1-\alpha\nu} + C_1 t^{\alpha-1/2} & E \ln |\varphi + b_0| = 0 \text{ with } P(b_0 = 0) = 1 \end{cases}$$

Proof. See Lemma 7.4 in Horváth and Trapani (2016). \square

Lemma 7 covers the cases $E \ln |\varphi + b_0| > 0$, $E \ln |\varphi + b_0| = 0$ with genuine random coefficient and $E \ln |\varphi + b_0| = 0$ with no randomness and $E|e_0|^{\nu'} < \infty$, with $\nu' > 2$. The next lemma is useful to study the case of a non-random unit root process with infinite variance.

Lemma 8. *Under Assumptions 1 and 4(ii), it holds that*

$$P(|X_t| \leq t^\alpha) \leq C_0 t^{1-\alpha\gamma-\gamma\epsilon} + C_1 t^{\alpha-1/\gamma},$$

for some $\epsilon > 0$.

Proof. By Theorem 1 in Berkes et al. (1986), on a suitably larger space we can construct two independent sequences of *i.i.d.* random variables, say $\{y_i, i \geq 1\}$ and $\{z_i, i \geq 1\}$ such that: y_i and z_i are both symmetric, the common characteristic function of the y_i is $\exp(-C_0 |x|^\gamma)$ with $C_0 > 0$, the z_i 's have common symmetric distribution function $F_z(x)$ satisfying $1 - F_z(x) = \zeta(x) x^{-\gamma}$ for $x \geq x_0$ and

$$\sum_{i=1}^t (e_i - y_i - z_i) = O\left(t^{1/\gamma-\epsilon'}\right), \quad (6.5)$$

for some $\epsilon' > 0$. Note also that, using equation (2.32) in Berkes and Dehling (1989) and Markov inequality, (6.5) entails the following estimate

$$P\left(\left|\sum_{i=1}^t (e_i - w_i)\right| \geq t^\alpha\right) \leq C_0 t^{1-\alpha\gamma-\gamma\epsilon}. \quad (6.6)$$

Let $w_i = y_i + z_i$; we can write

$$P(|X_t| \leq t^\alpha) \leq P\left(\left|\sum_{i=1}^t w_i\right| \leq 2t^\alpha\right) + C_0 t^{1-\alpha\gamma-\gamma\epsilon}.$$

Let now $Q(X; \lambda) = \sup_x P(x \leq X \leq x + \lambda)$ denote the concentration function of a random variable X (see [Petrov, 1995](#)), and let $Y = \sum_{i=1}^t y_i$ and $Z = \sum_{i=1}^t z_i$. Clearly

$$P\left(\left|\sum_{i=1}^t w_i\right| \leq t^\alpha\right) \leq 2Q(Y + Z; t^\alpha),$$

and by the independence between the y_i 's and the z_i 's,

$$Q(Y + Z; t^\alpha) \leq \min\{Q(Y; t^\alpha), Q(Z; t^\alpha)\} \leq Q(Y; t^\alpha).$$

Now, given that the y_i 's have a distribution belonging in the stable distribution family, we have

$$P\left(x \leq \left|\sum_{i=1}^t y_i\right| \leq x + t^\alpha\right) = P\left(xt^{-1/\gamma} \leq |y_1| \leq (x + t^\alpha)t^{-1/\gamma}\right); \quad (6.7)$$

further, since the characteristic function of the y_i s is integrable, the density of the y_i s is bounded with upper bound m_y . Hence

$$\sup_x P\left(xt^{-1/\gamma} \leq |y_1| \leq (x + t^\alpha)t^{-1/\gamma}\right) \leq \frac{1}{2}m_y t^{\alpha-1/\gamma},$$

which concludes the proof. \square

Lemma 9. *Let a_T be a positive, real-valued sequence diverging to positive infinity as $T \rightarrow \infty$. Under Assumptions 1, 3 and 4, it holds that*

$$P(v_p \geq a_T) \leq p^{-\min\{1, \nu/2\}} a_T^{-\nu/2} \sum_{i=1}^p E|X_i|^\nu,$$

for some $0 < C_0 < \infty$; ν is defined in Assumption 1, and, when Assumption 4(ii) holds, it is chosen so that $\nu < \gamma$.

