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Matteo Barigozzi, Giuseppe Cavaliere and Lorenzo Trapani

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Determining the rank of cointegration with infinite variance

Matteo Barigozzi* **Giuseppe Cavaliere**† **Lorenzo Trapani**‡

*University of Bologna
e-mail: matteo.barigozzi@unibo.it

†University of Bologna
e-mail: giuseppe.cavaliere@unibo.it

‡University of Nottingham
e-mail: lorenzo.trapani@nottingham.ac.uk

Abstract:

We study the issue of determining the rank of cointegration, R, in an N-variate time series y_t, allowing for the possible presence of heavy tails. Our methodology does not require any estimation of nuisance parameters such as the tail index, and indeed even knowledge as to whether certain moments (such as the variance) exist or not is not required. Our estimator of the rank is based on a sequence of tests on the eigenvalues of the sample second moment matrix of y_t. We derive the rates of such eigenvalues, showing that these do depend on the tail index, but also that there exists a gap in rates between the first $N-R$ and the remaining eigenvalues. The former ones, in particular, diverge at a rate which is faster than the latter ones by a factor T (where T denotes the sample size), irrespective of the tail index. We thus exploit this eigen-gap by constructing, for each eigenvalue, a test statistic which diverges to positive infinity or drifts to zero according as the relevant eigenvalue belongs in the set of the first $N-R$ eigenvalues or not. We then construct a randomised statistic based on this, using it as part of a sequential testing procedure. The resulting estimator of R is consistent, in that it picks the true value R with probability 1 as the sample size passes to infinity.

JEL codes: C30, C32.
Keywords and phrases: cointegration, heavy tails, randomized tests.

1. Introduction

Since the seminal contribution by Engle and Granger (1987), cointegration has become the prevalent tool for the analysis of multivariate time series, and its applications to economic and financial data have been ubiquitous. Thus, there is now a plethora of different contributions and methods to carry out inference in the context of a (possibly) cointegrated system (we refer, *inter alia*, to the books by Johansen (1995) and Juselius (2006) for comprehensive reviews). Although such techniques are very diverse, they all share the same starting point: the first step of cointegration analysis is, arguably, the determination of the rank of cointegration. Determining the rank of cointegration is, however, a delicate issue from both an empirical and a theoretical viewpoint. From an applied perspective, as well as constituting the preliminary step for the subsequent estimation of the cointegration vectors and the associated error correction model, knowing the number of cointegration relationships is of great importance *per se*, in that it corresponds to e.g. the presence or absence of integration among financial markets (Kasa, 1992), or the number of portfolios that can be constructed from a vector of assets (Alexander et al., 2002). Thus, it is of pivotal importance to have an accurate estimate of such number. As far as the theory is concerned, many estimators of the cointegration rank are now available, since the seminal paper by Johansen (1991), and are based either on sequential testing (as in the aforementioned paper by Johansen), or on information criteria (see Aznar and Salvador, 2002). Broadly speaking, a common feature of all available methodologies for the determination of the cointegration rank is the assumption that some moments (e.g. the variance, or the fourth moment in the case of ML-based methods) of the data exist. Violation of these assumptions may result in (possibly severe) incorrect estimation of the cointegration rank - see e.g. the simulations in Caner (1998), which show that such rank is often overstated, and also the corroborating empirical evidence presented in Falk and Wang (2003).
Datasets with heavy tails, which do not have finite second (or even first) moment, are often encountered in macroeconomics (see e.g. Ibragimov and Ibragimov, 2018, and the references therein) and finance (see Rachev, 2003) - we also refer to the textbook by Embrechts et al. (2013) for further examples and references. Although the literature has produced many contributions on inference in the presence of heavy tails (see De Haan and Ferreira, 2007 for a review), contributions which consider cointegration analysis in the presence of infinite variance are rare: to the best of our knowledge, only Caner (1998), Paulauskas and Rachev (1998), Fasen (2013) and, recently, She and Ling (2020), have studied this set-up. One arguable practical difficulty is that, in order to implement all the aforementioned methodologies, it is necessary to estimate the so-called "tail index", which is notoriously difficult (see Embrechts et al., 2013). In order to overcome such difficulty, a possible solution could be the bootstrap; using it in the case of cointegration analysis is certainly possible, although it is likely to be fraught with difficulties similar to the ones faced by Swensen (2006) and Cavaliere et al. (2012). Indeed, bootstrap methods have been employed to test for a unit root in the presence of infinite variance - see Cavaliere et al. (2018). Although the knowledge of the tail index is not needed in that case, extensions to multiple time series are not available in the literature and are likely to be extremely difficult to implement.

Contribution of this paper

In this paper, we propose an estimator for the rank R of an N-variate cointegrated system in the possible presence of heavy tails. Our procedure does not require any *a priori* knowledge

as to whether the variance is finite or not, or as to how many moments exist, thus avoiding having to estimate any nuisance parameters.
The details are spelt out in the remainder of the paper; here, we offer a heuristic preview of how the methodology works. The starting point of our analysis is a novel result concerning the properties of the sample second moment matrix of the data in levels. Specifically, if the rank of cointegration is R, the $N - R$ largest eigenvalues of the matrix diverge to positive infinity, as the sample size T passes to infinity, faster than the remaining eigenvalues by a factor (almost) equal to T. Crucially, this result always holds, irrespective of having finite or infinite variance. Building on this, for each eigenvalue we construct a statistic which diverges to infinity under the null that the eigenvalue is diverging at a "fast" rate, and drifts to zero under the alternative that the relevant eigenvalue diverges at a "slow" rate. Although the limiting distribution of our statistic is bound to depend on nuisance parameters such as the tail index, the relative rate of divergence between the null and the alternative does not depend on any nuisance parameters. In order to construct a test, we rely only on rates and randomise such statistic in a similar fashion to Corradi and Swanson (2006) and Trapani (2018). Thence, our estimator of the rank is based on running the tests in sequence; in this respect, it mimics the well-known sequential rank determination procedure advocated in Johansen (1995). The algorithm is constructed so that the rank estimate is consistent - that is, the estimated rank is equal to the population rank as the sample size grows to infinity for almost all realisations of the sample.
Thus, our methodology has at least three desirable features. Firstly, as mentioned above, our technique can be applied to data with infinite variance (and even infinite expectation), with no need to know this *a priori*. Secondly, our procedure does not require at any stage the estimation of the tail index of a distribution, which is a notoriously delicate issue; indeed, the applied user does not even need to know that the variance may actually not exist. This contrasts with the papers by Caner (1998), Paulauskas and Rachev (1998) and She and Ling (2020), where knowledge of the tail index is necessary in order to compute the relevant critical values. On a similar note, our procedure does not require the correct specification of the lag structure of the underlying VECM model. Thirdly, our results are based only on first order asymptotics, i.e. rates, which are easier to derive than the full blown distributional theory.

The remainder of the paper is organised as follows. Assumptions and all the relevant theory are spelt out in Section 2. We provide extensive Monte Carlo evidence in Section 3. Section 5 concludes.
NOTATION. For a given matrix $A \in \mathbb{R}^{n\times m}$, we denote its element in position (i,j) as $A_{i,j}$; we also let let $\lambda^{(j)}(A)$ denote the j-th largest eigenvalue of A. We denote with $c_0, c_1, ...$ positive, finite constants whose value can change from line to line. We denote the k-iterated logarithm of x (truncated at zero) as $\ln_k x$ - e.g. $\ln_2 x = \max\{\ln\ln x, 0\}$. Finally, we let χ^2_1 denote a chi-square with one degree of freedom.

2. Theory

Given the N-dimensional vector y_t, consider the $MA(\infty)$ representation

$$\Delta y_t = C(z)\,\varepsilon_t, \tag{2.1}$$

where $C(z) = \sum_{j=0}^{\infty} C_j z^j$. In (2.1), we assume that $y_0 = 0$ and $\varepsilon_t = 0$ for $t \leq 0$, merely for simplicity and with no loss of generality; upon inspecting the proofs, all the results derived

here can be extended to accommodate e.g. more general assumptions concerning the initial value y_0.
Standard arguments (see Watson, 1994) allow to represent (2.1) as

$$y_t = C \sum_{s=1}^{t} \varepsilon_s + C^* (z) \varepsilon_t, \tag{2.2}$$

having defined: $C = \sum_{j=0}^{\infty} C_j$, $C^*(z) = \sum_{j=0}^{\infty} C_j^* z^j$ and $C_j^* = \sum_{k=j+1}^{\infty} C_k$. Equation (2.2) is derived from the multivariate Beveridge-Nelson decomposition of the filter $C(z)$.
We assume that the system defined in (2.2) is cointegrated, viz. that the $N \times N$ matrix C can have reduced rank.

Assumption 1. *It holds that: (i) rank* $(C) = N - R$, *where* $0 \leq R \leq N$; *(ii)* $\|C_j\| = O(\rho^j)$ *for some* $0 < \rho < 1$.

Part *(ii)* of the assumption requires that the MA coefficients C_j decline geometrically. This assumption is similar to e.g. Assumption 1 in Caner (1998), where the C_js are, in essence, assumed to decline at a rate which increases as the tail index of the innovations ε_t decreases.

On account of the possible rank reduction of C, a different formulation of (2.2) may be helpful. Indeed, it is always possible to write $C = PQ$, where P and Q are full rank matrices of dimensions $N \times (N - R)$ and $(N - R) \times N$ respectively. Defining the $(N - R)$-dimensional process $x_t = Q \sum_{s=1}^{t} \varepsilon_s$, and using the short-hand notation $u_t = C^*(z) \varepsilon_t$, we rewrite (2.2) as

$$y_t = P x_t + u_t. \tag{2.3}$$

We now make some assumptions on the error term ε_t.

Assumption 2. *It holds that: (i)* ε_t *is an i.i.d. sequence; (ii) for all nonzero vectors* $l \in \mathbb{R}^N$, $l'\varepsilon_t$ *has distribution* $F_{l\varepsilon}$ *with strictly positive density, which belongs in the domain of attraction of a strictly stable law* G *with index* $0 < \eta \leq 2$.

As is generally the case in the analysis of time series with possibly infinite variance, we assume that ε_t is *i.i.d.*. Part *(ii)* of the assumption implicitly states that the vector ε_t has a multivariate distribution which belongs in the domain of attraction of a strictly stable, multivariate law (see Theorem 2.1.5(a) in Samorodnitsky and Taqqu, 1994) common tail index η, so that linear combinations of them can be constructed. The assumption implies that, when $\eta < 2$, $E|\varepsilon_{i,t}|^p < \infty$ for all $0 \leq p < \eta$, whereas $E|\varepsilon_{i,t}|^\eta = \infty$ (see Petrov, 1974). Also, by Property 1.2.15 in Samorodnitsky and Taqqu (1994), it holds that, as $x \to \infty$

$$\begin{aligned} F_{l\varepsilon}(-x) &= \frac{c_{l,1} + o(1)}{x^\eta} L(x), \\ 1 - F_{l\varepsilon}(x) &= \frac{c_{l,2} + o(1)}{x^\eta} L(x), \end{aligned}$$

where $L(x)$ is a slowly varying function in the sense of Karamata (see Seneta, 2006 for a review), and $c_{l,1}$, $c_{l,2} \geq 0$, $c_{l,1} + c_{l,2} > 0$. The condition that G is *strictly* stable entails that $c_{l,1} = c_{l,2}$ when $\eta = 1$, thus ruling out asymmetry in that case (see Property 1.2.8 in Samorodnitsky and Taqqu, 1994).

2.1. Asymptotics

Let

$$S_{11} = \sum_{t=1}^{T} y_t y_t', \tag{2.4}$$

$$S_{00} = \sum_{t=1}^{T} \Delta y_t \Delta y_t'. \tag{2.5}$$

We report a set of novel results for the eigenvalues of S_{11} and S_{00}, which we require for the construction of the test statistics.

Proposition 1. *We assume that Assumptions 1-2 are satisfied. Then there exists a random variable T_0 such that, for all $T \geq T_0$*

$$\lambda^{(j)}\left(S_{11}\right) \geq c_0 \frac{T^{1+2/\eta}}{\left(\ln \ln T\right)^{2/\eta}} \text{ for } j \leq N - R. \tag{2.6}$$

Also, for every $\epsilon > 0$, it holds that

$$\lambda^{(j)}\left(S_{11}\right) = o_{a.s.}\left(T^{2/p}\left(\ln T\right)^{2(2+\epsilon)/p}\right) \text{ for } j > N - R, \tag{2.7}$$

for every $0 \leq p < \eta$ when $\eta \leq 2$ with $E\left\|\varepsilon_t\right\|^{\eta} = \infty$, and $p = 2$ when $\eta = 2$ and $E\left\|\varepsilon_t\right\|^{\eta} < \infty$.

Proposition 1 states in essence, that the first $N - R$ eigenvalues of S_{11} diverge at a faster rate than the other ones (faster by an order of "almost" T). In this respect, the results in Proposition 1 entail that the spectrum of S_{11} has a "spiked" structure. This result is somewhat akin to the one typically found in the literature on large factor models (see e.g. Barigozzi and Trapani, 2020). However, in that case the spiked eigenvalues diverge due to the cross sectional dimension passing to infinity; in our case, such divergence is purely a time series phenomenon, due to the different rates of convergence of the partial sums of integrated versus non-integrated series.
Note that the rates in the proposition are - modulo the slowly varying functions - optimal, as can be seen by reading them in conjunction with the central limit theory presented e.g. in Samorodnitsky and Taqqu (1994).

In order to study S_{00} and its eigenvalues, we need the following assumption, which complements Assumption 1*(ii)*.

Assumption 3. *It holds that ε_t has density $p_{\varepsilon}(u)$ such that $\int \left|p_{\varepsilon}(u+y) - p_{\varepsilon}(u)\right| du \leq c_0 \left\|y\right\|$.*

The integral Lipschitz condition in Assumption 3 is a technical requirement needed for Δy_t to be strong mixing with geometrically declining mixing numbers, and it is a standard requirement in this literature (see e.g. Withers, 1981 and Pham and Tran, 1985; see also the discussion in Section 14.4 in Davidson, 1994, for possible relaxation of this condition).

Proposition 2. *We assume that Assumptions 1-3 are satisfied. Then*

$$\lambda^{(1)}\left(S_{00}\right) = o_{a.s.}\left(T^{2/\eta}\left(\prod_{i=1}^{j} \ln_i T\right)^{2/\eta}\left(\ln_{j+1} T\right)^{(2+\epsilon)/\eta}\right), \tag{2.8}$$

for every $\epsilon > 0$ and every integer j. Also, there exists a random variable T_0 such that, for all $T \geq T_0$

$$\lambda^{(N)}(S_{00}) \geq c_0 \frac{T^{2/\eta}}{(\ln T)^{(2/\eta-1)(2+\epsilon)}}, \tag{2.9}$$

for every $\epsilon > 0$.

Similarly to Proposition 1, Proposition 2 provides bounds for the spectrum of S_{00}. Part (2.8) has been shown in Trapani (2014), and it is a Chover-type Law of the Iterated Logarithm (Chover, 1966). As shown in Trapani (2014), the bound in (2.8) is almost sharp. The lower bound implied in (2.9) is also almost sharp.

2.2. The test statistics

The spectrum of S_{11} (and, in particular, the different rates of divergence of its eigenvalues) can - in principle - be employed in order to check whether there is cointegration or not, and to determine R. However, S_{11} has two main problems which make it unsuitable for direct usage. First, the matrix S_{11} and, consequently, its spectrum depend on the unit of measurement of the data, and thus they are not scale free. Secondly, as also highlighted by Proposition 1, the spectrum of S_{11} depends on the nuisance parameter η.
In order to construct scale-free and nuisance-free statistics, we propose to rescale S_{11} by S_{00}. Proposition 2 ensures that this is possible: by equation (2.9), the inverse of S_{00} cannot diverge too fast, and therefore the spectrum of the matrix $S_{00}^{-1}S_{11}$ should still have $N-R$ eigenvalues that diverge at a faster rate than the others.

Theorem 1. *We assume that Assumptions 1-3 are satisfied. Then there exists a random variable T_0 such that, for all $T \geq T_0$*

$$\lambda^{(j)}\left(S_{00}^{-1}S_{11}\right) \geq c_0 \frac{T^{1-\epsilon'}}{(\ln\ln T)^{2/\eta}(\ln T)^{2(2+\epsilon)/p}} \text{ for } 0 \leq j \leq N-R, \tag{2.10}$$

for every $\epsilon, \epsilon' > 0$, and every $0 \leq p < \eta$. Also, it holds that

$$\lambda^{(j)}\left(S_{00}^{-1}S_{11}\right) = o_{a.s.}\left(T^{\epsilon'}(\ln T)^{2(2+\epsilon)/\eta+(1+\epsilon)}\right) \text{ for } j > N-R, \tag{2.11}$$

for every $\epsilon, \epsilon' > 0$.

Theorem 1 states that the spectrum of $S_{00}^{-1}S_{11}$ has a similar structure to the spectrum of S_{11}. The first $N-R$ eigenvalues are spiked and their rate of divergence is faster than the rate of divergence of the remaining eigenvalues by a factor of almost T.
As mentioned above, using the eigenvalues of S_{11} is problematic, also in light of the fact that they depend on the tail index η - thus, any statistic based on them would not be free from this nuisance parameter. However, by normalising S_{11} by S_{00}, dependence on η disappears. In essence, apart from the slowly varying functions, equations (2.10) and (2.11) imply that the rates of divergence of the eigenvalues of $S_{00}^{-1}S_{11}$ are of order (arbitrarily close to) $O(T)$ for the spiked eigenvalues, and (arbitrarily close to) $O(1)$ for the other ones. This is the key property of $\lambda^{(j)}\left(S_{00}^{-1}S_{11}\right)$: dividing S_{11} by S_{00} washes out the impact of the tail index η, which therefore does not play any role (apart from appearing in the slowly varying functions) in determining the divergence or not of $\lambda^{(j)}\left(S_{00}^{-1}S_{11}\right)$. Although this result pertains to first order asymptotics

(i.e. to rates), it is possible to find an analogy between the result in Theorem 1 and approaches based on eliminating nuisance parameters using self-normalisation (see e.g. Shao, 2015).