Proof. The lemma follows immediately upon noting that $P(v_p \geq a_T) = P(v_p^{\nu/2} \geq a_T^{\nu/2})$, and applying convexity (when $\frac{\nu}{2} > 1$) or the C_r -inequality (otherwise) to $v_p^{\nu/2}$. \square

Lemma 10. *Under Assumptions 1 and 3-5, it holds that, in all cases considered*

$$\frac{1}{T} \sum_{t=1}^T E\left(\frac{v_p}{v_p + X_t^2}\right) = O(T^{-\epsilon}),$$

for some $\epsilon > 0$.

Proof. Let $k > 0$, and note that

$$\begin{aligned} E\left(\frac{v_p}{v_p + X_t^2}\right) &= E\left(\frac{v_p}{v_p + X_t^2} \Big| v_p \geq T^k\right) P(v_p \geq T^k) + E\left(\frac{v_p}{v_p + X_t^2} \Big| v_p < T^k\right) P(v_p < T^k) \\ &\leq P(v_p \geq T^k) + E\left(\frac{v_p}{v_p + X_t^2} \Big| v_p < T^k\right). \end{aligned}$$

On account of Lemmas 6 and 9, it is immediate to see that, in the worst case, $P(v_p \geq T^k) \leq C_0 T^{-k\nu/2} (\ln \ln T)^2 (\ln T)^\delta$ for some $\delta > 0$. Also

$$\begin{aligned} & E \left(\frac{v_p}{v_p + X_t^2} \middle| v_p < T^k \right) \\ & \leq E \left(\frac{v_p}{v_p + X_t^2} \middle| v_p < T^k, |X_t| > t^\alpha \right) P(|X_t| > t^\alpha) + E \left(\frac{v_p}{v_p + X_t^2} \middle| v_p < T^k, |X_t| \leq t^\alpha \right) P(|X_t| \leq t^\alpha) \\ & \leq T^k t^{-2\alpha} + P(|X_t| \leq t^\alpha); \end{aligned}$$

hence

$$\frac{1}{T} \sum_{t=1}^T E \left(\frac{v_p}{v_p + X_t^2} \right) \leq C_0 T^{-k\nu/2} (\ln \ln T)^2 (\ln T)^\delta + C_1 T^{k-2\alpha} + C_2 \frac{1}{T} \sum_{t=1}^T P(|X_t| \leq t^\alpha).$$

Using Lemmas 7 and 8, the desired result follows. \square

Finally, we need the following lemma.

Lemma 11. *Consider a sequence U_T for which $E|U_T| \leq a_T$, where a_T is a positive, monotonically non-decreasing sequence. Then there exists a $C_0 < \infty$ such that*

$$\limsup_{T \rightarrow \infty} \frac{|U_T|}{a_T (\ln T)^{2+\epsilon}} \leq C_0 \text{ a.s.}$$

Proof. By equation (2.3) in Serfling (1970), it holds that $E \max_{1 \leq t \leq T} |U_t| \leq C_1 a_T \ln T$. Therefore

$$\sum_{T=1}^{\infty} \frac{1}{T} P \left(\max_{1 \leq t \leq T} |U_t| \geq a_T (\ln T)^{2+\epsilon} \right) \leq C_1 \sum_{T=1}^{\infty} \frac{1}{T} \frac{a_T \ln T}{a_T (\ln T)^{2+\epsilon}} < \infty.$$

The desired result follows from the proof of Corollary 2.4 in Cai (2006). \square

7. Proofs

Proof of Theorem 1. We begin with equation (2.5), which holds for $E \ln |\varphi + b_0| < 0$. The constant C_0 in the statement of the theorem differs according as $E\bar{X}_0^2 = \infty$ or $E\bar{X}_0^2 < \infty$. In the latter case, an immediate consequence of Lemma 5 is

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{v_p}{v_p + X_t^2} = E \left(\frac{E\bar{X}_0^2}{E\bar{X}_0^2 + \bar{X}_0^2} \right) + o(1);$$

on account of Assumption 2(i), we also have

$$E \left(\frac{E\bar{X}_0^2}{E\bar{X}_0^2 + \bar{X}_0^2} \right) > 0.$$

When $E\bar{X}_0^2 = \infty$, Lemma 4 yields

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{v_p}{v_p + X_t^2} = 1 + o(1),$$

whence the desired result. Finally, equation (2.6) is an immediate consequence of Lemmas 10 and 11. \square

Proof of Theorem 2. The proof, and in particular condition (2.14), are a refinement of similar results in related papers, such as Bandi and Corradi (2014) and Horváth and Trapani (2017). We begin by considering equation (2.15); in this case, (2.5) entails $l_T = g(C_0\psi(T))$ a.s., so that, by (2.10), we can assume from now on that $\lim_{T \rightarrow \infty} l_T = \infty$. Let E^* denote the expected value with respect to P^* , and m_G the upper bound for the density of G . It holds that:

$$\begin{aligned} R^{-1/2} \sum_{i=1}^R [\zeta_i(u) - G(0)] &= R^{-1/2} \sum_{i=1}^R [I\{\xi_i \leq 0\} - G(0)] + R^{-1/2} \sum_{i=1}^R [G(ul_T^{-1}) - G(0)] \\ &\quad + R^{-1/2} \sum_{i=1}^R [I\{0 \leq \xi_i \leq ul_T^{-1}\} - (G(ul_T^{-1}) - G(0))] \\ &= I + II + III. \end{aligned} \quad (7.1)$$

Consider *III*, and note that the random variable $I\{0 \leq \xi_i \leq ul_T^{-1}\}$ has expected value $\Delta G(ul_T^{-1}) = G(ul_T^{-1}) - G(0)$ and variance $\Delta G(ul_T^{-1}) [1 - \Delta G(ul_T^{-1})]$. Since the ξ_i s are *i.i.d.*, it holds that

$$\begin{aligned} &E^* \int_{-\infty}^{\infty} \left| R^{-1/2} \sum_{i=1}^R [I\{0 \leq \xi_i \leq ul_T^{-1}\} - \Delta G(ul_T^{-1})] \right|^2 dF(u) \\ &= \int_{-\infty}^{\infty} E^* [I\{0 \leq \xi_1 \leq ul_T^{-1}\} - \Delta G(ul_T^{-1})]^2 dF(u) \\ &= \int_{-\infty}^{\infty} \Delta G(ul_T^{-1}) [1 - \Delta G(ul_T^{-1})] dF(u) \\ &\leq \int_{-\infty}^{\infty} \Delta G(ul_T^{-1}) dF(u) \leq \frac{m_G}{l_T} \int_{-\infty}^{\infty} |u| dF(u); \end{aligned}$$

thus, Assumption 6 and Markov inequality entail that *III* = $o_{P^*}(1)$. As far as *II* is concerned, it immediately follows that

$$\int_{-\infty}^{\infty} \left(R^{1/2} [G(ul_T^{-1}) - G(0)] \right)^2 dF(u) \leq \frac{R}{l_T^2} m_G \int_{-\infty}^{\infty} |u|^2 dF(u);$$

by Assumption 6 and (2.14), this also tends to zero. Hence

$$\begin{aligned} G(0) (1 - G(0)) \Theta_{T,R} &= \int_{-\infty}^{\infty} \left| \frac{1}{R^{1/2}} \sum_{i=1}^R [I\{\xi_i \leq 0\} - G(0)] \right|^2 dF(u) + o_{P^*}(1) \\ &= \left| \frac{1}{R^{1/2}} \sum_{i=1}^R [I\{\xi_i \leq 0\} - G(0)] \right|^2 + o_{P^*}(1), \end{aligned}$$

and therefore the result follows from the Central Limit Theorem for Bernoulli random variables (see [Chow and Teicher, 2012](#)).

We now turn to (2.16). In this case, note that

$$E^* \left[R^{-1/2} \sum_{i=1}^R (I\{\xi_i \leq ul_T^{-1}\} - G(0)) \right]^2 = E^* [I\{\xi_1 \leq ul_T^{-1}\} - G(ul_T^{-1})]^2 + R |G(ul_T^{-1}) - G(0)|^2.$$

Since $E^* [I\{\xi_1 \leq ul_T^{-1}\} - G(ul_T^{-1})]^2 < \infty$, and since (2.11) entails that we can assume that $\lim_{T \rightarrow \infty} l_T = 0$, by Markov inequality it holds that

$$\int_{-\infty}^{\infty} \left[R^{-1/2} \sum_{i=1}^R (I\{\xi_i \leq ul_T^{-1}\} - G(0)) \right]^2 dF(u) = R(1 - G(0)) + o(R) + O_{P^*}(1),$$

for almost all realisations of $\{b_t, e_t, -\infty < t < \infty\}$. Hence, (2.16) follows immediately. \square