Based on Theorem 1, we propose to use

$$\phi_T^{(j)} = \exp\left\{T^{-\kappa}\lambda^{(j)}\left(S_{00}^{-1}S_{11}\right)\right\} - 1, \tag{2.12}$$

where $\kappa \in (0,1)$ is a user-defined quantity whose choice we discuss in Section 3. On account of Theorem 1, by continuity it holds that, for any κ

$$\begin{aligned} P\left(\omega : \lim_{T\to\infty} \phi_T^{(j)} = \infty\right) &= 1 \text{ for } 0 \le j \le N-R, \\ P\left(\omega : \lim_{T\to\infty} \phi_T^{(j)} = 0\right) &= 1 \text{ for } j > N-R, \end{aligned}$$

so that we can assume from now on that

$$\lim_{T\to\infty} \phi_T^{(j)} = \infty \text{ for } 0 \le j \le N-R, \tag{2.13}$$

$$\lim_{T\to\infty} \phi_T^{(j)} = 0 \text{ for } j > N-R. \tag{2.14}$$

In essence, (2.13)-(2.14) state that $\phi_T^{(j)}$ diverges to positive infinity, or converges (to zero), according as $\lambda^{(j)}\left(S_{00}^{-1}S_{11}\right)$ is "large" (thereby corresponding to the presence of a stochastic trend) or "small" (thus suggesting that there is no stochastic trend). Based on the rates in (2.13)-(2.14), we propose (a sequence of) tests for

$$\left\{\begin{array}{l} H_0 : R \le N-j \\ H_A : R > N-j \end{array}\right. . \tag{2.15}$$

These tests will form the basis of our sequential procedure for the determination of the cointegration rank - see Section 2.3 below.

We do not know the limiting law of $\phi_T^{(j)}$; we therefore propose a randomised version of it. We present the construction of the test statistic as a three step algorithm.

Step 1 Generate an artificial sample $\left\{\xi_i^{(j)}, 1 \le i \le M\right\}$, with $\xi_i^{(j)} \sim i.i.d.N(0,1)$, independent across j and independent of the original data.
Step 2 For each $u \in U$, define the Bernoulli sequence $\zeta_i^{(j)}(u) = I\left(\phi_T^{(j)}\xi_i^{(j)} \le u\right)$, and let

$$\theta_{T,M}^{(j)}(u) = \frac{2}{\sqrt{M}}\sum_{i=1}^{M}\left(\zeta_i^{(j)}(u) - \frac{1}{2}\right).$$

Step 3 Compute

$$\Theta_{T,M}^{(j)} = \int_U \left[\theta_{T,M}^{(j)}(u)\right]^2 dF(u), \tag{2.16}$$

where $F(u)$ is a user-defined weight function.

In Step 2, the binary variable $\zeta_i^{(j)}(u)$ is created for several, user-specified, values of u, belongin to a support U. In Step 3, the resulting statistics $\theta_{T,M}^{(j)}(u)$ are averaged across u, through a weight function $F(u)$. This serves the purpose of attenuating the dependence of the test statistic on an arbitrary value u, a form of scale invariance which is also desirable when implementing the test (see also the discussion in Section 7 of Bandi and Corradi, 2014). The following assumption characterizes the weight function $F(\cdot)$.

Assumption 4. *It holds that (i)* $\int_U dF(u) = 1$*; (ii)* $\int_U u^2 dF(u) < \infty$.

In light of Assumption 4, a possible choice for $F(u)$ could be a distribution function with finite second moment; note that part *(ii)* of the assumption is trivially satisfied if U is bounded. Possible examples include a Rademacher distribution, based on choosing $U = \{-c, c\}$ with $F(c) = F(-c) = \frac{1}{2}$; or a standard normal distribution function, where $U = (-\infty, +\infty)$ and $F(u) = (2\pi)^{-1/2} \exp\left(-\frac{1}{2}u^2\right)$.
Let P^* denote the conditional probability with respect to the original sample; we use the notation "$\overset{D^*}{\to}$" and "$\overset{P^*}{\to}$" to define, respectively, conditional convergence in distribution and in probability according to P^*.

Theorem 2. *We assume that Assumptions 1-4 are satisfied. If* H_0 *holds, then, as* $\min(T, M) \to \infty$ *with*

$$M^{1/2} \exp\left(-T^{1-\kappa-\epsilon}\right) \to 0, \tag{2.17}$$

for any arbitrarily small $\epsilon > 0$*, it holds that*

$$\Theta_{T,M}^{(j)} \overset{D^*}{\to} \chi_1^2, \tag{2.18}$$

for each j*, for almost all realisations of* $\{\varepsilon_t, 0 < t < \infty\}$*. Under* H_A*, as* $\min(T, M) \to \infty$*, it holds that*

$$M^{-1}\Theta_{T,M}^{(j)} \overset{P^*}{\to} \frac{1}{4}, \tag{2.19}$$

for each j*, for almost all realisations of* $\{\varepsilon_t, 0 < t < \infty\}$*.*

Theorem 2 provides the limiting behaviour of $\Theta_{T,M}^{(j)}$. Indeed, an immediate consequence of the theorem is that - letting $0 < \alpha < 1$ denote the nominal level of the test and c_α such that $P\left(\chi_1^2 > c_\alpha\right) = \alpha$ - it holds that, under H_0

$$\lim_{\min(T,M)\to\infty} P^*\left(\Theta_{T,M}^{(j)} > c_\alpha\right) = \alpha, \tag{2.20}$$

for almost all realisations of $\{\varepsilon_t, 0 < t < \infty\}$; conversely, under H_A

$$\lim_{\min(T,M)\to\infty} P^*\left(\Theta_{T,M}^{(j)} > c_\alpha\right) = 1, \tag{2.21}$$

for almost all realisations of $\{\varepsilon_t, 0 < t < \infty\}$.
Although the primary purpose is to determine R, some comments on the individual tests are in order. Equation (2.21) states that, whenever using $\Theta_{T,M}^{(j)}$, a researcher will reject the null, when false, with probability one. This result is conditional on the sample, similar to the mode of convergence that one finds in the case of the bootstrap. Conversely, the result implied by equation (2.20) is subtler. As explained in Horváth and Trapani (2019), tests

based on $\Theta_{T,M}^{(j)}$ use some "added randomness" - the sequences $\left\{\xi_i^{(j)}, 1 \leq i \leq M\right\}$ - which do not vanish asymptotically. This entails that different researchers using the same dataset will obtain different p-values, which is a well-known feature of randomised tests (see e.g. Lehmann, 1951). This result may be viewed as undesirable, and Horváth and Trapani (2019) propose several ways of making the test unaffected by the added randomness.

2.3. Determining R

Bearing in mind the independence of the artificial random samples across j, the individual tests can be used, in a sequential procedure, in order to determine the cointegration rank R. The estimator of R (say $\widehat{R}$) is the output of the following algorithm:

Step 1 Run the test for $H_0 : R \leq N-1$ based on $\Theta_{T,M}^{(j)}$. If the null is rejected, set $\widehat{R} = N$ and stop, otherwise go to the next step.

Step 2 Starting from $j = 1$, run the test for $H_0 : R \leq N - j$ based on $\Theta_{T,M}^{(j+1)}$, constructed using an artificial sample $\left\{\xi_i^{(j+1)}\right\}_{i=1}^{M}$ generated independently of $\left\{\xi_i^{(1)}\right\}_{i=1}^{M}$, ..., $\left\{\xi_i^{(j)}\right\}_{i=1}^{M}$. If the null is rejected, set $\widehat{R} = N - j$ and stop; otherwise repeat the step until the null is rejected.

Theorem 3. *We assume that Assumptions 1-4 are satisfied. Define the level of each individual test as* $\alpha = \alpha(T)$*. As* $\min(T, M) \to \infty$ *under (2.17), if* $\alpha(T) \to 0$*, then it holds that* $P^*\left(\widehat{R} = R\right) = 1$ *for almost all realisations of* $\{\varepsilon_t, -\infty < t < \infty\}$.

Theorem 3 states that $\widehat{R}$ is consistent, as long as the nominal level α of the individual tests is chosen so as to drift to zero: no specific rates are required.

The way in which the sequential procedure works can be better understood upon inspecting the proof of the theorem; in (A.9), we show that, as $\min(T, M) \to \infty$, $P^*\left(\widehat{R} = R\right) = (1-\alpha)^R$ a.s. conditionally on the sample, whence the need to have $\alpha \to 0$. This ensures that $P^*\left(\widehat{R} = R\right) = 1$, a.s. conditionally on the sample. In fact, the proof of the theorem also states that $P^*\left(\widehat{R} = j\right) = \alpha(1-\alpha)^j$ for all $j \leq R-1$, as well as stating that $P^*\left(\widehat{R} = j\right) = 0$ whenever $j \geq R+1$, which may be helpful to construct a measure of confidence in the estimate $\widehat{R}$ for a given , fixed nominal level α (this result can be read in conjunction with Kong, 2020). Note that such probabilities depend on the unknown R, as well as being subject to the aforementioned issues relating to randomised tests. Conversely, the fact that $\alpha \to 0$ implies that all researchers using the same data will end up with the same estimate $\widehat{R}$.

Finally, the comment above can be read in conjunction with Johansen's procedure (Johansen, 1991), whose outcome is an estimate of R (say $\widetilde{R}$) such that, asymptotically, $P\left(\widetilde{R} = R\right) = 1-\alpha$ for a given nominal value α for the individual tests. On account of (A.9), in our case choosing a fixed nominal level α would yield, as mentioned above, $P^*\left(\widehat{R} = R\right) = (1-\alpha)^R$, which depends on the unknown R and is, for $R > 1$, worse than Johansen's procedure. On the other hand, in our procedure the individual tests are independent (conditional on the sample). Thus, in order to implement the procedure, a possible, "natural" approach could be based on using a Bonferroni correction, using $\frac{a}{N}$ as nominal level (where $0 < a < 1$ is user-defined) for each test. In this case, the same calculations as in the proof of Theorem 3 yield

$$P^{*}\left(\widehat{R}=R\right)=\left(1-a/N\right)^{R}\geq 1-\frac{R}{N}a\geq 1-a, \tag{2.22}$$

where the penultimate passage is based on Bernoulli's inequality.

As a final remark, we point out that all the results derived above are directly applicable in the presence of a (restricted) constant term in the VECM representation, along similar lines as in She and Ling (2020) (see also Yap and Reinsel, 1995). Indeed, consider the VAR and the $VECM$ representations

$$y_t = \alpha\rho_0 + \sum_{j=1}^{p} A_j y_{t-j} + \varepsilon_t, \tag{2.23}$$

$$\Delta y_t = \alpha\left(\beta' y_{t-1} + \rho_0\right) + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + \varepsilon_t, \tag{2.24}$$

where the constant in (2.23) is defined as the constrained quantity $\alpha\rho_0$ (with ρ_0 an $R \times 1$) vector. Under such a specification, it is well known that the series in y_t are $I(1)$ without drift and the cointegrating relations $\beta' y_{t-1}$ have non-zero means ρ_0. We claim that, in the presence of $\alpha\rho_0$, our procedure still yields the same results as without constant term.

Corollary 1. *We assume that (2.23) and (2.24) hold. Then, the conclusions of Theorems 2 and 3 still hold under the same assumptions.*

3. Monte Carlo evidence

In this section, we illustrate the properties of our procedure through a small scale Monte Carlo exercise; we also discuss the implementation/specification of our test.

As far as the data generating process (DGP) is concerned, along similar lines to She and Ling (2020) we simulate the following N-variate $VAR(1)$ model

$$y_t = A y_{t-1} + \varepsilon_t, \tag{3.1}$$

where the matrix A is constructed as $A = I_N - PP'$ and P is an orthonormal $N \times R$ matrix (i.e. such that $P'P = I_R$)[1], whence the $VECM$ representation

$$\Delta y_t = -PP' y_{t-1} + \varepsilon_t.$$

The innovations ε_t in (3.1) are generated as coordinate-wise independent. Although our theory is developed for *i.i.d.* errors, we consider the possible presence of serial dependence in the error term, through an $AR(1)$ specification

$$\varepsilon_{i,t} = \theta\varepsilon_{i,t-1} + e_{i,t} \tag{3.2}$$

[1]Specifically, in order to create P, we have used $P = D\left(D'D\right)^{-1/2}$, where $(M)^{-1/2}$ denotes the Choleski factor of a matrix M, and have set $D \sim \mathbf{1}_{N\times R} + d_{N\times R}$, where $\mathbf{1}_{N\times R}$ is an $N \times R$ matrix of ones; and $d_{N\times R}$ is an $N \times R$ matrix such that $vec\left(d_{N\times R}\right) \sim N\left(0, \mathbf{1}_{NR}\right)$.

for $1 \leq i \leq N$; we consider $\theta \in \{0, 0.5, -0.5\}$. The innovation $e_{i,t}$ is generated as $i.i.d.$ across t, with a power law distribution of tail index η, using $\eta \in \{0.5, 1, 1.5, 2\}$. As a benchmark, we have also considered the case $\varepsilon_t \sim i.i.d.N(0, I_N)$. In order to simulate the power law distribution, we follow the procedure proposed by Clauset et al. (2009), generating

$$e_{i,t} = (1 - v_{i,t})^{-1/\eta}, \tag{3.3}$$

for $1 \leq i \leq N$, where $v_{i,t}$ is generated as $i.i.d.U[0, 1]$; $e_{i,t}$ is subsequently centered. The first $1,000$ datapoints are discarded in order to avoid dependence on initial conditions.

As far as our testing procedure is concerned, we note, from preliminary, unreported experiments, that it is not particularly sensitive to the choice of the various specifications.
In our experiments, we have used $M = 100$ to speed up the computational time, but we note that results do not change when setting e.g. $M = T$, $M = T/2$ or $M = T/4$. In (2.12), we have used $\kappa = 10^{-4}$. This is a conservative choice, whose rationale follows from the fact that, in (2.12), dividing by T^{κ} serves the purpose of making the non-spiked eigenvalues drift to zero. The upper bound provided in (2.11) for such non-spiked eigenvalues is given by slowly varying functions, which indicates that even a very small value of κ should suffice. Indeed, altering the value of κ has virtually no consequence. In order to compute the integral in (2.16), we use the Gauss-Hermite quadrature[2]
Finally, as far as the family-wise detection procedure is concerned, we have set the level of the individual tests as $\alpha(T) = \frac{0.05}{T}$, as also used in Barigozzi and Trapani (2020) - this corresponds to having a critical value c_{α}, in (2.20) and (2.21), which grows logarithmically with T. All routines are based on $1,000$ iterations and are written using GAUSS 10.0.0.

By way of comparison, we also considered rank estimation via information criteria. Whilst we have used the three most popular ones (namely, AIC, BIC and HQ), in the tables, we only report the results corresponding to BIC, which is consistently the best out of the three for all experiments[3]. We point out that the use of information criteria in the determination of the rank of cointegration is well-studied in the finite variance case (see e.g. Aznar and Salvador, 2002 and Kapetanios, 2004); conversely, the case of infinite variance has not been studied in the case of cointegrated systems (see however the contribution by Knight et al., 1989). In our

[2] In our case, we have used

$$\Theta^{j}_{T,M} = \frac{1}{\sqrt{\pi}} \sum_{s=1}^{n_S} w_s \left(\theta^{(j)}_{T,M} \left(\sqrt{2} z_s \right) \right)^2, \tag{3.4}$$

where the z_ss, $1 \leq s \leq n_S$, are the zeros of the Hermite polynomial $H_{n_S}(z)$ and the weights w_s are defined as

$$w_s = \frac{\sqrt{\pi} 2^{n_S - 1} (n_S - 1)!}{n_S \left[H_{n_S - 1}(z_s) \right]^2}. \tag{3.5}$$

Thus, when computing $\theta^{(j)}_{T,M}(u)$ in Step 2 of the algorithm, we construct n_S of these statistics, each using $u = \pm\sqrt{2} z_s$. The values of the roots z_s, and of the corresponding weights w_s, are tabulated e.g. in Salzer et al. (1952).

In our case, we have used $n_S = 2$, which corresponds to $u = \pm 1$ with equal weight $\frac{1}{2}$; we note that in unreported experiments we tried $n_S = 4$ with the corresponding weights, but there were no changes up to the 4-th decimal in the empirical rejection frequencies.

[3] We note that AIC and HQ tend to grossly overstate R in all experiments, with this tendency worsening for lower values of η. Results are available upon request.

experiments, we use an "infeasible" version of each information criterion, assuming that the lag structure in (3.1) is known, so that the only quantity to be determined is the rank of cointegration R.

For all rank estimates, we report three measures of performance. Denoting the estimate of R at iteration $1 \leq mc \leq 1,000$ as $\widehat{R}_{mc}$, we define

$$ME = \frac{1}{1000} \sum_{mc=1}^{1000} \widehat{R}_{mc}, \tag{3.6}$$

$$STD = \frac{1}{1000} \sum_{mc=1}^{1000} \left(\widehat{R}_{mc} - ME \right)^2, \tag{3.7}$$

$$PCW = \frac{1}{1000} \sum_{mc=1}^{1000} I\left(\widehat{R}_{mc} \neq R \right), \tag{3.8}$$

Results are reported in Tables 4 and 5, where we analyse the properties of our estimator of R for a small-scale VAR with $N \in \{3, 4, 5\}$, and for the simple case of *i.i.d.* errors. Broadly speaking, as far as the PCW and ME indicators are concerned, the tables show that our procedure is very good on average at estimating R - and better than the best information criterion, BIC - for all values of N and T, even for the (rather extreme) case $\eta = 0.5$. This is true across all values of R, including the case of stationary vector ($R = N$) and lack of cointegration ($R = 0$). Such evidence is further corroborated by considering the STD indicator, which shows that our procedure is rarely wrong or very wrong, and systematically better than BIC in this respect. Indeed, the BIC performs (very) marginally better when $R = N$, but this is more than offset when considering that BIC tends to overstate R in general (as the values of ME indicate), and especially when $R = 0$ and $R = 1$; in particular, BIC seems to perform puzzlingly badly when errors are Gaussian. This is even more remarkable when thinking that information criteria are used under the (infeasible) assumption that the lag structure is known. In line with the theory, results are better for larger T; also, results do improve as η increases, although this becomes less evident as T increases. The impact of N is less clear, although generally speaking larger values of N worsen the performance of both BIC and our procedure when $R = 0$, whereas it has virtually no impact for other values of R. When errors are serially correlated (see Tables 6-9), results are affected, albeit marginally, but the relative performance between our procedure and BIC remains as described above. In particular, when $\theta = 0.5$ in (3.2), our procedure fares better than in the *i.i.d.* case when $R = 0$, worsening instead when $R = N$. In the latter case, as T increases (and as η becomes closer to 2), the impact of serial correlation is mitigated - see Tables 6 and 7. Conversely, the case of negative autocorrelation (Tables 8 and 9), results worsen in a more pronounced way, especially for small values of T and/or as N increases. Even in this case, however, our procedure is better than BIC for the case $R = 0$ (and in general for small R), with an almost indistinguishable performance when $R = N$.