Proof of Corollary 1. Let $Y_s = I(\Theta_{R,T}^{(s)} \leq c_\alpha)$, and note that this is an i.i.d. sequence with all moments finite. Equation (2.23) follows upon noting that

$$\frac{1}{S} \sum_{s=1}^S Y_s = \frac{1}{S} \sum_{s=1}^S (Y_s - E^* Y_s) + E^* Y_s;$$

the first term converges to zero on account of the strong LLN (conditional on the sample), whereas the second one, $E^* Y_s$, drifts to zero in the ordinary limit sense by Theorem 2, again conditional on the sample. As far as (2.22) is concerned, by the Law of the Iterated Logarithm, there exists a large, random S_0 such that, for all $S \geq S_0$

$$\sqrt{\frac{S}{\ln \ln S}} \frac{1}{S} \sum_{s=1}^S Y_s \geq -\sqrt{2\alpha(1-\alpha)} + \sqrt{\frac{S}{\ln \ln S}} E^* Y_s;$$

also, by Theorem 2, under H_0 it holds that $E^* Y_s = 1 - \alpha + o(1)$, conditional on the sample. Equation (2.22) follows immediately. \square

Proof of Theorem 3. The proof repeats the arguments above, and therefore we only report its main passages. We let d_t^* be equal to d_t or $\hat{d}_t - d_t$, depending on which case is considered. We begin with the case of H_0 , where we aim to show that

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{Y_t^2}{v_p + Y_t^2} < 1 \text{ a.s.} \quad (7.2)$$

Consider first the case where $E|\bar{X}_0|^2 < \infty$. In this case, it is immediate to see that $v_p = p^{-1} \sum_{t=1}^p E(\bar{X}_0 + d_t^*)^2 + o(1)$, by the ergodic theorem. When $d_t^* = d_t$, Assumption 7(ii) ensures that $v_p > 0$; otherwise, Assumptions 2(i) and 7(i)(b) imply the same result - note that it does not matter what limit v_p converges

to, as long as it holds that such limit is positive. Then the same arguments as in the proof of Lemma 5 yield (7.2). Let $\bar{Y}_t = \bar{X}_t + d_t^*$. When $E|\bar{X}_0|^2 = \infty$, we have

$$\left| v_p - \frac{1}{p} \sum_{t=1}^p \bar{Y}_t^2 \right| \leq C_0 \frac{1}{p} \sum_{t=1}^p |\bar{X}_t + d_t^*| |\bar{X}_t - X_t|.$$

We already know that $\sum_{t=1}^p |\bar{X}_t| |\bar{X}_t - X_t| = O(1)$. Also, since $\sum_{t=1}^p |\bar{X}_t - X_t|^2 = O(1)$, by the Cauchy-Schwartz inequality we have $\sum_{t=1}^p |d_t^*| |\bar{X}_t - X_t| \leq C_o \left(\sum_{t=1}^p |d_t^*|^2 \right)^{1/2}$; under Assumption 7(i)(a), this entails that $\sum_{t=1}^p |d_t^*| |\bar{X}_t - X_t| = O(p^{1/2})$. Conversely, when Assumption 7(i)(b) holds, by using Lemma 11 we have $\sum_{t=1}^p |d_t^*|^2 = o(1)$. Thus, in both cases

$$v_p = \frac{1}{p} \sum_{t=1}^p \bar{X}_t^2 + \frac{1}{p} \sum_{t=1}^p d_t^{*2} + \frac{2}{p} \sum_{t=1}^p d_t^* \bar{X}_t + o(1);$$

the second term is at most $O(1)$, and therefore the third one can be shown to be dominated by applying the Cauchy-Schwartz inequality. Lemma 3 now entails that $v_p \rightarrow \infty$ a.s.; also, by the same arguments as in (5)

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{Y_t^2}{v_p + Y_t^2} = \frac{1}{T-p} \sum_{t=p+1}^T \frac{\bar{Y}_t^2}{v_p + \bar{Y}_t^2} + o(1).$$

Thus, for every C_0 there is a random p_0 such that, when $p \geq p_0$

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{\bar{Y}_t^2}{v_p + \bar{Y}_t^2} \leq \frac{1}{T-p} \sum_{t=p+1}^T \frac{\bar{Y}_t^2}{C_0 + \bar{Y}_t^2} \text{ a.s.}$$

Now, for $0 < \delta < 2$ such that $E|\bar{X}_0|^\delta < \infty$ we have

$$\begin{aligned} \frac{1}{T-p} \sum_{t=p+1}^T \frac{\bar{Y}_t^2}{C_0 + \bar{Y}_t^2} &\leq \frac{1}{T-p} \sum_{t=p+1}^T \left| \frac{\bar{Y}_t^2}{C_0 + \bar{Y}_t^2} \right|^{\delta/2} \leq C_0^{-\delta/2} \frac{1}{T-p} \sum_{t=p+1}^T |\bar{Y}_t|^\delta \\ &\leq C_1 C_0^{-\delta/2} \frac{1}{T-p} \sum_{t=p+1}^T \left(|\bar{X}_t|^\delta + |d_t^*|^\delta \right) \\ &= C_1 C_0^{-\delta/2} E|\bar{X}_0|^\delta + C_1 C_0^{-\delta/2} \frac{1}{T-p} \sum_{t=p+1}^T |d_t^*|^\delta \leq C_2 C_0^{-\delta/2}, \end{aligned} \quad (7.3)$$

where $C_2 < \infty$. This follows directly when Assumption 7(i)(a) holds; under Assumption 7(i)(b), it can be shown by elementary arguments that $E \left| \sum_{t=p+1}^T |d_t^*|^\delta \right| \leq \sum_{t=p+1}^T E |d_t^*|^2 \leq C_0 T^{1-\epsilon_1}$, so that Lemma 11 ensures that $(T-p)^{-1} \sum_{t=p+1}^T |d_t^*|^\delta = o(1)$, whence (7.3). In conclusion, for any $\epsilon > 0$ there is a random p_0 such that for all $p \geq p_0$

$$\frac{1}{T-p} \sum_{t=p+1}^T \frac{Y_t^2}{v_p + Y_t^2} \leq \epsilon \text{ a.s.},$$

which proves (7.2) even in the case of infinite variance. Thus, the same passages as above yield that, under H_0 , $D_T^* = C_0 + o(1)$ for some $0 < C_0 < \infty$.

Under H_A , the proof that $D_T^* = O(T^{-\epsilon})$ follows immediately if we show that Lemmas 7 and 8 hold. However, this can be easily verified by noting that $P(|Y_t| \leq t^\alpha) \leq P(|X_t| \leq t^\alpha + |d_t^*|)$. \square