In Tables 10-12, we report results from a further experiment for a medium-large VAR with $N \in \{10, 15, 20\}$, in order to evaluate the impact of N on our procedure. Our estimator delivers a superior performance with respect to information criteria for $R = 0$ and $R = 1$, especially for small values of η, when BIC grossly overstates R, even for large values of T. Conversely (and as a consequence), BIC is (albeit marginally) superior when $R = N$.

4. Real data examples

We illustrate our methodology by applying it to three datasets. We firstly reconsider the same dataset as in She and Ling (2020), namely a three-dimensional vector containing US interest rates (Section 4.1). We then revisit, in a shorter example, the (weak form of the) PPP, by using the same dataset as Falk and Wang (2003), comprising the three-dimensional vectors of exchange rates and CPIs for 12 countries (Section 4.2). Finally, we consider a portfolio of 6 cryptocurrency (Section 4.3)

4.1. Application to interest rate data

We illustrate our methodology through a small scale application, which is based on the same dataset as in She and Ling (2020), consisting of $N = 3$ monthly, seasonally unadjusted series: the 3-month Treasury Bill rate, the 1-year Treasury Bill rate (both from sales on the secondary market) and the Effective Federal Fund rate, spanning the period between February 1974 and February 1999 (thus corresponding to $T = 301$)[4]. Following She and Ling (2020), we consider the logs of the original data, say $Y_t = (y_{1,t}, y_{2,t}, y_{3,t})'$ where $y_{1,t}$ is the log of the 3-month Treasury Bill rate, $y_{2,t}$ is the log of the 1-year Treasury Bill rate, and $y_{3,t}$ represents the log of the Fed Fund rate. As suggested in Corollary 1, we do not demean the data and subtract the initial value from each series.

She and Ling (2020) carry out their analysis using the $VECM$ specification

$$\Delta Y_t = \alpha \left(\beta' Y_{t-1} + \rho_0 \right) + \sum_{j=1}^{p} \Phi_j \Delta Y_{t-j} + \varepsilon_t, \tag{4.1}$$

with $p = 1$ - i.e., the consider under a restricted constant specification like the one in (2.24). As Corollary 1 states, our set-up can be applied with no modifications in this case. Further, we point out that, in our analysis, we do not need to specify any model or lag structure for ΔY_t. In order to eliminate the effect of the initial condition, we subtract this from our original data.

As demonstrated above, the main feature of our contribution is that we do not require any prior knowledge of the index η. This advantage can be further understood upon observing the estimates of η obtained in She and Ling (2020) using Hill's estimator (see e.g. De Haan and Ferreira, 2007) - see in particular, Figure 6 in their paper. As is typical, the Hill's estimator does not produce a single value, but rather an interval of possible values: in the case of our dataset, η should range between 1 and 1.5 (thereby implying that the data have infinite variance and, possibly, infinite mean, and a homogeneous η across units, as customarily assumed). On account of this information, She and Ling (2020) report the critical values for the Likelihood Ratio tests developed by Caner (1998) for both the case $\eta = 1$ and $\eta = 1.5$, using both set of values as a decision rule.

We have applied our methodology using exactly the same specifications as described in Section 3, namely we have set $\kappa = 10^{-4}$ and $M = 100$. Also, we have implemented (3.4) using $n_S = 2$. In order to assess the robustness of our results to these specifications, we have also considered other values for M (including $M = T$) and $n_S = 4$.

[4]The data have been downloaded from the Federal Reserve Economic Data website, https://fred.stlouisfed.org

Table 1
Estimated rank

Panel A: results

	$\widehat{R}$	*Notes*
BCT	1	
She and Ling (2020)	2	($\alpha = 5\%$; $\eta = 1$)
She and Ling (2020)	2	($\alpha = 5\%$; $\eta = 1.5$)
She and Ling (2020)	0	($\alpha = 1\%$; $\eta = 1$)
She and Ling (2020)	2	($\alpha = 1\%$; $\eta = 1.5$)
Johansen's trace test	1	($\alpha = 5\%$; the same result is found for $\alpha = 1\%$)
Johansen's $\lambda_{\max}$ test	2	($\alpha = 5\%$; $\widehat{R} = 1$ when using $\alpha = 1\%$)

Panel B: sensitivity analysis

$(\kappa = 10^{-4}; n_S = 2)$	$\frac{0.05}{T}$	$\frac{0.05}{\ln T}$	$\frac{0.05}{N}$	$(\kappa = 10^{-2}; n_S = 2)$	$\frac{0.05}{T}$	$\frac{0.05}{\ln T}$	$\frac{0.05}{N}$
M				M			
100	1	1	1	100	1	1	1
$T/2$	1	1	1	$T/2$	1	1	1
T	1	1	1	T	1	1	1
$2T$	1	1	1	$2T$	1	1	1
$(\kappa = 10^{-4}; n_S = 4)$	$\frac{0.05}{T}$	$\frac{0.05}{\ln T}$	$\frac{0.05}{N}$	$(\kappa = 10^{-2}; n_S = 4)$	$\frac{0.05}{T}$	$\frac{0.05}{\ln T}$	$\frac{0.05}{N}$
M				M			
100	1	1	1	100	1	1	1
$T/2$	1	1	1	$T/2$	1	1	1
T	1	1	1	T	1	1	1
$2T$	1	1	1	$2T$	1	1	1

The top part of the table summarizes the findings using various procedures. The BCT procedure has been implemented with $M = 100$, nominal level for individual tests equal to $\frac{0.05}{T}$, $\kappa = 10^{-4}$ and $n_S = 4$; see also Panel B of the table for results obtained using different specifications.
In Panel A of the table, the results based on She and Ling (2020) are taken from Table 4 in their paper. The results corresponding to Johansen's procedure (Johansen, 1991) have been derived setting $p = 4$ in the specification of the VAR based on BIC.
In Panel B, we report results on $\widehat{R}$ obtained using different specifications as written in the table. In particular, in each sub-panel, the columns contain different values of the nominal level of the family-wise procedure, set equal to $\frac{0.05}{T}$, $\frac{0.05}{\ln(T)}$ and $\frac{0.05}{N}$.

Results in Table 1 show that our test finds one cointegration relationship (i.e., $\widehat{R} = 1$). We point out that - as the results reported in the table for different values of M, κ, etc... show - this result is robust to the various test specifications.
Our findings can be read in conjunction with the example in She and Ling (2020), and with a related contribution by Yap and Reinsel (1995). In particular, She and Ling (2020) conclude that there is some evidence in favour of two cointegration relationships (i.e., $R = 2$). Upon inspecting their results, this emerges clearly when using a 5% nominal level for the individual tests; however, evidence is much weaker when considering a 1% nominal level. In particular, in the latter case, should one use the critical values computed for $\eta = 1$, there would be no evidence of cointegration at all (i.e., $R = 0$), whereas using the critical values computed for the case $\eta = 1.5$, one would conclude that $R = 2$. We believe that this illustrates the dependence of tests based on second order asymptotics on the nuisance parameter η. Similarly, Yap and Reinsel (1995) considered the same three monthly series, albeit with a different time period (namely, between 1960 and 1979) and using critical values for the case $\eta = 2$. Interestingly, their paper finds mixed evidence in favour of either $R = 1$ or $R = 2$, concluding that a clear-cut answer "may depend on considerations beyond the test results".

4.2. Testing for the weak form of the PPP

In this exercise, we check the validity of the weak form of the PPP, using the same dataset as in Falk and Wang (2003). In particular, for $1 \leq c \leq 12$ countries, we individually verify the presence of cointegration in the vectors $(p_{c,t}, p_{US,t}, FX_{c,t})'$, where $p_{c,t}$ is the (log of the) CPI of country c at time t, $p_{US,t}$ is the (log of the) US CPI index, and $FX_{c,t}$ is the log of the exchange rate of country c currency vis-a-vis the dollar. The series are monthly, spanning from January 1973 to December 1999, corresponding to $T = 324$ for each country. Falk and Wang (2003) find evidence of heavy tails in this dataset (see their Table III), with series typically having finite mean but infinite variance. Thence, the authors test for cointegration using critical values based on allowing for heavy tails; even in this case, the authors require prior knowledge of the tail index η.

As in the previous sections, we subtract the initial value from each series: no other transformations are applied. The specifications of the test are broadly in line with the section above - i.e., we use $M = T$, $\kappa = 10^{-2}$ and a level for the individual tests given by $\frac{0.05}{\ln T}$.

TABLE 2
Estimated rank

	FW - Gaussian trace test		*FW - Gaussian* $\lambda_{\max}$ test		*FW - heavy tailed* trace test		*FW - heavy tailed* $\lambda_{\max}$ test		*BCT*
Country	*5% level*	*10% level*	*5% level*	*10% level*	*5% level*	*10% level*	*5% level*	*10% level*	
Belgium	Y	Y	Y	Y	Y	Y	Y	Y	N
Canada	N	N	N	N	N	N	N	N	Y
Denmark	Y	Y	Y	Y	N	Y	N	N	N
France	Y	Y	Y	Y	N	Y	N	Y	Y
Germany	N	N	N	N	N	N	N	N	N
Italy	N	Y	N	Y	N	N	N	N	Y
Japan	Y	Y	Y	Y	Y	Y	Y	Y	Y
Netherlands	Y	Y	Y	Y	Y	Y	Y	Y	N
Norway	Y	Y	N	N	N	Y	N	N	N
Spain	Y	Y	Y	Y	N	Y	N	N	N
Sweden	N	Y	N	N	N	N	N	N	N
UK	Y	Y	Y	Y	Y	Y	N	Y	Y

The entries in the table correspond to having found cointegration, and therefore the weak form of the PPP holding (denoted as "Y"), or having found no evidence of cointegration, and consequently of PPP (denoted as "N")
Results in the table corresponding to the paper by Falk and Wang (2003) are taken from their Tables VII (for the Gaussian case) and VIII (for the case with heavy tails). The test statistics employed are the maximum eigenvalue (λ_{max} column) and the trace test, as developed by Johansen (1991) for the Gaussian case. We refer to Table V in Falk and Wang (2003) for the critical values for both tests computed under the assumption of heavy-tailed data,

Results in Table 2 show that a minority of countries satisfy the weak form of the PPP; this is in stark contrast with what would be found if one employed the critical values valid for Gaussian data. This result can be read in conjunction with the findings in Falk and Wang (2003); results are indeed broadly similar, except for the case of Italy, where our test finds evidence of the PPP holding in weak form. Overall, we find that the PPP holds for 5 countries out of 12, which is closely in line with Falk and Wang (2003), where, when using critical values corresponding to heavy tailed distributions, the number of countries where cointegration is found ranges between 3 (when using a maximum eigenvalue test at the 5% nominal level) and 8 (when using a trace test at the 10% nominal level).

4.3. Application to a medium-sized portfolio of cryptocurrencies

In this application, we consider a portfolio comprising $N = 6$ cryptocurrencies, chosen among the ones with the largest market capitalisation. In the context of asset pricing, cointegration has played a prominent role since the seminal contribution by Diebold et al. (1994). In particular, evidence of cointegration relationships, in this literature, is linked to the possibility of constructing portfolios which are mean reverting, so that a profit can be made when such portfolios depart from their mean - the so-called "statistical arbitrage". Although this literature usually focuses on portfolios based on two assets which cointegrate (whence the expression "pairs trading"), it is possible to generalise the notion of statistical arbitrage to the multi-asset case (see Alexander et al., 2002). In such a case, having e.g. R cointegrating relationships entails that it is possible to construct R mean reverting portoflios, each having weights given by the relevant cointegration vector (suitably normalised to have coefficients adding up to 1). Studying cryptocurrency data in general, and investigating in particular the possibility of statistical arbitrage within this field, has been paid a growing level of attention (Makarov and Schoar, 2020). On the other hand, there is strong evidence that returns on cryptocurrencies may exhibit heavy tails (see Fry, 2018 and Pele et al., 2020).
In our analysis, we consider the following cryptocurrencies: Cardano (ADA), Bitcoin Cash (BCH), BitCoin (BTC), EOS, Ethereum (ETH), and XRP[5]. We use daily data from October 18, 2017 until September 18, 2020, which is equivalent to a sample of size $T = 1,066$ observations. As in the previous applications, each series has been rebased by subtracting its initial observation. In Table 3, we also report some evidence on the tail index of each series: all cryptocurrencies seem to exhibit heavy-tailed behaviour and have infinite variance, indeed with tail index floating around 1. Note that, despite some apparent heterogeneity in the tail indices, confidence intervals for the Hill's estimator show that the variables seem to satisfy the common tail index assumption which characterizes stable distributions.

The results indicate that $R = 1$ for all cases considered- that is, there is one cointegration relationship among the series. Note that Johansen's tests broadly indicate $R = 2$, with one exception where the estimate $R = 3$ is found. However, we note that, as the results by Caner (1998) show, in the presence of heavy tails Johansen's procedures have a tendency to overstate R. Our finding entails that there exists the possibility of statistical arbitrage in the cryptocurrency market, since one mean-reverting portfolio can be constructed. Whilst this is only the first step of the analysis, and a more thorough investigation goes beyond the scope of this section, we point out that estimating the weights required to construct such portfolio is possible by simply applying OLS: although the limiting distribution of such estimates obviously depends on the tail index, the results in e.g. She and Ling (2020) show that estimators are consistent for every value of η.

5. Conclusions

In this paper, we have proposed a methodology to estimate the rank of a cointegrated system, R, in the possible presence of heavy tails. Our procedure does not require the estimation of

[5]Data have been downloaded from https://www.coingecko.com/

TABLE 3
Estimated rank

Results and sensitivity analysis

Currency	*Tail index*
ADA	0.912 (0.60,1.22)
BCH	0.833 (0.55,1.11)
BTC	1.284 (0.85,1.71)
EOS	1.126 (0.74,1.50)
ETH	1.165 (0.77,1.55)
XRP	1.464 (0.97,1.95)

nominal level	1%	5%
Johansen's trace test	2	2
Johansen's $\lambda_{\max}$ test	2	3

$\left(\kappa = 10^{-4}; n_S = 2\right)$ M	$\frac{0.05}{T}$	$\frac{0.05}{\ln T}$	$\frac{0.05}{N}$	$\left(\kappa = 10^{-2}; n_S = 2\right)$ M	$\frac{0.05}{T}$	$\frac{0.05}{\ln T}$	$\frac{0.05}{N}$
$T/4$	1	1	1	$T/4$	1	1	1
$T/2$	1	1	1	$T/2$	1	1	1
T	1	1	1	T	1	1	1
$2T$	1	1	1	$2T$	1	1	1

$\left(\kappa = 10^{-4}; n_S = 4\right)$ M	$\frac{0.05}{T}$	$\frac{0.05}{\ln T}$	$\frac{0.05}{N}$	$\left(\kappa = 10^{-2}; n_S = 4\right)$ M	$\frac{0.05}{T}$	$\frac{0.05}{\ln T}$	$\frac{0.05}{N}$
$T/4$	1	1	1	$T/4$	1	1	1
$T/2$	1	1	1	$T/2$	1	1	1
T	1	1	1	T	1	1	1
$2T$	1	1	1	$2T$	1	1	1

In the top part of the table we report the estimated values of the tail index using the Hill's estimator - the package 'ptsuite' in R has been employed, using a number of order statistics equal to $k_T = 32$, which corresponds to $O(T^{1/2})$. We also report the number of cointegration relationships found by Johansen's procedure; this has been implemented using $p = 3$ lags in the VAR specification, as suggested using BIC.
We report the outcome of our test under various specifications in the bottom half of the table, similarly to 1.

nuisance parameters, or of any prior knowledge as to whether the data have finite variance or not; indeed, the procedure can be applied even in the case of finite second moments.
The proposed estimation technique is based on a sequence of tests. The tests use the eigenvalues of the sample second moment matrix of the data in levels, the largest R of which are shown to diverge at a higher rate, as T increases, than the remaining ones. Based on such rates, we propose a randomised statistic whose limiting distribution is a chi-squared under the null, and which diverges to positive infinity under the alternative. Each test is then carried out with a critical value tending to infinity, thus ensuring consistency of the estimated rank. Our simulations show that the estimator has good finite sample properties, and it is particularly good at estimating the rank when this is greater than zero under virtually any circumstances.

6. Technical lemmas

Henceforth, we denote the L_p-norm of a random variable X as $|X|_p$; also, the subscript "$\perp$" denotes the orthogonal complement of a matrix.

Lemma A.1. *Consider a sequence U_T for which $E|U_T|^\delta \leq a_T$, where a_T is a positive, monotonically non-decreasing sequence, and let $d_\delta = 1$ if $\delta \leq 2$ and zero otherwise. Then there exists a $C_0 < \infty$ such that*

$$\limsup_{T\to\infty} \frac{|U_T|}{a_T^{1/\delta} (\ln T)^{(1+d_\delta+\epsilon)/\delta}} \leq C_0 \ a.s.$$

Proof. It holds that

$$E \max_{1\leq t\leq T} |U_T|^\delta \leq a_T (\ln T)^{d_\delta},$$

having used Theorem B in Serfling (1970) when $\delta > 2$, and the first Theorem in the same paper when $\delta \leq 2$. Thus

$$\sum_{T=1}^{\infty} \frac{1}{T} P\left(\max_{1\leq t\leq T} |U_T| > a_T^{1/\delta} (\ln T)^{(1+d_\delta+\epsilon)/\delta}\right) \leq \sum_{T=1}^{\infty} \frac{E\max_{1\leq t\leq T} |U_T|^\delta}{T a_T (\ln T)^{1+d_\delta+\epsilon}} = \sum_{T=1}^{\infty} \frac{1}{T(\ln T)^{1+\epsilon}} < \infty,$$

having used Markov inequality. By the proof of Corollary 2.4 in Cai (2006) (see also Lemma A.5 in Horváth and Trapani, 2019), it follows that

$$\sum_{T=1}^{\infty} P\left[|U_T| > a_T^{1/\delta} (\ln T)^{1+d_\delta+\epsilon}\right] < \infty,$$

which entails that the sequence U_T converges completely (see Hsu and Robbins, 1947). The desired result now follows immediately from the Borel-Cantelli lemma. □

Lemma A.2. *We assume that Assumption 2(ii) is satisfied. Then there exists a random T_0 and a positive definite matrix D such that, for all $T \geq T_0$*

$$\sum_{t=1}^{T} x_t x_t' \geq D \frac{T^{1+2/\eta}}{(\ln\ln T)^{2/\eta}}.$$

Proof. The lemma follows immediately upon noting that, by Assumption 2*(ii)* and Theorem 1.3 in Jain (1982), it holds a.s. that, for all $b \in \mathbb{R}^{N-R}$, $b \neq 0$

$$\liminf_{T\to\infty} \frac{(\ln\ln T)^{2/\eta}}{T^{1+2/\eta}} \sum_{t=1}^{T} (b' x_t)^2 = c_0 > 0.$$

□

Lemma A.3. *Let Assumptions 1(i) and 2(ii) be satisfied. Then there exists a random variable T_0 and a constant $0 < c_0 < \infty$ such that, for all $T \geq T_0$*

$$\lambda^{(j)}\left(P\sum_{t=1}^{T} x_t x_t' P'\right) \geq c_0 \frac{T^{1+2/\eta}}{(\ln\ln T)^{2/\eta}},$$

for all $1 \leq j \leq N-R$.

Proof. The lemma is an immediate consequence of Lemma A.2. Indeed, by the multiplicative Weyl's inequality (see Theorem 7 in Merikoski and Kumar, 2004), it holds that

$$\lambda^{(j)}\left(P\sum_{t=1}^{T}x_t x_t' P'\right) \geq \lambda^{(\min)}\left(P'P\right)\lambda^{(j)}\left(\sum_{t=1}^{T}x_t x_t'\right),$$

and Lemma A.2 readily implies that, for every j

$$\liminf_{T\to\infty}\frac{(\ln\ln T)^{2/\eta}}{T^{1+2/\eta}}\lambda^{(j)}\left(\sum_{t=1}^{T}x_t x_t'\right) > 0 \quad \text{a.s.}$$

Assumption 1*(i)* entails that $\lambda^{(\min)}\left(P'P\right) > 0$, whence the desired result. □

Lemma A.4. *Let Assumption 2 be satisfied. Then, for all $\epsilon > 0$ it holds that*

$$\lambda^{(\max)}\left(\sum_{t=1}^{T}u_t u_t'\right) = o_{a.s.}\left(T^{2/p}\left(\ln T\right)^{2(2+\epsilon)/p}\right),$$

where $0 \leq p < \eta$ when $\eta \leq 2$ with $E\left\|\varepsilon_t\right\|^{\eta} = \infty$, and $p = 2$ when $\eta = 2$ with $E\left\|\varepsilon_t\right\|^{\eta} < \infty$.

Proof. The proof does not require any distinction between $p < 2$ and $p = 2$. We have

$$\lambda^{(\max)}\left(\sum_{t=1}^{T}u_t u_t'\right) \leq \sum_{i=1}^{N}\sum_{t=1}^{T}u_{i,t}^2.$$

Also, given that $u_{i,t} = \sum_{k=1}^{N}\sum_{m=0}^{\infty}C_{m,ik}^{*}\varepsilon_{k,t-m}$

$$\sum_{t=1}^{T}u_{i,t}^2 = \sum_{t=1}^{T}\left(\sum_{m=0}^{t-1}C_{m,ik}^{*}\varepsilon_{k,t-m}\right)^2 = \sum_{k,k'=1}^{N}\sum_{t=1}^{T}\sum_{m=0}^{t-1}\sum_{n=0}^{t-1}C_{m,ik}^{*}C_{n,ik'}^{*}\varepsilon_{k,t-m}\varepsilon_{k',t-n}.$$

It holds that

$$\begin{aligned}
\left|\sum_{t=1}^{T}u_{i,t}^2\right|_{p/2}^{p/2} &\leq \sum_{k,k'=1}^{N}\sum_{t=1}^{T}\sum_{m=0}^{t-1}\sum_{n=0}^{t-1}\left|C_{m,ik}^{*}\right|^{p/2}\left|C_{n,ik'}^{*}\right|^{p/2}\left|\varepsilon_{k,t-m}\varepsilon_{k',t-n}\right|_{p/2}^{p/2} \\
&\leq \sum_{k,k'=1}^{N}\sum_{t=1}^{T}\sum_{m=0}^{t-1}\sum_{n=0}^{t-1}\left|C_{m,ik}^{*}\right|^{p/2}\left|C_{n,ik'}^{*}\right|^{p/2}\left|\varepsilon_{k,t-m}\right|_{p}^{p/2}\left|\varepsilon_{k',t-n}\right|_{p}^{p/2} \\
&\leq c_0\sum_{k,k'=1}^{N}\sum_{t=1}^{T}\left(\sum_{m=0}^{t-1}\left|C_{m,ik}^{*}\right|^{p/2}\right)\left(\sum_{n=0}^{t-1}\left|C_{n,ik'}^{*}\right|^{p/2}\right) \\
&\leq c_0 T,
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality and the fact that $C_{m,ik}^{*} = O\left(\exp\left(-c_0 m\right)\right)$ on account of Assumption 1*(ii)*. Using Lemma A.1, the desired result follows immediately. □

Lemma A.5. *Let Assumptions 1-2 be satisfied. Then, for all $\epsilon > 0$, it holds that*

$$\lambda^{(\max)}\left(P\sum_{t=1}^{T}x_t u_t' + \sum_{t=1}^{T}u_t x_t' P'\right) = o_{a.s.}\left(T^{2/p}\left(\ln T\right)^{2(2+\epsilon)/p}\right),$$

where $0 \leq p < \eta$ when $\eta \leq 2$ with $E\left\|\varepsilon_t\right\|^{\eta} = \infty$, and $p = 2$ when $\eta = 2$ with $E\left\|\varepsilon_t\right\|^{2} < \infty$.

Proof. Recall that $C_{h,k}$ indicates the element of C in position h, k. Using the definition of Frobenius norm and the L_p-norm inequality, it follows that

$$\lambda^{(\max)}\left(B\sum_{t=1}^{T}x_t u_t' + \sum_{t=1}^{T}u_t x_t' B'\right) \leq 2\sum_{k,h=1}^{N}\left|\sum_{t=1}^{T}C_{h,k}\sum_{s=1}^{t}\varepsilon_{k,t}\sum_{m=0}^{\infty}C^*_{m,jh}\varepsilon_{h,t-m}\right|.$$

Thus, using the C_r-inequality, after some algebra it holds that

$$E\left|\sum_{k,h=1}^{N}\left|\sum_{t=1}^{T}C_{h,k}\sum_{s=1}^{t}\varepsilon_{k,t}\sum_{m=0}^{\infty}C^*_{m,jh}\varepsilon_{h,t-m}\right|\right|^{p} \leq c_0\sum_{k,h=1}^{N}E\left|\sum_{t=1}^{T}\sum_{s=1}^{t}\varepsilon_{k,t}\sum_{m=0}^{\infty}C^*_{m,jh}\varepsilon_{h,t-m}\right|^{p},$$

where the constant $c_0 < \infty$ depends only on p, N and R. It is convenient to consider separately the cases where $0 \leq p < \eta$ when $\eta \leq 2$ with $E\left\|\varepsilon_t\right\|^{\eta} = \infty$, and $p = 2$ when $\eta = 2$ with $E\left\|\varepsilon_t\right\|^{2} < \infty$. In the former case, let $p < 2$. In order to lighten up the notation, we omit the subscripts j, h and k, and study

$$\sum_{m=0}^{\infty}C^*_m\sum_{t=1}^{T}\sum_{s=1}^{t}\varepsilon_s\varepsilon_{t-m}.$$

It holds that

$$\begin{aligned}\sum_{m=0}^{\infty}C^*_m\sum_{t=1}^{T}\sum_{s=1}^{t}\varepsilon_s\varepsilon_{t-m} &= \sum_{m=0}^{\infty}C^*_m\sum_{t=1}^{T}\sum_{s=1}^{t}\varepsilon_s\varepsilon_t + \sum_{m=0}^{\infty}C^*_m\sum_{j=1}^{m}\sum_{t=1}^{T-m-j}\varepsilon_t\varepsilon_{t+j}\\ &\quad - \sum_{m=0}^{\infty}C^*_m\sum_{j=1}^{m}\varepsilon_{T+1-j}\sum_{t=1}^{T+1-j}\varepsilon_t + \sum_{m=0}^{\infty}C^*_m\sum_{j=1}^{m}\varepsilon_{T+1-j}\sum_{i=1}^{j}\varepsilon_{T-m-i}\\ &= I + II + III + IV,\end{aligned}$$

with the convention that $\sum_{j=1}^{0} = 0$. Consider I

$$I \leq c_0\left|\sum_{t=1}^{T}\sum_{s=1}^{t}\varepsilon_s\varepsilon_t\right|,$$

by the summability of C^*_m. Also

$$\left|\sum_{t=1}^{T}\sum_{s=1}^{t}\varepsilon_s\varepsilon_t\right| \leq \left|\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varepsilon_s\varepsilon_t\right| + \left|\sum_{t=1}^{T}\varepsilon_t^2\right| = I_a + I_b.$$

Similar passages as above yield

$$\left|\sum_{t=1}^{T}\varepsilon_t^2\right|_{p/2}^{p/2} \leq \sum_{t=1}^{T}|\varepsilon_t|_p^p \leq c_0 T,$$

so that Lemma A.1 entails that $I_b = o_{a.s.}\left(T^{2/p}(\ln T)^{2(1+\epsilon)/p}\right)$ for all $\epsilon > 0$. Turning to I_a, when $p \leq 1$

$$\left|\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varepsilon_s\varepsilon_t\right|_p^p \leq \sum_{t=1}^{T}\sum_{s=1}^{t-1}|\varepsilon_s\varepsilon_t|_p^p = \sum_{t=1}^{T}\sum_{s=1}^{t-1}|\varepsilon_s|_p^p\,|\varepsilon_t|_p^p \leq c_0 T^2.$$

When $1 < p \leq 2$, the von Bahr-Esseen inequality (von Bahr and Esseen, 1965) can be applied, giving

$$\left|\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varepsilon_s\varepsilon_t\right|_p^p \leq c_0\sum_{t=1}^{T}\left|\sum_{s=1}^{t-1}\varepsilon_s\varepsilon_t\right|_p^p = c_0\sum_{t=1}^{T}\left|\sum_{s=1}^{t-1}\varepsilon_s\right|_p^p|\varepsilon_t|_p^p \leq c_1\sum_{t=1}^{T}\left|\sum_{s=1}^{t-1}\varepsilon_s\right|_p^p \leq c_2\sum_{t=1}^{T}\sum_{s=1}^{t-1}|\varepsilon_s|_p^p \leq c_3 T^2.$$

By Lemma A.1, it follows that, for all $0 < p \leq 2$, $I_a = o_{a.s.}\left(T^{2/p}(\ln T)^{(1+\epsilon)/p}\right)$. Turning to II, note that

$$\left|\sum_{m=1}^{\infty}C_m^*\sum_{j=1}^{m}\sum_{t=1}^{T-j}\varepsilon_t\varepsilon_{t+j}\right|_{p/2}^{p/2} \leq \sum_{m=1}^{\infty}|C_m^*|^{p/2}\sum_{j=1}^{m}\sum_{t=1}^{T-j}|\varepsilon_t\varepsilon_{t+j}|_{p/2}^{p/2} \leq c_0 T\sum_{m=1}^{\infty}m\,|C_m^*|^{p/2} \leq c_0 T,$$

so that $II = o_{a.s.}\left(T^{2/p}(\ln T)^{2(1+\epsilon)/p}\right)$. Finally, as far as III is concerned, we apply the same logic as above to get

$$\left|\sum_{m=1}^{\infty}C_m^*\sum_{j=0}^{m-1}\varepsilon_{T-j}\sum_{t=1}^{T}\varepsilon_t\right|_{p/2}^{p/2} \leq c_0\sum_{m=1}^{\infty}|C_m^*|^{p/2}\sum_{j=0}^{m-1}\sum_{t=1}^{T}|\varepsilon_{T-j}\varepsilon_t|_{p/2}^{p/2} \leq c_0 T,$$

which yields $III = o_{a.s.}\left(T^{2/p}(\ln T)^{2(1+\epsilon)/p}\right)$ as above. Finally, note that, as above

$$\begin{aligned}\left|\sum_{m=0}^{\infty}C_m^*\sum_{j=1}^{m}\varepsilon_{T+1-j}\sum_{i=1}^{j}\varepsilon_{T-m-i}\right|_{p/2}^{p/2} &\leq \sum_{m=0}^{\infty}C_m^*\sum_{j=1}^{m}\left|\varepsilon_{T+1-j}\sum_{i=1}^{j}\varepsilon_{T-m-i}\right|_{p/2}^{p/2}\\ &= \sum_{m=0}^{\infty}C_m^*\sum_{j=1}^{m}|\varepsilon_{T+1-j}|_{p/2}^{p/2}\sum_{i=1}^{j}|\varepsilon_{T-m-i}|_{p/2}^{p/2}\\ &\leq c_0\sum_{m=0}^{\infty}C_m^*\sum_{j=1}^{m}j \leq c_1,\end{aligned}$$

again by the exponential summability of C_m^*; this entails that $IV = o_{a.s.}\left((\ln T)^{2(1+\epsilon)/p}\right)$. The lemma follows by putting everything together.

The case $\eta = 2$ with finite variance follows from exactly the same calculations, with $p = 2$. □

Lemma A.6. *We assume that Assumptions 1-3 are satisfied. Then it holds that $z_t = l'\Delta y_t$ is a strongly mixing sequence with mixing numbers $\alpha_m = O(\rho^m)$, for all l and some $0 < \rho < 1$.*

Proof. In order to prove the lemma, recall the three following equivalent representations (see Johansen, 1995):

$$\Delta y_t = C(z)\varepsilon_t \tag{A.1}$$
$$= C\varepsilon_t + C^*(z)\varepsilon_t - C^*(z)\varepsilon_{t-1} \tag{A.2}$$
$$= ab'y_{t-1} + \sum_{j=1}^{q}\Gamma_j\Delta y_{t-j} + \varepsilon_t, \tag{A.3}$$

where $C^*(z)$ is an invertible filter, a and b are $N\times R$ matrices of full rank R such that $b'C = 0$, and the autoregressive polynomial $I_N - \sum_{j=1}^{q}\Gamma_j L^j$ is invertible (we refer to Johansen, 1995 for the full details).

We prove the lemma by considering separately the cases $R = 0$, $R = N$ and $0 < R < N$. When $R = 0$, (A.2) boils down to

$$\Delta y_t = C^*(z)\varepsilon_t - C^*(z)\varepsilon_{t-1}.$$

We start by showing that $C^*(z)\varepsilon_t$ is strong mixing with geometrically declining mixing numbers, by verifying the conditions of Lemma 2.2 and Theorem 2.1 in Pham and Tran (1985). Firstly, Assumption 3 ensures that the error term ε_t satisfies condition (i) in Lemma 2.2 in Pham and Tran (1985). Also, $C^*(z)$ is invertible[6], and, using Assumption 1*(ii)*, it is easy to see that $\sum_{j=0}^{\infty}\left\|C_j^*\right\| < \infty$. Hence, Theorem 2.1 in Pham and Tran (1985) can be applied, yielding that $C^*(z)\varepsilon_t$ is a strongly mixing sequence with geometrically declining mixing numbers. Using Theorem 14.1 in Davidson (1994), it follows that both Δy_t and $l'\Delta y_t$ are also strongly mixing sequence with geometrically declining mixing numbers. When $R = N$, exactly the same arguments can be applied to (A.1), noting that in this case $C(z)$ is invertible, and that the condition $\sum_{j=0}^{\infty}\|C_j\| < \infty$ follows immediately from Assumption 1*(ii)*.

Finally, consider the case $0 < R < N$. By Lemma A.1*(iv)* in Hansen (2005), the $N\times N$ matrix $(b, a_\perp)$ has full rank. Thus, any $l \in \mathbb{R}^N$ can be expressed as $l = bv_1 + a_\perp v_2$, where v_1 is $R\times 1$ and v_2 is $(N-R)\times 1$. Therefore it holds that

$$z_t = l'\Delta y_t = v_1'b'\Delta y_t + v_2'a_\perp'\Delta y_t. \tag{A.4}$$

Using (A.2) and (A.3) respectively, this entails that

$$z_t = v_1'b'C^*(z)\varepsilon_t - v_1'b'C^*(z)\varepsilon_{t-1} + v_2'a_\perp'\Delta y_t.$$

Note also that, by (A.3), $v_2'a_\perp'\Delta y_t$ is the cross-sectional aggregation of a finite order, stable AR model. This entails that $v_2'a_\perp'\Delta y_t$ can also be expressed as a finite order, stable AR model (Lütkepohl, 1984), and therefore it can be inverted into an $MA(\infty)$ representation whose are linear combinations of ε_t. By Corollary 1.2 in Teräsvirta (1977),[7], z_t is therefore an invertible

[6] We point out that in the paper by Pham and Tran (1985), condition (ii) in their Lemma 2.2 contains a typo, as it requires (using our notation) that $\sum_{j=0}^{\infty}C_j^*z^j \neq 0$ for all $|z| \leq 1$ - upon inspecting the proof, this is required in order to invert the polynomial $C^*(L)$, and thus it should instead read $\det\left(\sum_{j=0}^{\infty}C_j^*z^j\right) \neq 0$ for all $|z| \leq 1$, where $\det(A)$ is the determinant of matrix A.

[7] We note - see also footnote 5 in Smeekes and Urbain (2014) - that the results in the paper by Teräsvirta apply to finite order MA processes, but can be readily extended to the case of infinite order ones.

infinite order MA process. Further, it is easy to see that, by Assumption 1 *(ii)*, the coefficients of such MA process are summable. Hence, z_t satisfies all the assumptions of Corollary 4(b) in Withers (1981), and it therefore is a strongly mixing sequence with geometrically declining mixing numbers. □

7. Proofs

Henceforth, we let E^* and V^* denote the expected value and the variance with respect to P^* respectively.