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φ	σ_b^2	$e_t \sim N(0, 1)$				$e_t \sim t_2$				$e_t \sim t_1$			
		250	500	1000	2000	250	500	1000	2000	250	500	1000	2000
1.05	0	0.989 (0.00)	0.981 (0.00)	0.998 (0.00)	1.000 (0.00)	0.889 (0.00)	0.990 (0.00)	0.998 (0.00)	1.000 (0.00)	0.647 (0.00)	0.995 (0.00)	1.000 (0.00)	1.000 (0.00)
	0.10	0.389 (0.12)	0.410 (0.09)	0.519 (0.07)	0.628 (0.05)	0.473 (0.11)	0.413 (0.08)	0.556 (0.07)	0.619 (0.05)	0.266 (0.12)	0.398 (0.09)	0.434 (0.07)	0.579 (0.04)
	0.25	0.071 (0.73)	0.060 (0.93)	0.059 (0.94)	0.051 (0.96)	0.065 (0.64)	0.055 (0.81)	0.057 (0.90)	0.052 (0.99)	0.055 (0.50)	0.050 (0.78)	0.060 (0.91)	0.055 (0.96)
1	0	0.484 (0.06)	0.525 (0.05)	0.687 (0.03)	0.754 (0.01)	0.350 (0.06)	0.564 (0.04)	0.711 (0.01)	0.796 (0.02)	0.432 (0.04)	0.537 (0.04)	0.781 (0.01)	0.857 (0.00)
	0.10	0.068 (0.90)	0.059 (0.93)	0.059 (0.94)	0.052 (0.97)	0.074 (0.80)	0.061 (0.82)	0.057 (0.89)	0.053 (0.90)	0.090 (0.81)	0.056 (0.77)	0.059 (0.87)	0.051 (0.92)
	0.25	0.050 (0.93)	0.056 (0.93)	0.059 (0.97)	0.051 (0.99)	0.060 (0.78)	0.053 (0.84)	0.057 (0.88)	0.051 (0.91)	0.052 (0.78)	0.047 (0.82)	0.058 (0.90)	0.051 (0.92)
0.95	0	0.054 (0.79)	0.056 (0.88)	0.060 (0.90)	0.051 (0.92)	0.051 (0.60)	0.053 (0.71)	0.057 (0.79)	0.055 (0.90)	0.053 (0.52)	0.054 (0.66)	0.056 (0.70)	0.060 (0.85)
	0.10	0.049 (0.90)	0.056 (0.90)	0.058 (0.94)	0.051 (0.96)	0.058 (0.64)	0.053 (0.73)	0.057 (0.82)	0.051 (0.90)	0.052 (0.55)	0.047 (0.57)	0.058 (0.67)	0.051 (0.95)
	0.25	0.048 (0.91)	0.055 (0.91)	0.058 (0.94)	0.051 (0.96)	0.052 (0.71)	0.053 (0.81)	0.057 (0.86)	0.051 (0.94)	0.051 (0.58)	0.047 (0.59)	0.058 (0.76)	0.051 (0.99)
0.5	0	0.048 (0.90)	0.055 (0.91)	0.058 (0.95)	0.051 (1.00)	0.050 (0.92)	0.053 (0.92)	0.057 (0.96)	0.051 (0.99)	0.051 (0.92)	0.047 (0.90)	0.057 (0.93)	0.051 (0.98)
	0.10	0.048 (0.91)	0.055 (0.96)	0.058 (0.95)	0.051 (0.99)	0.050 (0.95)	0.053 (0.91)	0.057 (0.95)	0.051 (0.99)	0.051 (0.95)	0.047 (0.91)	0.058 (0.99)	0.051 (0.99)
	0.25	0.048 (0.93)	0.055 (0.95)	0.058 (0.98)	0.051 (0.99)	0.050 (0.93)	0.053 (0.92)	0.057 (0.96)	0.051 (0.99)	0.051 (0.95)	0.047 (0.95)	0.058 (0.99)	0.051 (0.99)
0.0	0	0.048 (0.92)	0.056 (0.93)	0.058 (0.96)	0.051 (0.99)	0.050 (0.95)	0.053 (0.93)	0.057 (0.96)	0.051 (0.99)	0.051 (0.94)	0.047 (0.94)	0.057 (0.95)	0.051 (0.99)
	0.10	0.048 (0.93)	0.055 (0.95)	0.058 (0.98)	0.051 (0.99)	0.050 (0.95)	0.053 (0.94)	0.057 (0.95)	0.051 (0.99)	0.051 (0.95)	0.047 (0.92)	0.058 (0.99)	0.051 (0.99)
	0.25	0.048 (0.93)	0.055 (0.96)	0.058 (0.98)	0.051 (0.99)	0.050 (0.95)	0.053 (0.95)	0.057 (0.96)	0.051 (0.99)	0.051 (0.95)	0.047 (0.93)	0.058 (0.99)	0.051 (0.99)

L. Trupani/Testing for stationarity

Table 1. Empirical rejection frequencies for the test for $H_0 : X_t$ is stationary.

T	$e_t \sim N(0, 1)$				$e_t \sim t_2$				$e_t \sim t_1$			
	250	500	1000	2000	250	500	1000	2000	250	500	1000	2000
(φ, σ_b^2)												
(0.2, 3.6190)	0.574 (0.12)	0.657 (0.09)	0.789 (0.07)	0.913 (0.02)	0.994 (0.12)	1.000 (0.06)	1.000 (0.02)	1.000 (0.01)	0.997 (0.10)	1.000 (0.06)	1.000 (0.03)	1.000 (0.01)
(0.3, 3.5556)	0.556 (0.12)	0.662 (0.10)	0.781 (0.08)	0.903 (0.02)	0.985 (0.11)	0.999 (0.08)	1.000 (0.03)	1.000 (0.02)	0.994 (0.09)	1.000 (0.07)	1.000 (0.02)	1.000 (0.01)
(0.4, 3.4460)	0.551 (0.11)	0.624 (0.10)	0.756 (0.07)	0.964 (0.02)	0.988 (0.10)	0.999 (0.07)	1.000 (0.02)	1.000 (0.02)	0.998 (0.09)	0.999 (0.06)	1.000 (0.03)	1.000 (0.00)
(0.5, 3.3390)	0.553 (0.12)	0.614 (0.09)	0.747 (0.07)	0.957 (0.02)	0.994 (0.10)	0.999 (0.07)	1.000 (0.02)	1.000 (0.01)	0.974 (0.10)	0.999 (0.06)	1.000 (0.03)	1.000 (0.01)
(0.6, 3.2245)	0.556 (0.10)	0.614 (0.10)	0.754 (0.08)	0.968 (0.01)	0.996 (0.11)	0.999 (0.07)	1.000 (0.04)	1.000 (0.01)	0.998 (0.09)	1.000 (0.07)	1.000 (0.04)	1.000 (0.00)
(0.7, 3.1310)	0.553 (0.11)	0.670 (0.10)	0.792 (0.07)	0.906 (0.02)	0.993 (0.11)	0.998 (0.06)	1.000 (0.03)	1.000 (0.02)	0.919 (0.09)	0.999 (0.06)	1.000 (0.02)	1.000 (0.01)
(0.8, 2.8650)	0.537 (0.12)	0.687 (0.09)	0.701 (0.08)	0.900 (0.01)	0.991 (0.10)	0.998 (0.06)	1.000 (0.02)	1.000 (0.02)	0.999 (0.09)	0.996 (0.07)	1.000 (0.02)	1.000 (0.01)
(0.9, 2.6815)	0.545 (0.12)	0.619 (0.09)	0.731 (0.08)	0.924 (0.01)	0.992 (0.10)	0.998 (0.07)	1.000 (0.02)	1.000 (0.01)	0.998 (0.10)	1.000 (0.06)	1.000 (0.03)	1.000 (0.01)
(1.0, 2.4440)	0.554 (0.12)	0.638 (0.09)	0.748 (0.07)	0.930 (0.02)	0.989 (0.10)	0.999 (0.06)	1.000 (0.03)	1.000 (0.01)	0.994 (0.10)	1.000 (0.06)	1.000 (0.02)	1.000 (0.00)