Proof of Proposition 1. The result follows from Weyl's inequality (see e.g. Horn and Johnson, 2012) and Lemmas A.3, A.4 and A.5. Indeed, it holds that

$$\lambda^{(j)}\left(S_{11}\right) \geq \lambda^{(j)}\left(P\sum_{t=1}^{T}x_t x_t' P'\right)+\lambda^{(\min)}\left(\sum_{t=1}^{T}u_t u_t' + P\sum_{t=1}^{T}x_t u_t' + \sum_{t=1}^{T}u_t x_t' P'\right), \quad (A.1)$$

$$\lambda^{(j)}\left(S_{11}\right) \leq \lambda^{(j)}\left(P\sum_{t=1}^{T}x_t x_t' P'\right)+\lambda^{(\max)}\left(\sum_{t=1}^{T}u_t u_t' + P\sum_{t=1}^{T}x_t u_t' + \sum_{t=1}^{T}u_t x_t' P'\right). \quad (A.2)$$

Further, it holds that

$$\lambda^{(\max)}\left(\sum_{t=1}^{T}u_t u_t' + P\sum_{t=1}^{T}x_t u_t' + \sum_{t=1}^{T}u_t x_t' P'\right)$$
$$\leq \lambda^{(\max)}\left(\sum_{t=1}^{T}u_t u_t'\right)+\lambda^{(\max)}\left(P\sum_{t=1}^{T}x_t u_t' + \sum_{t=1}^{T}u_t x_t' P'\right). \quad (A.3)$$

In light of (A.3), Lemmas A.4 and A.5 entail that

$$\lambda^{(\max)}\left(\sum_{t=1}^{T}u_t u_t' + P\sum_{t=1}^{T}x_t u_t' + \sum_{t=1}^{T}u_t x_t' P'\right) = o_{a.s.}\left(T^{2/p}\left(\ln T\right)^{2(2+\epsilon)/p}\right),$$

for all $0 \leq p < \eta$ when $\eta \leq 2$ with $E\left\|\varepsilon_t\right\|^{\eta} = \infty$, and $p = 2$ when $\eta = 2$ with $E\left\|\varepsilon_t\right\|^{\eta} < \infty$. Consider (A.1); when $0 \leq j \leq N-R$, the term that dominates is $\lambda^{(j)}\left(P\sum_{t=1}^{T}x_t x_t' P'\right)$, and (2.6) follows immediately from Lemma A.3. When $j > N-R$, $\lambda^{(j)}\left(P\sum_{t=1}^{T}x_t x_t' P'\right) = 0$ by construction, and therefore (2.7) follows from (A.2). □

Proof of Proposition 2. Lemma A.6 entails that, for all $l \neq 0$, $z_t = l'\Delta y_t$ is strong mixing with geometrically declining mixing numbers. Thence, (2.8) follows immediately from Theorem 2.1 in Trapani (2014). Consider now (2.9), and define the sequence

$$\varphi_T = \frac{T^{1/\eta}}{\left(\ln T\right)^{(2+\epsilon)/\eta}}.$$

It holds that

$$\sum_{t=1}^{T} z_t^2 = \sum_{t=1}^{T} z_t^2 I\left(|z_t| > \varphi_T\right) + \sum_{t=1}^{T} z_t^2 I\left(|z_t| \leq \varphi_T\right)$$
$$\geq \sum_{t=1}^{T} z_t^2 I\left(|z_t| > \varphi_T\right) \geq \varphi_T^2 \sum_{t=1}^{T} I\left(|z_t| > \varphi_T\right).$$

Define the array $\omega_{T,t} = I\left(|z_t| > \varphi_T\right)$. We define the sigma-field ${}^{\omega}\mathcal{F}_{T,0}^{t} = \sigma\left(\omega_{Tt}, .., \omega_{T0}\right) = \bigcup_{i=0}^{t} \sigma\left(\omega_{Ti}\right)$. Note that, by construction, $\sigma\left(\omega_{Ti}\right) = \left\{\varnothing, \Omega, \left[-\varphi_T, \varphi_T\right], \left\{\left(-\infty, -\varphi_T\right) \cup \left(\varphi_T, \infty\right)\right\}\right\}$, so that $\sup_T \sigma\left(\omega_{Ti}\right) \subset \sigma\left(z_i\right)$. This entails that $\sup_T \left({}^{\omega}\mathcal{F}_{T,0}^{t}\right) \subset \bigcup_{i=0}^{t} \sigma\left(z_i\right)$, so that

$$\alpha_k^{\omega} = \sup_T \sup_t \alpha\left({}^{\omega}\mathcal{F}_{T,0}^{t}, {}^{\omega}\mathcal{F}_{T,t+k}^{\infty}\right) \leq \alpha_k,$$

where $\alpha\left({}^{\omega}\mathcal{F}_{T,0}^{t}, {}^{\omega}\mathcal{F}_{T,t+k}^{\infty}\right)$ is the mixing number of order k, and α_k is the mixing number of order k associated to the sequence z_t. Since $\alpha_k = O\left(\rho^k\right)$, $0 < \rho < 1$, it also follows that the sequence ω_{Tt} is strong mixing with geometrically declining mixing numbers. Note also that $\omega_{T,t}$ has finite moments up to any order; also, by Assumption 2*(i)*, it holds that $E\omega_{T,t} = \left(c_{l,1} + c_{l,2}\right) \varphi_T^{-\eta} L\left(\varphi_T\right)\left(1 + o\left(1\right)\right)$. Letting $\overline{\omega}_{T,t} = \omega_{T,t} - E\omega_{T,t}$, equation (5.1) in Rio (1995) ensures that

$$P\left(\max_{1 \leq s \leq T} \left|\sum_{t=1}^{s} \varphi_T^{\eta} \overline{\omega}_{T,t}\right| > 2x_T\right) \leq \frac{c_0}{x_T^2} T \int_0^1 \left(\alpha_k^{\omega}\right)^{-1}\left(\frac{u}{2}\right) Q_w^2\left(u\right) du, \tag{A.4}$$

where

$$x_T = \sum_{t=1}^{T} E\left(\varphi_T^{\eta} \omega_{T,t}\right),$$

and $Q_w\left(u\right)$ is the quantile function of $\varphi_T^{\eta} \overline{\omega}_{T,t}$. Consider now the function $f\left(x\right) = x \ln\left(1 + x\right)$, and let

$$f^*\left(y\right) = \sup_{x > 0}\left(xy - f\left(x\right)\right)$$

be its Young dual function (see Appendix A in Rio, 1995); as $x \to \infty$, $f^*\left(y\right) = \exp\left(y\right)$. Then it holds that

$$\int_0^1 \left(\alpha_k^{\omega}\right)^{-1}\left(\frac{u}{2}\right) Q_w^2\left(u\right) du \leq \left\|\left(\alpha_k^{\omega}\right)^{-1}\left(U\right)\right\|_{f^*} \left\|\left(\varphi_T^{\eta} \overline{\omega}_{T,t}\right)^2\right\|_f,$$

where U is a random variable with a uniform distribution on $[0, 1]$ and $\|X\|_f$ is the Luxemburg norm (Luxemburg, 1955) of a random variable X with respect to the function f, i.e.

$$\|X\|_f = \inf\left\{c > 0 : Ef\left(c^{-1}|X|\right) \leq 1\right\}.$$

Equation (1.29) in Rio (1995) ensures that $\left\|\alpha^{-1}\left(U\right)\right\|_{f^*} < \infty$. Also

$$\left\|\left(\varphi_T^{\eta} \overline{\omega}_{T,t}\right)^2\right\|_f = \inf\left\{c > 0 : Ef\left(c^{-1}\left|\varphi_T^{\eta} \overline{\omega}_{T,t}\right|^2\right) \leq 1\right\}.$$

Note that we have

$$Ef\left(c^{-1}\left|\varphi_T^{\eta}\overline{\omega}_{T,t}\right|^2\right)=c^{-1}\left(\varphi_T^{\eta}-1\right)^2\varphi_T^{-\eta}\ln\left(1+c^{-1}\left(\varphi_T^{\eta}-1\right)^2\right)+c^{-1}\left(1-\varphi_T^{-\eta}\right)\ln\left(1+\frac{1}{c}\right);$$

the choice $c=\varphi_T^{\eta}\ln\left(\varphi_T^{2\eta}\right)$ can be shown to correspond to $Ef\left(c^{-1}\left|\varphi_T^{\eta}\overline{\omega}_{T,t}\right|^2\right)<1$ after some algebra. Therefore

$$\int_0^1\left(\alpha_k^{\omega}\right)^{-1}\left(\frac{u}{2}\right)Q_w^2\left(u\right)du=O\left(\varphi_T^{\eta}\ln\left(\varphi_T\right)\right).$$

Noting that, by definition, $x_T=T$, equation (A.4) yields

$$P\left(\max_{1\leq s\leq T}\left|\sum_{t=1}^{s}\varphi_T^{\eta}\overline{\omega}_{T,t}\right|>2x_T\right)\leq c_0\frac{1}{\left(\ln T\right)^{1+\epsilon}},$$

which entails that

$$\sum_{T=1}^{\infty}\frac{1}{T}P\left(\max_{1\leq s\leq T}\left|\sum_{t=1}^{s}\varphi_T^{\eta}\overline{\omega}_{T,t}\right|>2x_T\right)<\infty.$$

By Lemma A.1, this entails that

$$\limsup_{T\to\infty}\frac{\sum_{t=1}^{T}\varphi_T^{\eta}\overline{\omega}_{T,t}}{\sum_{t=1}^{T}E\left(\varphi_T^{\eta}\omega_{T,t}\right)}=0\text{ a.s.},$$

so that we can write

$$\frac{\sum_{t=1}^{T}\varphi_T^{\eta}\omega_{T,t}}{\sum_{t=1}^{T}E\left(\varphi_T^{\eta}\omega_{T,t}\right)}=1+o_{a.s.}\left(1\right).$$

Putting all together, (2.9) obtains.

□

Proof of Theorem 1. Consider (2.10). By Theorem 7 in Merikoski and Kumar (2004)

$$\lambda^{(j)}\left(S_{00}^{-1}S_{11}\right)\geq\lambda^{(j)}\left(S_{11}\right)\lambda^{(\min)}\left(S_{00}^{-1}\right)=\frac{\lambda^{(j)}\left(S_{11}\right)}{\lambda^{(\max)}\left(S_{00}\right)};$$

combining (2.6) and (2.8), we obtain (2.10). Turning to (2.11), applying again Theorem 7 in Merikoski and Kumar (2004), it holds that

$$\lambda^{(j)}\left(S_{00}^{-1}S_{11}\right)\leq\lambda^{(j)}\left(S_{11}\right)\lambda^{(\max)}\left(S_{00}^{-1}\right)=\frac{\lambda^{(j)}\left(S_{11}\right)}{\lambda^{(\min)}\left(S_{00}\right)};$$

thence, (2.11) follows from (2.7) and (2.9). □

Proof of Theorem 2. The proof follows exactly the same lines as in Horváth and Trapani (2019) and we therefore omit most passages. Under H_0, it follows from Theorem 1 that

$$P\left(\omega:\lim_{T\to\infty}\exp\left\{-T^{1-\kappa-\epsilon}\right\}\phi_T^{(j)}=\infty\right)=1,\tag{A.5}$$

for every j and any $\epsilon > 0$. Let $\Phi(\cdot)$ denote the standard normal distribution, and note that

$$E^*\zeta_i^{(j)} = \Phi\left(u/\phi_T^{(j)}\right) \quad \text{and} \quad E^*\left(\zeta_i^{(j)} - E^*\zeta_i^{(j)}\right)^2 = \Phi\left(u/\phi_T^{(j)}\right)\left[1 - \Phi\left(u/\phi_T^{(j)}\right)\right].$$

We can write

$$M^{-1/2}\sum_{i=1}^{M}\left(\zeta_i^{(j)} - \frac{1}{2}\right) = M^{-1/2}\sum_{i=1}^{M}\left[I\left(\xi_i^{(j)} \leq 0\right) - \frac{1}{2}\right] + M^{-1/2}\sum_{i=1}^{M}\left(\Phi\left(u/\phi_T^{(j)}\right) - \frac{1}{2}\right)$$
$$+ M^{-1/2}\sum_{i=1}^{M}\left[I\{\left(\xi_i^{(j)} \leq u/\phi_T^{(j)}\right) - I\left(\xi_i^{(j)} \leq 0\right) - \left(\Phi\left(u/\phi_T^{(j)}\right) - \frac{1}{2}\right)\right].$$

It holds that

$$E^*\left(M^{-1/2}\sum_{i=1}^{M}\left[I\left(\xi_i^{(j)} \leq u/\phi_T^{(j)}\right) - I\left(\xi_i^{(j)} \leq 0\right) - \left(\Phi\left(u/\phi_T^{(j)}\right) - \frac{1}{2}\right)\right]\right)^2$$
$$= E^*\left[I\left(\xi_1^{(j)} \leq u/\phi_T^{(j)}\right) - I\left(\xi_1^{(j)} \leq 0\right) - \left(\Phi\left(u/\phi_T^{(j)}\right) - \frac{1}{2}\right)\right]^2,$$

on account of the independence of the $\xi_i^{(j)}$s across i. Given that

$$E^*\left(I\left(\xi_1^{(j)} \leq u/\phi_T^{(j)}\right) - I\left(\xi_1^{(j)} \leq 0\right)\right) = \Phi\left(u/\phi_T^{(j)}\right) - \frac{1}{2}, \tag{A.6}$$

$$V^*\left(I\left(\xi_1^{(j)} \leq u/\phi_T^{(j)}\right) - I\left(\xi_1^{(j)} \leq 0\right)\right) \leq \Phi\left(u/\phi_T^{(j)}\right) - \frac{1}{2}, \tag{A.7}$$

we have

$$E^*\left[I\left(\xi_1^{(j)} \leq u/\phi_T^{(j)}\right) - I\left(\xi_1^{(j)} \leq 0\right) - \left(\Phi\left(u/\phi_T^{(j)}\right) - \frac{1}{2}\right)\right]^2 \leq \Phi\left(u/\phi_T^{(j)}\right) - \frac{1}{2} \leq \frac{|u|}{\sqrt{2\pi}\phi_T^{(j)}} < \infty.$$

Thus

$$\int_U E^*\left[I\left(\xi_1^{(j)} \leq u/\phi_T^{(j)}\right) - I\left(\xi_1^{(j)} \leq 0\right) - \left(\Phi\left(u/\phi_T^{(j)}\right) - \frac{1}{2}\right)\right]^2 dF(u) \leq \int_U \frac{|u|}{\sqrt{2\pi}\phi_T^{(j)}} dF(u).$$

Also, it follows immediately that

$$M^{-1/2}\sum_{i=1}^{M}\left(\Phi\left(u/\phi_T^{(j)}\right) - \frac{1}{2}\right) \leq \left(\frac{M^{1/2}|u|}{\sqrt{2\pi}\phi_T^{(j)}}\right)^2,$$

so that

$$\int_U \left[M^{-1/2}\sum_{i=1}^{M}\left(\Phi\left(u/\phi_T^{(j)}\right) - \frac{1}{2}\right)\right]^2 dF(u) \leq \int_U \left(\frac{M^{1/2}|u|}{\sqrt{2\pi}\phi_T^{(j)}}\right)^2 dF(u).$$

Using Assumption **??**, (A.5), (2.17) and Markov's inequality, it is easy to see that

$$\Theta_{T,M}^{(j)} = \left\{\frac{2}{\sqrt{M}}\sum_{i=1}^{M}\left[I\left(\xi_i^{(j)} \leq 0\right) - \frac{1}{2}\right]\right\}^2 + o_{P^*}(1),$$

and therefore the desired result follows from the Central Limit Theorem. Under H_A, Theorem 1 entails that

$$P\left(\omega:\lim_{T\to\infty}\phi_T^{(j)}=0\right)=1. \tag{A.8}$$

Noting that

$$\zeta_i^{(j)}-\frac{1}{2}=I\left(\xi_1^{(j)}\leq u/\phi_T^{(j)}\right)\pm\Phi\left(u/\phi_T^{(j)}\right)-\frac{1}{2},$$

and we have

$$E^*\left[M^{-1/2}\sum_{i=1}^{M}\left(I\left(\xi_1^{(j)}\leq u/\phi_T^{(j)}\right)-\frac{1}{2}\right)\right]^2=E^*\left[\left(I\left(\xi_1^{(j)}\leq u/\phi_T^{(j)}\right)-\Phi\left(u/\phi_T^{(j)}\right)\right)\right]^2+M\left[\Phi\left(u/\phi_T^{(j)}\right)-\frac{1}{2}\right]^2.$$

Since $E^*\left[\left(I\left(\xi_1^{(j)}\leq u/\phi_T^{(j)}\right)-\Phi\left(u/\phi_T^{(j)}\right)\right)\right]^2<\infty$, by the Markov inequality it follows that

$$M^{-1/2}\sum_{i=1}^{M}\left(I\left(\xi_1^{(j)}\leq u/\phi_T^{(j)}\right)-\Phi\left(u/\phi_T^{(j)}\right)\right)=O_{P^*}(1).$$

The desired result now follows from (A.8).

□

Proof of Theorem 3. The proof is the same as the proof of Theorem 3 in Trapani (2018), and therefore we only report the main passages. Consider the events $\left\{\widehat{R}=j\right\}$, $0\leq j\leq N$, and recall that the test statistics $\Theta_{T,M}^{(j)}$ are independent across j conditional on the sample. Thus, for all $j\leq R-1$, we have $P^*\left(\widehat{R}=j\right)=\alpha\left(1-\alpha\right)^j$, where we omit dependence on N and T from the nominal level α for the sake of the notation. Letting π be the power of the test, it easy to see that $P^*\left(\widehat{R}=j\right)=\left(1-\alpha\right)^R\pi\left(1-\pi\right)^{j-R}$, whenever $j\geq R+1$. Thus

$$P^*\left(\widehat{R}=R\right)=1-\sum_{j\neq R}P^*\left(\widehat{R}=j\right)=\left(1-\alpha\right)^R\left[\pi+\left(1-\pi\right)^{N-R+1}\right].$$

Given that, in light of (2.21), $\pi\overset{P^*}{\to}1$ as $\min\left(T,M\right)\to\infty$, it follows that

$$\lim_{\min(T,M)\to\infty}P^*\left(\widehat{R}=R\right)=\left(1-\alpha\right)^R, \tag{A.9}$$

for almost all realisations of $\{\varepsilon_t,0<t<\infty\}$. The desired result follows by specialising (A.9) to the case where $\lim_{\min(T,M)\to\infty}\alpha=0$. □

Proof of Corollary 1. The result follows from the theory developed in Hansen (2005). Indeed, using the results in Section 3 in his paper, it follows that, under (2.23), equation (2.2) still holds with

$$y_t=C\sum_{s=1}^{t}\varepsilon_s+C^*\left(z\right)\varepsilon_t,$$

and therefore the conclusions of Theorem 1 remain unaltered. Similarly, (2.1) also holds, and therefore Theorem 2 also holds. The final results now follows immediately. □

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TABLE 4

Comparison between BCT and BIC - i.i.d. case, Part 1

$\theta = 0$	n	3						4						5					
	T	100		200		400		100		200		400		100		200		400	
		BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC
r	η																		
		0.054	0.182	0.015	0.136	0.004	0.105	0.120	0.250	0.043	0.208	0.012	0.162	0.228	0.309	0.051	0.218	0.012	0.162
	0.5	(0.243)	(0.432)	(0.121)	(0.379)	(0.063)	(0.325)	(0.354)	(0.497)	(0.202)	(0.443)	(0.108)	(0.433)	(0.454)	(0.598)	(0.220)	(0.492)	(0.108)	(0.407)
		[0.051]	[0.166]	[0.015]	[0.123]	[0.004]	[0.099]	[0.113]	[0.224]	[0.043]	[0.193]	[0.012]	[0.139]	[0.216]	[0.248]	[0.051]	[0.185]	[0.012]	[0.147]
		0.025	0.095	0.005	0.056	0.000	0.044	0.060	0.111	0.009	0.089	0.001	0.056	0.114	0.138	0.013	0.082	0.001	0.052
	1	(0.180)	(0.303)	(0.070)	(0.238)	(0.000)	(0.210)	(0.280)	(0.326)	(0.094)	(0.298)	(0.031)	(0.250)	(0.362)	(0.375)	(0.113)	(0.288)	(0.031)	(0.230)
		[0.022]	[0.092]	[0.005]	[0.054]	[0.000]	[0.043]	[0.053]	[0.107]	[0.009]	[0.085]	[0.001]	[0.051]	[0.106]	[0.128]	[0.013]	[0.078]	[0.001]	[0.050]
		0.013	0.056	0.004	0.031	0.000	0.015	0.035	0.056	0.002	0.036	0.000	0.014	0.095	0.064	0.004	0.029	0.000	0.015
	1.5	(0.144)	(0.246)	(0.063)	(0.173)	(0.000)	(0.121)	(0.236)	(0.234)	(0.044)	(0.196)	(0.000)	(0.125)	(0.360)	(0.268)	(0.063)	(0.179)	(0.000)	(0.121)
		[0.010]	[0.052]	[0.004]	[0.031]	[0.000]	[0.015]	[0.028]	[0.055]	[0.002]	[0.034]	[0.000]	[0.013]	[0.085]	[0.059]	[0.004]	[0.027]	[0.000]	[0.015]
		0.008	0.059	0.002	0.021	0.000	0.001	0.025	0.046	0.000	0.018	0.000	0.006	0.080	0.042	0.003	0.010	0.000	0.005
0	2	(0.126)	(0.325)	(0.044)	(0.143)	(0.000)	(0.099)	(0.215)	(0.282)	(0.000)	(0.140)	(0.000)	(0.089)	(0.340)	(0.257)	(0.054)	(0.099)	(0.000)	(0.070)
		[0.005]	[0.041]	[0.002]	[0.021]	[0.000]	[0.001]	[0.018]	[0.037]	[0.000]	[0.017]	[0.000]	[0.005]	[0.071]	[0.035]	[0.003]	[0.010]	[0.000]	[0.005]
		0.001	1.336	0.003	1.226	0.000	1.230	0.009	1.526	0.000	1.391	0.006	1.409	0.045	1.726	0.000	1.460	0.000	1.399
	∞	(0.031)	(1.314)	(0.070)	(1.326)	(0.000)	(1.346)	(0.122)	(1.594)	(0.000)	(1.599)	(0.141)	(1.649)	(0.212)	(1.746)	(0.000)	(1.761)	(0.000)	(1.809)
		[0.001]	[0.580]	[0.002]	[0.523]	[0.000]	[0.510]	[0.007]	[0.599]	[0.000]	[0.541]	[0.002]	[0.501]	[0.044]	[0.667]	[0.000]	[0.544]	[0.000]	[0.501]
		1.007	1.104	1.004	1.083	1.000	1.065	1.032	1.192	1.013	1.144	1.000	1.101	1.075	1.255	1.021	1.171	1.007	1.121
	0.5	(0.104)	(0.324)	(0.063)	(0.286)	(0.000)	(0.250)	(0.197)	(0.444)	(0.163)	(0.376)	(0.063)	(0.317)	(0.278)	(0.486)	(0.150)	(0.407)	(0.113)	(0.352)
		[0.008]	[0.098]	[0.004]	[0.080]	[0.000]	[0.065]	[0.032]	[0.177]	[0.016]	[0.138]	[0.004]	[0.099]	[0.080]	[0.233]	[0.020]	[0.159]	[0.005]	[0.115]
		1.004	1.066	1.004	1.053	1.000	1.032	1.009	1.081	1.002	1.080	1.000	1.046	1.049	1.125	1.003	1.060	1.000	1.035
	1	(0.077)	(0.264)	(0.077)	(0.237)	(0.000)	(0.176)	(0.094)	(0.294)	(0.044)	(0.285)	(0.000)	(0.214)	(0.301)	(0.359)	(0.054)	(0.249)	(0.000)	(0.183)
		[0.003]	[0.066]	[0.003]	[0.050]	[0.000]	[0.032]	[0.009]	[0.076]	[0.002]	[0.077]	[0.000]	[0.045]	[0.040]	[0.115]	[0.003]	[0.057]	[0.000]	[0.035]
		1.003	1.053	1.000	1.024	1.000	1.019	1.008	1.060	1.000	1.036	1.000	1.012	1.022	1.052	1.001	1.023	1.000	1.012
	1.5	(0.070)	(0.237)	(0.000)	(0.153)	(0.000)	(0.143)	(0.126)	(0.237)	(0.000)	(0.186)	(0.000)	(0.108)	(0.165)	(0.235)	(0.031)	(0.156)	(0.000)	(0.108)
		[0.002]	[0.050]	[0.000]	[0.024]	[0.000]	[0.018]	[0.005]	[0.060]	[0.000]	[0.036]	[0.000]	[0.012]	[0.020]	[0.049]	[0.001]	[0.022]	[0.000]	[0.012]
		1.003	1.051	1.000	1.027	1.000	1.012	1.008	1.034	1.000	1.022	1.000	1.005	1.015	1.029	1.001	1.007	1.000	1.006
1	2	(0.070)	(0.257)	(0.000)	(0.179)	(0.000)	(0.117)	(0.126)	(0.238)	(0.000)	(0.146)	(0.000)	(0.070)	(0.121)	(0.173)	(0.031)	(0.083)	(0.000)	(0.077)
		[0.002]	[0.051]	[0.000]	[0.024]	[0.000]	[0.011]	[0.005]	[0.039]	[0.000]	[0.022]	[0.000]	[0.005]	[0.015]	[0.031]	[0.001]	[0.007]	[0.000]	[0.006]
		1.000	1.406	1.001	1.513	1.000	1.631	1.002	1.350	1.000	1.595	1.004	1.740	1.011	1.446	1.000	1.556	1.000	1.712
	∞	(0.000)	(0.883)	(0.031)	(0.886)	(0.000)	(0.899)	(0.063)	(0.986)	(0.000)	(1.135)	(0.099)	(1.194)	(0.113)	(1.109)	(0.000)	(1.185)	(0.000)	(1.350)
		[0.000]	[0.371]	[0.001]	[0.341]	[0.000]	[0.355]	[0.001]	[0.310]	[0.000]	[0.301]	[0.002]	[0.323]	[0.010]	[0.302]	[0.000]	[0.296]	[0.000]	[0.288]
		1.987	2.062	1.998	2.040	1.998	2.034	1.995	2.129	2.000	2.078	1.999	2.069	2.014	2.191	2.004	2.171	2.004	2.107
	0.5	(0.129)	(0.249)	(0.044)	(0.196)	(0.044)	(0.181)	(0.113)	(0.363)	(0.044)	(0.279)	(0.070)	(0.257)	(0.154)	(0.469)	(0.118)	(0.426)	(0.099)	(0.334)
		[0.017]	[0.066]	[0.002]	[0.040]	[0.002]	[0.034]	[0.013]	[0.119]	[0.002]	[0.075]	[0.005]	[0.068]	[0.024]	[0.161]	[0.006]	[0.152]	[0.007]	[0.105]
		1.996	2.099	2.000	2.071	2.000	2.058	2.004	2.092	2.000	2.044	1.999	2.034	2.009	2.112	2.002	2.063	2.000	2.029
	1	(0.063)	(0.298)	(0.000)	(0.256)	(0.000)	(0.233)	(0.077)	(0.315)	(0.000)	(0.205)	(0.031)	(0.192)	(0.113)	(0.348)	(0.063)	(0.255)	(0.000)	(0.179)
		[0.004]	[0.099]	[0.000]	[0.071]	[0.000]	[0.058]	[0.006]	[0.084]	[0.000]	[0.044]	[0.001]	[0.032]	[0.013]	[0.102]	[0.001]	[0.060]	[0.000]	[0.027]
		2.001	2.119	2.000	2.081	2.000	2.073	1.999	2.058	2.000	2.028	2.000	2.023	2.008	2.066	2.002	2.026	2.000	2.005
	1.5	(0.031)	(0.323)	(0.000)	(0.272)	(0.000)	(0.260)	(0.031)	(0.262)	(0.000)	(0.176)	(0.000)	(0.156)	(0.099)	(0.275)	(0.063)	(0.165)	(0.000)	(0.070)
		[0.001]	[0.119]	[0.000]	[0.081]	[0.000]	[0.073]	[0.001]	[0.051]	[0.000]	[0.026]	[0.000]	[0.022]	[0.007]	[0.060]	[0.001]	[0.025]	[0.000]	[0.005]
		2.001	2.114	2.000	2.092	2.000	2.074	2.002	2.057	2.000	2.020	2.000	2.014	2.004	2.034	2.002	2.013	2.000	2.002
2	2	(0.031)	(0.330)	(0.000)	(0.289)	(0.000)	(0.261)	(0.044)	(0.256)	(0.000)	(0.153)	(0.000)	(0.117)	(0.063)	(0.192)	(0.063)	(0.121)	(0.000)	(0.044)
		[0.001]	[0.122]	[0.000]	[0.092]	[0.000]	[0.074]	[0.002]	[0.054]	[0.000]	[0.018]	[0.000]	[0.014]	[0.004]	[0.032]	[0.001]	[0.012]	[0.000]	[0.002]
		2.001	2.262	2.000	2.283	2.000	2.331	2.001	2.199	2.000	2.292	2.000	2.369	2.000	2.162	2.000	2.217	2.000	2.363
	∞	(0.031)	(0.534)	(0.000)	(0.493)	(0.000)	(0.470)	(0.031)	(0.675)	(0.000)	(0.710)	(0.000)	(0.750)	(0.000)	(0.672)	(0.000)	(0.735)	(0.000)	(0.922)
		[0.001]	[0.354]	[0.000]	[0.323]	[0.000]	[0.331]	[0.001]	[0.228]	[0.000]	[0.194]	[0.000]	[0.210]	[0.000]	[0.155]	[0.000]	[0.137]	[0.000]	[0.165]

The table XXX

TABLE 5
Comparison between BCT and BIC - i.i.d. case, Part 2

$\theta = 0$	n	3						4						5					
	T	100		200		400		100		200		400		100		200		400	
		BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC
r	η																		
	0.5	2.987 (0.113) [0.013]	2.992 (0.089) [0.008]	2.988 (0.108) [0.012]	2.995 (0.070) [0.005]	2.993 (0.083) [0.007]	2.998 (0.044) [0.002]	2.986 (0.125) [0.016]	3.049 (0.262) [0.063]	2.992 (0.089) [0.008]	3.028 (0.212) [0.038]	2.994 (0.077) [0.006]	3.025 (0.156) [0.025]	2.991 (0.122) [0.012]	3.105 (0.390) [0.106]	2.996 (0.063) [0.004]	3.083 (0.303) [0.088]	2.997 (0.054) [0.003]	3.047 (0.266) [0.054]
	1	2.992 (0.089) [0.008]	2.996 (0.063) [0.004]	2.997 (0.054) [0.003]	2.999 (0.031) [0.001]	2.999 (0.031) [0.001]	3.000 (0.000) [0.000]	2.994 (0.089) [0.008]	3.081 (0.276) [0.083]	3.001 (0.031) [0.001]	3.075 (0.263) [0.075]	2.999 (0.031) [0.001]	3.031 (0.173) [0.031]	3.001 (0.094) [0.006]	3.086 (0.310) [0.077]	3.002 (0.063) [0.001]	3.059 (0.256) [0.054]	3.000 (0.000) [0.000]	3.032 (0.176) [0.032]
	1.5	2.998 (0.044) [0.002]	3.000 (0.000) [0.000]	2.999 (0.031) [0.001]	3.000 (0.000) [0.000]	2.999 (0.031) [0.001]	3.000 (0.000) [0.000]	3.003 (0.054) [0.003]	3.121 (0.326) [0.121]	3.001 (0.031) [0.001]	3.091 (0.287) [0.091]	3.000 (0.000) [0.000]	3.054 (0.226) [0.054]	3.000 (0.000) [0.000]	3.058 (0.246) [0.055]	3.002 (0.063) [0.001]	3.030 (0.170) [0.030]	3.000 (0.000) [0.000]	3.012 (0.108) [0.012]
3	2	3.000 (0.000) [0.000]	2.998 (0.063) [0.001]	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	2.999 (0.031) [0.001]	3.000 (0.000) [0.000]	3.003 (0.054) [0.003]	3.121 (0.326) [0.121]	3.001 (0.031) [0.001]	3.094 (0.291) [0.094]	3.000 (0.000) [0.000]	3.064 (0.244) [0.064]	2.999 (0.031) [0.001]	3.058 (0.265) [0.050]	3.002 (0.063) [0.001]	3.027 (0.162) [0.027]	3.000 (0.000) [0.000]	3.008 (0.089) [0.008]
	∞	3.000 (0.000) [0.000]	2.953 (0.211) [0.047]	3.000 (0.000) [0.000]	2.981 (0.136) [0.019]	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	3.232 (0.475) [0.280]	3.000 (0.000) [0.000]	3.242 (0.455) [0.266]	3.000 (0.000) [0.000]	3.239 (0.431) [0.243]	3.001 (0.031) [0.001]	3.121 (0.500) [0.127]	3.000 (0.000) [0.000]	3.159 (0.515) [0.102]	3.000 (0.000) [0.000]	3.176 (0.541) [0.102]
	0.5							3.967 (0.178) [0.033]	3.982 (0.133) [0.018]	3.989 (0.104) [0.011]	3.997 (0.054) [0.003]	3.991 (0.094) [0.009]	3.997 (0.054) [0.003]	3.977 (0.156) [0.025]	4.043 (0.298) [0.070]	3.982 (0.133) [0.018]	4.017 (0.273) [0.037]	3.993 (0.083) [0.007]	4.014 (0.194) [0.023]
	1							3.988 (0.108) [0.012]	3.998 (0.044) [0.002]	3.998 (0.044) [0.002]	4.000 (0.000) [0.000]	3.999 (0.031) [0.001]	4.000 (0.000) [0.000]	3.996 (0.063) [0.004]	4.101 (0.301) [0.101]	4.001 (0.031) [0.001]	4.067 (0.250) [0.067]	4.000 (0.000) [0.000]	4.044 (0.205) [0.044]
	1.5							3.996 (0.063) [0.004]	4.000 (0.000) [0.000]	3.998 (0.044) [0.002]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	3.999 (0.031) [0.001]	4.114 (0.317) [0.114]	4.001 (0.031) [0.001]	4.079 (0.269) [0.079]	4.000 (0.000) [0.000]	4.045 (0.207) [0.045]
4	2							3.997 (0.031) [0.001]	3.999 (0.031) [0.001]	3.999 (0.031) [0.001]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.001 (0.031) [0.001]	4.147 (0.354) [0.147]	4.001 (0.031) [0.001]	4.094 (0.291) [0.094]	4.000 (0.000) [0.000]	4.058 (0.233) [0.058]
	∞							4.000 (0.000) [0.000]	3.983 (0.129) [0.017]	4.000 (0.000) [0.000]	3.989 (0.104) [0.011]	4.000 (0.000) [0.000]	3.998 (0.044) [0.002]	4.000 (0.000) [0.000]	4.229 (0.450) [0.252]	4.000 (0.000) [0.000]	4.175 (0.385) [0.179]	4.000 (0.000) [0.000]	4.177 (0.384) [0.179]
	0.5													4.946 (0.226) [0.054]	4.980 (0.140) [0.020]	4.984 (0.125) [0.016]	4.993 (0.083) [0.007]	4.990 (0.099) [0.010]	4.998 (0.044) [0.002]
	1													4.990 (0.099) [0.010]	4.999 (0.031) [0.001]	4.993 (0.083) [0, 007]	4.999 (0.031) [0.001]	4.997 (0.054) [0.003]	5.000 (0.000) [0.000]
	1.5													4.998 (0.044) [0.002]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]
5	2													4.998 (0.044) [0.002]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]
	∞													5.000 (0.000) [0.000]	4.993 (0.083) [0.007]	5.000 (0.000) [0.000]	4.996 (0.063) [0.004]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]