Table 2. Empirical rejection frequencies for the test for $H_0 : X_t$ is stationary. The cases considered in the table are boundary cases in which $E \ln |\varphi + b_0| = 0$ - thus, X_t is nonstationary, and the figures in the table should be interpreted as an assessment of the size of the test.

φ	σ_b^2	$e_t \sim N(0, 1)$				$e_t \sim t_2$				$e_t \sim t_1$			
		250	500	1000	2000	250	500	1000	2000	250	500	1000	2000
1.05	0	0.052	0.052	0.049	0.051	0.053	0.054	0.052	0.058	0.053	0.055	0.053	0.060
	0.10	0.056	0.064	0.055	0.062	0.054	0.062	0.053	0.063	0.054	0.108	0.057	0.073
	0.25	0.096	0.219	0.290	0.465	0.102	0.180	0.240	0.483	0.064	0.129	0.345	0.489
1	0	0.055	0.057	0.055	0.055	0.053	0.054	0.052	0.058	0.053	0.055	0.052	0.060
	0.10	0.096	0.306	0.469	0.639	0.091	0.209	0.345	0.622	0.169	0.260	0.442	0.630
	0.25	0.144	0.459	0.536	0.746	0.138	0.495	0.579	0.723	0.147	0.463	0.509	0.712
0.95	0	0.222	0.497	0.579	0.780	0.159	0.421	0.549	0.886	0.154	0.467	0.582	0.754
	0.10	0.229	0.505	0.600	0.813	0.221	0.593	0.464	0.829	0.218	0.522	0.548	0.763
	0.25	0.238	0.535	0.630	0.833	0.233	0.541	0.674	0.831	0.256	0.505	0.630	0.802
0.5	0	0.467	0.841	0.907	0.986	0.557	0.733	0.977	0.837	0.404	0.805	0.871	1.000
	0.10	0.457	0.833	0.900	0.982	0.533	0.770	0.854	0.925	0.402	0.876	0.974	0.929
	0.25	0.430	0.815	0.877	0.975	0.509	0.748	0.881	0.925	0.489	0.898	0.799	0.926
0.0	0	0.541	0.907	0.939	0.995	0.407	0.902	1.000	0.930	0.546	0.874	0.954	1.000
	0.10	0.530	0.896	0.933	0.992	0.497	0.967	0.995	0.982	0.398	0.873	0.940	0.998
	0.25	0.503	0.866	0.910	0.989	0.519	0.905	0.894	0.962	0.668	0.870	1.000	0.993

Table 3. Empirical rejection frequencies for the test for $H_0 : X_t$ is nonstationary.