The table XXX

TABLE 6

Comparison between BCT and BIC - positive autocorrelation case, Part 1

$\theta = 0.5$	n	3						4						5					
	T	100		200		400		100		200		400		100		200		400	
		BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC
r	η																		
		0.016	0.192	0.006	0.151	0.001	0.115	0.045	0.308	0.015	0.261	0.003	0.194	0.078	0.415	0.013	0.291	0.005	0.215
	0.5	(0.154)	(0.428)	(0.077)	(0.392)	(0.031)	(0.340)	(0.255)	(0.524)	(0.121)	(0.495)	(0.054)	(0.445)	(0.357)	(0.645)	(0.113)	(0.542)	(0.070)	(0.454)
		[0.013]	[0.179]	[0.006]	[0.138]	[0.001]	[0.108]	[0.038]	[0.280]	[0.015]	[0.237]	[0.003]	[0.173]	[0.065]	[0.341]	[0.013]	[0.250]	[0.005]	[0.196]
		0.009	0.106	0.003	0.076	0.000	0.058	0.019	0.166	0.001	0.127	0.000	0.085	0.025	0.217	0.002	0.122	0.000	0.078
	1	(0.130)	(0.317)	(0.054)	(0.279)	(0.000)	(0.233)	(0.210)	(0.382)	(0.031)	(0.359)	(0.000)	(0.322)	(0.257)	(0.469)	(0.044)	(0.351)	(0.000)	(0.282)
		[0.006]	[0.103]	[0.003]	[0.072]	[0.000]	[0.058]	[0.012]	[0.162]	[0.001]	[0.119]	[0.000]	[0.073]	[0.016]	[0.194]	[0.002]	[0.114]	[0.000]	[0.074]
		0.006	0.073	0.001	0.039	0.000	0.029	0.013	0.107	0.000	0.065	0.000	0.034	0.019	0.129	0.001	0.067	0.000	0.034
	1.5	(0.118)	(0.271)	(0.031)	(0.198)	(0.000)	(0.167)	(0.186)	(0.328)	(0.000)	(0.258)	(0.000)	(0.211)	(0.246)	(0.363)	(0.031)	(0.261)	(0.000)	(0.181)
		[0.003]	[0.070]	[0.001]	[0.038]	[0.000]	[0.029]	[0.006]	[0.101]	[0.000]	[0.062]	[0.000]	[0.029]	[0.010]	[0.120]	[0.001]	[0.064]	[0.000]	[0.034]
		0.005	0.050	0.001	0.029	0.000	0.017	0.012	0.075	0.000	0.038	0.000	0.015	0.014	0.081	0.000	0.031	0.000	0.010
0	2	(0.113)	(0.222)	(0.031)	(0.173)	(0.000)	(0.129)	(0.184)	(0.295)	(0.000)	(0.196)	(0.000)	(0.137)	(0.236)	(0.283)	(0.000)	(0.173)	(0.000)	(0.099)
		[0.002]	[0.049]	[0.001]	[0.028]	[0.000]	[0.017]	[0.005]	[0.069]	[0.000]	[0.037]	[0.000]	[0.013]	[0.005]	[0.078]	[0.000]	[0.031]	[0.000]	[0.010]
		0.000	0.181	0.003	0.058	0.000	0.029	0.003	0.107	0.000	0.035	0.006	0.010	0.002	0.106	0.000	0.024	0.000	0.006
	∞	(0.000)	(0.512)	(0.070)	(0.246)	(0.000)	(0.173)	(0.094)	(0.345)	(0.000)	(0.189)	(0.141)	(0.099)	(0.044)	(0.380)	(0.000)	(0.153)	(0.000)	(0.077)
		[0.000]	[0.138]	[0.002]	[0.055]	[0.000]	[0.028]	[0.001]	[0.096]	[0.000]	[0.034]	[0.002]	[0.010]	[0.002]	[0.093]	[0.000]	[0.024]	[0.000]	[0.006]
		0.994	1.093	0.995	1.084	0.994	1.063	1.010	1.225	0.999	1.159	1.002	1.115	1.024	1.310	0.995	1.236	1.001	1.155
	0.5	(0.118)	(0.297)	(0.070)	(0.288)	(0.089)	(0.247)	(0.178)	(0.467)	(0.083)	(0.387)	(0.134)	(0.334)	(0.171)	(0.565)	(0.083)	(0.512)	(0.122)	(0.388)
		[0.011]	[0.091]	[0.005]	[0.081]	[0.008]	[0.062]	[0.013]	[0.211]	[0.007]	[0.151]	[0.010]	[0.110]	[0.027]	[0.264]	[0.007]	[0.206]	[0.007]	[0.145]
		0.992	1.070	0.992	1.039	0.995	1.035	1.003	1.111	0.988	1.076	0.993	1.054	1.005	1.148	0.994	1.101	0.990	1.054
	1	(0.126)	(0.270)	(0.089)	(0.208)	(0.070)	(0.183)	(0.137)	(0.329)	(0.108)	(0.268)	(0.083)	(0.239)	(0.151)	(0.424)	(0.077)	(0.338)	(0.099)	(0.226)
		[0.013]	[0.066]	[0.008]	[0.036]	[0.005]	[0.035]	[0.080]	[0.106]	[0.012]	[0.075]	[0.007]	[0.051]	[0.008]	[0.127]	[0.006]	[0.090]	[0.010]	[0.054]
		0.995	1.060	0.990	1.027	0.993	1.019	1.000	1.068	0.985	1.043	0.990	1.016	1.006	1.073	0.994	1.053	0.991	1.025
	1.5	(0.113)	(0.253)	(0.099)	(0.168)	(0.083)	(0.143)	(0.148)	(0.267)	(0.121)	(0.212)	(0.099)	(0.133)	(0.189)	(0.322)	(0.077)	(0.228)	(0.094)	(0.156)
		[0.010]	[0.056]	[0.010]	[0.026]	[0.007]	[0.018]	[0.011]	[0.064]	[0.015]	[0.041]	[0.010]	[0.015]	[0.006]	[0.092]	[0.006]	[0.052]	[0.009]	[0.025]
		0.993	1.048	0.992	1.027	0.993	1.009	1.000	1.048	0.988	1.028	0.991	1.014	1.005	1.026	0.996	1.027	0.994	1.011
1	2	(0.122)	(0.240)	(0.089)	(0.179)	(0.083)	(0.104)	(0.148)	(0.244)	(0.108)	(0.182)	(0.094)	(0.125)	(0.192)	(0.292)	(0.063)	(0.162)	(0.077)	(0.104)
		[0.012]	[0.042]	[0.008]	[0.024]	[0.007]	[0.008]	[0.011]	[0.050]	[0.012]	[0.025]	[0.009]	[0.013]	[0.007]	[0.080]	[0.004]	[0.027]	[0.006]	[0.011]
		0.991	1.166	0.995	1.108	0.999	1.059	0.995	1.076	0.996	1.045	1.002	1.018	0.994	0.909	0.996	1.031	0.998	1.013
	∞	(0.094)	(0.496)	(0.083)	(0.369)	(0.031)	(0.252)	(0.104)	(0.390)	(0.063)	(0.234)	(0.109)	(0.133)	(0.077)	(0.516)	(0.063)	(0.173)	(0.044)	(0.113)
		[0.009]	[0.145]	[0.007]	[0.088]	[0.001]	[0.055]	[0.008]	[0.117]	[0.004]	[0.040]	[0.004]	[0.018]	[0.006]	[0.245]	[0.004]	[0.031]	[0.002]	[0.013]
		1.962	2.082	1.955	2.044	1.971	2.039	1.960	2.120	1.974	2.077	1.979	2.061	1.984	2.199	1.983	2.173	1.989	2.112
	0.5	(0.196)	(0.274)	(0.207)	(0.214)	(0.167)	(0.198)	(0.220)	(0.357)	(0.159)	(0.270)	(0.143)	(0.247)	(0.178)	(0.457)	(0.129)	(0.425)	(0.122)	(0.334)
		[0.040]	[0.082]	[0.045]	[0.048]	[0.029]	[0.041]	[0.047]	[0.118]	[0.026]	[0.076]	[0.021]	[0.065]	[0.029]	[0.182]	[0.017]	[0.155]	[0.015]	[0.112]
		1.956	2.093	1.956	2.031	1.972	2.027	1.972	2.057	1.972	2.055	1.976	2.027	1.978	2.086	1.984	2.066	1.991	2.057
	1	(0.210)	(0.290)	(0.205)	(0.173)	(0.165)	(0.162)	(0.182)	(0.244)	(0.165)	(0.240)	(0.153)	(0.162)	(0.198)	(0.369)	(0.125)	(0.271)	(0.104)	(0.256)
		[0.046]	[0.093]	[0.044]	[0.031]	[0.028]	[0.027]	[0.031]	[0.057]	[0.028]	[0.052]	[0.024]	[0.027]	[0.029]	[0.112]	[0.016]	[0.061]	[0.011]	[0.052]
		1.959	2.059	1.976	2.022	1.979	2.028	1.971	2.045	1.975	2.026	1.975	2.019	1.973	2.065	1.989	2.026	1.981	2.014
	1.5	(0.203)	(0.235)	(0.153)	(0.146)	(0.143)	(0.165)	(0.173)	(0.225)	(0.156)	(0.159)	(0.156)	(0.136)	(0.200)	(0.284)	(0.029)	(0.159)	(0.157)	(0.117)
		[0.040]	[0.059]	[0.024]	[0.022]	[0.021]	[0.028]	[0.028]	[0.041]	[0.025]	[0.026]	[0.025]	[0.019]	[0.033]	[0.073]	[0.018]	[0.026]	[0.022]	[0.014]
		1.963	2.050	1.971	2.014	1.975	2.016	1.963	2.044	1.974	2.022	1.983	2.014	1.983	2.020	1.982	2.019	1.985	2.010
2	2	(0.188)	(0.235)	(0.173)	(0.117)	(0.156)	(0.125)	(0.188)	(0.232)	(0.159)	(0.165)	(0.129)	(0.125)	(0.186)	(0.252)	(0.153)	(0.136)	(0.121)	(0.099)
		[0.037]	[0.055]	[0.031]	[0.014]	[0.025]	[0.016]	[0.037]	[0.044]	[0.026]	[0.019]	[0.017]	[0.013]	[0.024]	[0.058]	[0.021]	[0.019]	[0.015]	[0.010]
		1.952	2.198	1.968	2.165	1.983	2.149	1.968	2.131	1.982	2.110	1.990	2.049	1.971	2.004	1.984	2.047	1.989	2.024
	∞	(0.213)	(0.427)	(0.181)	(0.376)	(0.129)	(0.356)	(0.176)	(0.453)	(0.133)	(0.384)	(0.099)	(0.258)	(0.184)	(0.555)	(0.166)	(0.221)	(0.104)	(0.159)
		[0.048]	[0.222]	[0.031]	[0.169]	[0.017]	[0.149]	[0.032]	[0.118]	[0.018]	[0.085]	[0.010]	[0.039]	[0.032]	[0.190]	[0.020]	[0.045]	[0.011]	[0.023]

The table XXX

TABLE 7

Comparison between BCT and BIC - positive autocorrelation, Part 2

$\theta = 0.5$	n	3						4						5					
	T	100		200		400		100		200		400		100		200		400	
r	η	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC
	0.5	2.890 (0.316) [0.109]	3.000 (0.000) [0.000]	2.922 (0.272) [0.077]	2.999 (0.031) [0.001]	2.945 (0.228) [0.055]	3.000 (0.000) [0.000]	2.876 (0.332) [0.123]	3.052 (0.239) [0.060]	2.919 (0.272) [0.081]	3.085 (0.282) [0.087]	2.946 (0.234) [0.052]	3.038 (0.224) [0.044]	2.922 (0.279) [0.081]	3.118 (0.397) [0.126]	2.941 (0.252) [0.064]	3.084 (0.339) [0.091]	2.958 (0.215) [0.045]	3.071 (0.313) [0.075]
	1	2.892 (0.316) [0.106]	2.999 (0.031) [0.001]	2.925 (0.267) [0.074]	2.999 (0.031) [0.001]	2.952 (0.218) [0.047]	3.000 (0.000) [0.000]	2.895 (0.306) [0.105]	3.093 (0.294) [0.095]	2.926 (0.265) [0.073]	3.065 (0.246) [0.065]	2.956 (0.205) [0.044]	3.041 (0.198) [0.041]	2.933 (0.269) [0.074]	3.093 (0.326) [0.085]	2.955 (0.212) [0.044]	3.062 (0.265) [0.056]	2.953 (0.211) [0.047]	3.028 (0.171) [0.027]
	1.5	2.907 (0.290) [0.093]	2.999 (0.031) [0.001]	2.925 (0.263) [0.075]	3.000 (0.000) [0.000]	2.957 (0.202) [0.043]	3.000 (0.000) [0.000]	2.916 (0.281) [0.083]	3.101 (0.301) [0.101]	2.934 (0.252) [0.065]	3.030 (0.170) [0.030]	2.956 (0.205) [0.044]	3.037 (0.188) [0.037]	2.922 (0.282) [0.083]	3.052 (0.247) [0.046]	2.934 (0.248) [0.066]	3.039 (0.203) [0.037]	2.959 (0.198) [0.041]	3.018 (0.140) [0.017]
3	2	2.906 (0.295) [0.093]	2.999 (0.031) [0.001]	2.935 (0.246) [0.065]	3.000 (0.000) [0.000]	2.957 (0.202) [0.043]	3.000 (0.000) [0.000]	2.903 (0.299) [0.096]	3.079 (0.294) [0.085]	2.920 (0.275) [0.079]	3.041 (0.198) [0.041]	2.957 (0.202) [0.043]	3.026 (0.159) [0.026]	2.928 (0.266) [0.070]	3.043 (0.217) [0.040]	2.949 (0.220) [0.051]	3.022 (0.165) [0.019]	2.961 (0.193) [0.039]	3.012 (0.117) [0.011]
	∞	2.861 (0.349) [0.138]	2.958 (0.224) [0.037]	2.946 (0.226) [0.054]	2.989 (0.104) [0.011]	2.960 (0.210) [0.039]	2.997 (0.054) [0.003]	2.879 (0.332) [0.122]	3.215 (0.423) [0.225]	2.937 (0.243) [0.063]	3.166 (0.377) [0.170]	2.957 (0.202) [0.043]	3.124 (0.329) [0.124]	2.921 (0.284) [0.081]	3.162 (0.473) [0.133]	2.956 (0.210) [0.046]	3.103 (0.352) [0.087]	2.976 (0.153) [0.024]	3.058 (0.273) [0.048]
	0.5							3.812 (0.401) [0.184]	3.992 (0.089) [0.008]	3.870 (0.339) [0.129]	3.997 (0.054) [0.003]	3.911 (0.291) [0.087]	3.999 (0.031) [0.001]	3.826 (0.384) [0.172]	4.071 (0.387) [0.107]	3.883 (0.327) [0.115]	4.065 (0.258) [0.071]	3.932 (0.259) [0.066]	4.036 (0.206) [0.041]
	1							3.830 (0.381) [0.168]	3.997 (0.054) [0.003]	3.856 (0.365) [0.139]	4.000 (0.000) [0.000]	3.921 (0.269) [0.079]	4.000 (0.000) [0.000]	3.827 (0.391) [0.171]	4.133 (0.342) [0.135]	3.883 (0.327) [0.115]	4.057 (0.231) [0.057]	3.925 (0.267) [0.074]	4.061 (0.239) [0.061]
	1.5							3.803 (0.405) [0.194]	4.000 (0.000) [0.000]	3.875 (0.339) [0.122]	4.000 (0.000) [0.000]	3.906 (0.295) [0.093]	4.000 (0.000) [0.000]	3.847 (0.371) [0.152]	4.096 (0.294) [0.096]	3.868 (0.344) [0.130]	4.068 (0.251) [0.068]	3.921 (0.269) [0.079]	4.035 (0.183) [0.035]
4	2							3.806 (0.400) [0.192]	3.996 (0.099) [0.002]	3.871 (0.341) [0.127]	4.000 (0.000) [0.000]	3.913 (0.285) [0.086]	4.000 (0.000) [0.000]	3.835 (0.382) [0.164]	4.080 (0.271) [0.080]	3.875 (0.336) [0.123]	4.052 (0.222) [0.052]	3.939 (0.243) [0.063]	4.026 (0.159) [0.026]
	∞							3.725 (0.462) [0.268]	3.975 (0.195) [0.019]	3.874 (0.335) [0.125]	3.996 (0.063) [0.004]	3.920 (0.275) [0.079]	4.000 (0.000) [0.000]	3.787 (0.414) [0.211]	4.211 (0.417) [0.219]	3.892 (0.313) [0.107]	4.173 (0.378) [0.173]	3.942 (0.233) [0.058]	4.131 (0.337) [0.131]
	0.5													4.700 (0.492) [0.284]	4.994 (0.077) [0.006]	4.800 (0.422) [0.191]	4.997 (0.054) [0.003]	4.860 (0.352) [0.138]	4.999 (0.031) [0.001]
	1													4.703 (0.488) [0.282]	4.998 (0.044) [0.002]	4.817 (0.414) [0.172]	4.999 (0.031) [0.001]	4.870 (0.339) [0.129]	5.000 (0.000) [0.000]
	1.5													4.676 (0.507) [0.305]	5.000 (0.000) [0.000]	4.816 (0.402) [0.178]	5.000 (0.000) [0.000]	4.869 (0.352) [0.126]	5.000 (0.000) [0.000]
5	2													4.677 (0.500) [0.307]	5.000 (0.000) [0.000]	4.791 (0.416) [0.205]	5.000 (0.000) [0.000]	4.880 (0.343) [0.114]	5.000 (0.000) [0.000]
	∞													4.563 (0.553) [0.408]	4.999 (0.031) [0.001]	4.781 (0.437) [0.209]	5.000 (0.000) [0.000]	4.888 (0.321) [0.110]	5.000 (0.000) [0.000]