T	$e_t \sim N(0, 1)$				$e_t \sim t_2$				$e_t \sim t_1$			
	250	500	1000	2000	250	500	1000	2000	250	500	1000	2000
(φ, σ_b^2)												
(0.2, 3.6190)	0.051	0.058	0.053	0.053	0.054	0.053	0.059	0.053	0.054	0.054	0.058	0.060
(0.3, 3.5556)	0.053	0.058	0.053	0.053	0.056	0.060	0.057	0.053	0.054	0.056	0.057	0.053
(0.4, 3.4460)	0.054	0.058	0.056	0.055	0.057	0.053	0.060	0.060	0.057	0.051	0.051	0.055
(0.5, 3.3390)	0.053	0.062	0.054	0.055	0.055	0.056	0.054	0.054	0.056	0.057	0.051	0.055
(0.6, 3.2245)	0.052	0.062	0.052	0.057	0.055	0.060	0.058	0.053	0.054	0.060	0.054	0.057
(0.7, 3.1310)	0.052	0.059	0.053	0.054	0.055	0.059	0.058	0.054	0.053	0.054	0.059	0.054
(0.8, 2.8650)	0.055	0.065	0.056	0.057	0.058	0.060	0.056	0.057	0.052	0.057	0.052	0.059
(0.9, 2.6815)	0.053	0.062	0.054	0.058	0.057	0.060	0.053	0.052	0.054	0.057	0.060	0.060
(1.0, 2.4440)	0.054	0.059	0.056	0.059	0.055	0.058	0.058	0.053	0.057	0.058	0.060	0.053

Table 4. Empirical rejection frequencies for the test for $H_0 : X_t$ is nonstationary. The cases considered in the table are boundary cases in which $E \ln |\varphi + b_0| = 0$ - thus, X_t is nonstationary, and the figures in the table should be interpreted as the power of the test.

Series	Unit	Frequency	SA	Sample	<i>T</i>
GDP	Billions of dollars	Quarterly	YES	Q1 1947-Q3 2017	283
CPI	Percentage	Monthly	NO	Feb 1913-Nov 2017	1259
Ind. Prod.	Index (2012=100)	Monthly	YES	Jan 1919-Nov 2017	1187
M2	Billions of dollars	Weekly	YES	03/11/80-11/12/17	1937
Unemployment	Percentage	Monthly	YES	Jan 1948-Nov 2017	839
T-Bill	Percentage	Daily	NO	21/12/12-20/12/17	1304

Table 5. Data description of the series employed in the empirical exercise; the column headed SA refers to whether data are seasonally adjusted or not.

Series	$\hat{\varphi}$	$\hat{\sigma}_b^2$	ERS	KPSS	Q (0.05)		Q (0.05)	
					$H_0 : X_t$ stationary		$H_0 : X_t$ nonstationary	
					(level)	(first-diff)	(level)	(first-diff)
GDP	0.9988	-1.38×10^{-4}	96.10 (*)	0.293 (*)	0.0000 (*)	0.9514	0.9526	0.0000 (*)
CPI	1.0002	8.19×10^{-3}	45.66 (*)	0.709 (*)	0.0000 (*)	0.9460	0.9528	0.0000 (*)
Ind. Prod.	0.9995	2.32×10^{-4} (*)	N/A	N/A	0.0000 (*)	0.9448	0.9468	0.0000 (*)
M2	0.9999	-6.64×10^{-7}	133.82 (*)	0.356 (*)	0.0000 (*)	0.9486	0.9482	0.0000 (*)
Unemployment	0.9884	2.39×10^{-3} (*)	N/A	N/A	0.0000 (*)	0.9472	0.9444	0.0000 (*)
T-Bill	1.0021	7.18×10^{-5}	66.17 (*)	0.861 (*)	0.0000 (*)	0.9500	0.9500	0.0000 (*)

Table 6. Outcomes of estimation and testing for the null of strict stationarity. We have also reported the WLS estimators of φ and of σ_b^2 ($\hat{\varphi}$ and $\hat{\sigma}_b^2$), as studied in [Horváth and Trapani \(2017\)](#); the symbol “(*)” next to the values of $\hat{\sigma}_b^2$ denotes rejection of the null of no coefficient randomness expressed as $H_0 : \sigma_b^2 = 0$ (we refer to ? for the theory of estimation and the test). As mentioned in the paper, we have computed two popular unit root tests: the Elliot-Rothenberg-Stock test (reported in the column ERS) and the KPSS test. The former has been carried out under the hypothesis of a constant and a trend, using the Bartlett kernel to estimate the spectral density and choosing the related bandwidth via the criteria discussed in [Andrews \(1991\)](#). The same specifications were used for the KPSS test. In the last four columns, we have reported the values of $Q(\alpha)$ (for $\alpha = 0.05$) for the data in levels and first differences, considering both testing for the null of stationarity and the null of nonstationarity. In all tests considered, the symbol “(*)” indicates rejection of the null hypothesis.