The table XXX

TABLE 8

Comparison between BCT and BIC - negative autocorrelation, Part 1

$\theta=-0.5$	n	3						4						5					
	T	100		200		400		100		200		400		100		200		400	
		BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC
r	η																		
		0.618	1.210	0.135	1.066	0.037	0.897	1.102	1.005	0.316	0.983	0.089	0.852	1.666	0.849	0.599	0.979	0.141	0.912
	0.5	(0.523)	(0.966)	(0.341)	(0.887)	(0.194)	(0.836)	(0.545)	(0.994)	(0.484)	(0.890)	(0.288)	(0.804)	(0.557)	(0.982)	(0.576)	(0.922)	(0.365)	(0.857)
		[0.599]	[0.730]	[0.135]	[0.701]	[0.036]	[0.639]	[0.898]	[0.628]	[0.307]	[0.667]	[0.088]	[0.622]	[0.996]	[0.544]	[0.554]	[0.650]	[0.135]	[0.640]
		0.662	1.308	0.099	1.236	0.006	1.007	1.146	1.133	0.284	1.132	0.036	1.011	1.738	0.943	0.591	1.058	0.053	1.015
	1	(0.529)	(0.980)	(0.298)	(0.921)	(0.077)	(0.864)	(0.557)	(1.008)	(0.457)	(0.925)	(0.191)	(0.854)	(0.575)	(1.013)	(0.572)	(0.951)	(0.224)	(0.873)
		[0.634]	[0.766]	[0.099]	[0.772]	[0.006]	[0.697]	[0.909]	[0.687]	[0.281]	[0.723]	[0.035]	[0.698]	[0.998]	[0.595]	[0.550]	[0.681]	[0.053]	[0.690]
		0.685	1.360	0.098	1.302	0.000	1.126	1.159	1.257	0.278	1.244	0.012	1.106	1.782	1.033	0.633	1.174	0.024	1.091
	1.5	(0.538)	(0.985)	(0.297)	(0.962)	(0.000)	(0.919)	(0.540)	(1.048)	(0.452)	(1.007)	(0.108)	(0.868)	(0.580)	(1.012)	(0.581)	(0.972)	(0.153)	(0.897)
		[0.649]	[0.781]	[0.098]	[0.779]	[0.000]	[0.729]	[0.924]	[0.731]	[0.276]	[0.732]	[0.012]	[0.739]	[0.989]	[0.636]	[0.586]	[0.726]	[0.024]	[0.726]
		0.718	1.409	0.101	1.356	0.000	1.173	1.197	1.298	0.276	1.277	0.006	1.140	1.760	1.086	0.646	1.224	0.022	1.146
0	2	(0.504)	(0.970)	(0.301)	(0.969)	(0.000)	(0.945)	(0.538)	(1.058)	(0.451)	(1.051)	(0.077)	(0.898)	(0.600)	(1.039)	(0.614)	(0.985)	(0.146)	(0.932)
		[0.692]	[0.816]	[0.101]	[]	[0.000]	[0.732]	[0.936]	[0.744]	[0.274]	[0.733]	[0.006]	[0.746]	[0.996]	[0.661]	[0.590]	[0.751]	[0.022]	[0.737]
		0.523	0.986	0.047	0.836	0.000	0.772	1.014	0.985	0.191	0.916	0.008	0.777	1.543	0.906	0.489	0.923	0.012	0.855
	∞	(0.528)	(0.892)	(0.216)	(0.803)	(0.000)	(0.742)	(0.564)	(0.951)	(0.395)	(0.823)	(0.148)	(0.750)	(0.592)	(0.941)	(0.536)	(0.843)	(0.108)	(0.805)
		[0.508]	[0.662]	[0.046]	[0.615]	[0.000]	[0.603]	[0.857]	[0.650]	[0.190]	[0.658]	[0.004]	[0.600]	[0.981]	[0.601]	[0.470]	[0.653]	[0.012]	[0.632]
		1.110	1.666	1.016	1.627	1.011	1.542	1.420	1.739	1.113	1.700	1.026	1.593	1.916	1.754	1.258	1.805	1.057	1.706
	0.5	(0.322)	(0.710)	(0.125)	(0.684)	(0.104)	(0.640)	(0.521)	(0.803)	(0.349)	(0.735)	(0.159)	(0.695)	(0.541)	(0.884)	(0.451)	(0.817)	(0.244)	(0.741)
		[0.116]	[0.525]	[0.016]	[0.510]	[0.011]	[0.461]	[0.409]	[0.539]	[0.108]	[0.548]	[0.026]	[0.489]	[0.808]	[0.517]	[0.255]	[0.585]	[0.055]	[0.549]
		1.142	1.875	1.008	1.863	1.002	1.753	1.487	1.936	1.070	1.888	1.003	1.726	1.979	1.921	1.240	2.009	1.018	1.843
	1	(0.352)	(0.793)	(0.089)	(0.784)	(0.044)	(0.776)	(0.525)	(0.900)	(0.255)	(0.806)	(0.054)	(0.780)	(0.559)	(0.943)	(0.443)	(0.865)	(0.178)	(0.816)
		[0.141]	[0.615]	[0.008]	[0.615]	[0.002]	[0.545]	[0.475]	[0.626]	[0.070]	[0.645]	[0.003]	[0.552]	[0.837]	[0.611]	[0.233]	[0.698]	[0.017]	[0.613]
		1.220	2.178	1.012	1.994	1.000	1.936	1.576	2.140	1.074	2.081	1.001	1.997	2.020	2.034	1.234	2.050	1.007	1.994
	1.5	(0.414)	(0.799)	(0.108)	(0.771)	(0.000)	(0.787)	(0.551)	(0.965)	(0.261)	(0.922)	(0.031)	(0.872)	(0.570)	(0.984)	(0.439)	(0.948)	(0.083)	(0.892)
		[0.220]	[0.754]	[0.012]	[0.700]	[0.000]	[0.656]	[0.547]	[0.705]	[0.074]	[0.702]	[0.001]	[0.683]	[0.853]	[0.655]	[0.227]	[0.674]	[0.007]	[0.681]
		1.230	2.177	1.012	2.110	1.000	2.027	1.613	2.222	1.073	2.152	1.000	2.073	2.110	2.176	1.256	2.145	1.004	2.115
1	2	(0.423)	(0.793)	(0.108)	(0.777)	(0.000)	(0.799)	(0.532)	(0.992)	(0.260)	(0.954)	(0.000)	(0.923)	(0.591)	(1.065)	(0.447)	(0.995)	(0.063)	(0.951)
		[0.229]	[0.761]	[0.012]	[0.747]	[0.000]	[0.694]	[0.591]	[0.727]	[0.073]	[0.715]	[0.000]	[0.703]	[0.880]	[0.683]	[0.252]	[0.714]	[0.004]	[0.715]
		1.148	1.811	1.007	1.701	1.000	1.646	1.500	1.888	1.043	1.835	1.004	1.694	1.971	1.982	1.201	1.923	1.000	1.797
	∞	(0.360)	(0.744)	(0.083)	(0.701)	(0.000)	(0.673)	(0.525)	(0.852)	(0.202)	(0.786)	(0.099)	(0.710)	(0.546)	(0.927)	(0.400)	(0.846)	(0.000)	(0.777)
		[0.146]	[0.611]	[0.007]	[0.560]	[0.000]	[0.534]	[0.488]	[0.624]	[0.043]	[0.620]	[0.002]	[0.556]	[0.836]	[0.641]	[0.201]	[0.651]	[0.000]	[0.601]
		2.001	2.512	1.998	2.326	2.000	2.278	2.040	2.442	2.015	2.387	2.001	2.297	2.276	2.542	2.058	2.515	2.007	2.421
	0.5	(0.070)	(0.502)	(0.044)	(0.468)	(0.000)	(0.448)	(0.215)	(0.653)	(0.121)	(0.625)	(0.054)	(0.537)	(0.469)	(0.714)	(0.246)	(0.691)	(0.104)	(0.606)
		[0.005]	[0.514]	[0.002]	[0.326]	[0.000]	[0.278]	[0.048]	[0.355]	[0.015]	[0.319]	[0.003]	[0.263]	[0.270]	[0.434]	[0.061]	[0.419]	[0.011]	[0.374]
		2.014	2.568	2.001	2.489	2.000	2.475	2.079	2.645	2.009	2.656	1.999	2.532	2.328	2.734	2.044	2.703	2.004	2.635
	1	(0.117)	(0.495)	(0.031)	(0.500)	(0.000)	(0.499)	(0.269)	(0.742)	(0.094)	(0.739)	(0.031)	(0.696)	(0.473)	(0.866)	(0.205)	(0.774)	(0.063)	(0.722)
		[0.014]	[0.568]	[0.001]	[0.489]	[0.000]	[0.475]	[0.079]	[0.484]	[0.009]	[0.496]	[0.001]	[0.414]	[0.326]	[0.507]	[0.044]	[0.703]	[0.004]	[0.504]
		2.021	2.655	2.000	2.573	2.000	2.577	2.144	2.979	2.007	2.849	2.000	2.730	2.428	2.932	2.053	2.968	2.001	2.832
	1.5	(0.143)	(0.475)	(0.000)	(0.494)	(0.000)	(0.494)	(0.354)	(0.798)	(0.083)	(0.787)	(0.000)	(0.767)	(0.507)	(0.935)	(0.237)	(0.894)	(0.031)	(0.844)
		[0.021]	[0.655]	[0.000]	[0.573]	[0.000]	[0.577]	[0.143]	[0.671]	[0.007]	[0.603]	[0.000]	[0.534]	[0.422]	[0.609]	[0.051]	[0.654]	[0.001]	[0.591]
		2.025	2.687	2.001	2.663	2.000	2.626	2.224	3.183	2.016	3.080	2.000	2.936	2.536	3.109	2.071	3.028	2.002	3.018
2	2	(0.156)	(0.463)	(0.031)	(0.472)	(0.000)	(0.484)	(0.417)	(0.773)	(0.133)	(0.776)	(0.000)	(0.794)	(0.539)	(0.961)	(0.256)	(0.934)	(0.044)	(0.910)
		[0.025]	[0.687]	[0.001]	[0.663]	[0.000]	[0.626]	[0.224]	[0.776]	[0.015]	[0.736]	[0.000]	[0.651]	[0.517]	[0.690]	[0.071]	[0.671]	[0.002]	[0.670]
		2.010	2.520	2.002	2.482	2.000	2.439	2.123	2.748	2.005	2.695	2.000	2.592	2.507	2.895	2.048	2.858	2.000	2.717
	∞	(0.099)	(0.501)	(0.044)	(0.499)	(0.000)	(0.496)	(0.331)	(0.735)	(0.070)	(0.700)	(0.000)	(0.679)	(0.523)	(0.809)	(0.213)	(0.796)	(0.000)	(0.746)
		[0.010]	[0.522]	[0.002]	[0.482]	[0.000]	[0.439]	[0.122]	[0.575]	[0.005]	[0.556]	[0.000]	[0.482]	[0.495]	[0.650]	[0.048]	[0.632]	[0.000]	[0.561]

The table XXX

TABLE 9

Comparison between BCT and BIC - negative autocorrelation, Part 2

$\theta = -0.5$	n	3						4						5					
	T	100		200		400		100		200		400		100		200		400	
		BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC
r	η																		
	0.5	2.992 (0.089) [0.008]	2.996 (0.063) [0.004]	2.999 (0.031) [0.001]	3.000 (0.000) [0.000]	2.999 (0.031) [0.001]	2.999 (0.031) [0.001]	2.996 (0.063) [0.004]	3.391 (0.492) [0.395]	2.997 (0.070) [0.005]	3.310 (0.481) [0.317]	2.999 (0.031) [0.001]	3.229 (0.422) [0.231]	3.030 (0.202) [0.042]	3.358 (0.637) [0.303]	3.005 (0.104) [0.011]	3.300 (0.551) [0.262]	2.999 (0.031) [0.001]	3.250 (0.515) [0.227]
	1	2.998 (0.044) [0.002]	2.999 (0.031) [0.001]	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	2.999 (0.031) [0.001]	3.000 (0.000) [0.000]	3.004 (0.063) [0.004]	3.474 (0.333) [0.873]	3.001 (0.031) [0.001]	3.435 (0.496) [0.435]	3.000 (0.000) [0.000]	3.438 (0.496) [0.438]	3.066 (0.252) [0.068]	3.693 (0.768) [0.504]	3.006 (0.089) [0.005]	3.621 (0.705) [0.490]	3.001 (0.031) [0.001]	3.542 (0.696) [0.424]
	1.5	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	2.999 (0.031) [0.001]	3.000 (0.000) [0.000]	3.022 (0.146) [0.022]	3.620 (0.485) [0.620]	3.001 (0.031) [0.001]	3.616 (0.486) [0.616]	3.000 (0.000) [0.000]	3.537 (0.498) [0.537]	3.128 (0.334) [0.128]	3.837 (0.795) [0.589]	3.009 (0.094) [0.009]	3.905 (0.791) [0.635]	3.001 (0.031) [0.001]	3.767 (0.763) [0.565]
3	2	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	2.999 (0.031) [0.001]	3.000 (0.000) [0.000]	3.017 (0.129) [0.017]	3.682 (0.465) [0.682]	3.002 (0.044) [0.002]	3.647 (0.478) [0.647]	3.000 (0.000) [0.000]	3.628 (0.483) [0.628]	3.132 (0.344) [0.130]	3.997 (0.807) [0.673]	3.004 (0.063) [0.004]	3.943 (0.797) [0.652]	3.000 (0.000) [0.000]	3.881 (0.792) [0.620]
	∞	3.000 (0.000) [0.000]	2.997 (0.054) [0.003]	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	3.000 (0.000) [0.000]	3.105 (0.121) [0.015]	3.517 (0.499) [0.517]	3.000 (0.000) [0.000]	3.495 (0.500) [0.495]	3.000 (0.000) [0.000]	3.439 (0.496) [0.439]	3.133 (0.339) [0.133]	3.776 (0.748) [0.583]	3.005 (0.070) [0.005]	3.692 (0.719) [0.540]	3.000 (0.000) [0.000]	3.560 (0.656) [0.468]
	0.5							3.991 (0.094) [0.009]	3.995 (0.070) [0.005]	3.999 (0.031) [0.001]	4.000 (0.000) [0.000]	3.997 (0.054) [0.003]	3.998 (0.044) [0.002]	3.997 (0.083) [0.007]	4.362 (0.3484) [0.366]	3.993 (0.094) [0.006]	4.259 (0.488) [0.272]	3.996 (0.063) [0.004]	4.208 (0.408) [0.210]
	1							3.998 (0.044) [0.002]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.003 (0.054) [0.003]	4.509 (0.500) [0.509]	4.001 (0.031) [0.001]	4.448 (0.497) [0.448]	4.000 (0.000) [0.000]	4.407 (0.491) [0.407]
	1.5							3.999 (0.031) [0.001]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.008 (0.089) [0.008]	4.607 (0.488) [0.607]	4.001 (0.031) [0.001]	4.562 (0.496) [0.562]	4.000 (0.000) [0.000]	4.552 (0.497) [0.552]
4	2							4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	3.999 (0.031) [0.001]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.018 (0.133) [0.018]	4.697 (0.459) [0.697]	4.001 (0.031) [0.001]	4.634 (0.481) [0.634]	4.000 (0.000) [0.000]	4.624 (0.484) [0.624]
	∞							4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.000 (0.000) [0.000]	4.006 (0.077) [0.006]	4.526 (0.499) [0.526]	4.000 (0.000) [0.000]	4.476 (0.499) [0.476]	4.000 (0.000) [0.000]	4.466 (0.499) [0.466]
	0.5													4.988 (0.108) [0.012]	4.995 (0.070) [0.005]	4.998 (0.044) [0.002]	5.000 (0.000) [0.000]	4.994 (0.077) [0.006]	4.998 (0.044) [0.002]
	1													5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]
	1.5													5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]
5	2													5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]
	∞													5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]	5.000 (0.000) [0.000]

The table XXX

TABLE 10

Comparison between BCT and BIC - large N, i.i.d. case

$\theta = 0$	n	10						15						20					
	T	1000		1500		2000		1000		1500		2000		1000		1500		2000	
		BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC
r	η																		
		0.027	0.277	0.020	0.235	0.004	0.210	0.083	0.418	0.030	0.352	0.030	0.341	0.159	0.567	0.074	0.441	0.038	0.422
	0.5	(0.185)	(0.544)	(0.306)	(0.515)	(0.063)	(0.458)	(0.537)	(0.660)	(0.170)	(0.580)	(0.266)	(0.577)	(0.703)	(0.747)	(0.585)	(0.665)	(0.191)	(0.640)
		[0.024]	[0.231]	[0.011]	[0.196]	[0.004]	[0.189]	[0.068]	[0.334]	[0.030]	[0.299]	[0.024]	[0.292]	[0.133]	[0.434]	[0.057]	[0.359]	[0.038]	[0.349]
		0.006	0.068	0.007	0.069	0.000	0.049	0.025	0.095	0.004	0.091	0.010	0.077	0.036	0.129	0.005	0.111	0.005	0.103
	1	(0.134)	(0.267)	(0.192)	(0.268)	(0.000)	(0.215)	(0.486)	(0.309)	(0.063)	(0.298)	(0.227)	(0.277)	(0.396)	(0.349)	(0.070)	(0.332)	(0.070)	(0.317)
		[0.003]	[0.064]	[0.002]	[0.065]	[0.000]	[0.049]	[0.010]	[0.090]	[0.004]	[0.088]	[0.004]	[0.074]	[0.023]	[0.124]	[0.005]	[0.105]	[0.005]	[0.099]
		0.004	0.010	0.007	0.009	0.000	0.006	0.018	0.015	0.001	0.015	0.008	0.016	0.017	0.022	0.000	0.014	0.000	0.016
	1.5	(0.126)	(0.099)	(0.192)	(0.094)	(0.000)	(0.077)	(0.479)	(0.121)	(0.031)	(0.121)	(0.223)	(0.125)	(0.372)	(0.146)	(0.000)	(0.117)	(0.000)	(0.125)
		[0.001]	[0.010]	[0.002]	[0.009]	[0.000]	[0.006]	[0.003]	[0.015]	[0.001]	[0.015]	[0.002]	[0.016]	[0.004]	[0.022]	[0.000]	[0.014]	[0.000]	[0.016]
		0.004	0.002	0.007	0.001	0.000	0.001	0.017	0.002	0.000	0.002	0.008	0.004	0.016	0.004	0.000	0.004	0.000	0.004
0	2	(0.126)	(0.044)	(0.192)	(0.031)	(0.000)	(0.031)	(0.478)	(0.044)	(0.000)	(0.044)	(0.223)	(0.063)	(0.371)	(0.063)	(0.000)	(0.063)	(0.000)	(0.063)
		[0.001]	[0.002]	[0.002]	[0.001]	[0.000]	[0.001]	[0.002]	[0.002]	[0.000]	[0.002]	[0.002]	[0.004]	[0.003]	[0.004]	[0.000]	[0.004]	[0.000]	[0.004]
		0.002	1.332	0.000	1.394	0.000	1.437	0.003	1.358	0.000	1.350	0.000	1.322	0.005	1.565	0.000	1.454	0.016	1.410
	∞	(0.063)	(1.567)	(0.000)	(1.696)	(0.000)	(1.862)	(0.094)	(1.234)	(0.000)	(1.340)	(0.000)	(1.463)	(0.113)	(1.182)	(0.000)	(1.219)	(0.505)	(1.261)
		[0.001]	[0.581]	[0.000]	[0.570]	[0.000]	[0.549]	[0.001]	[0.695]	[0.000]	[0.648]	[0.000]	[0.592]	[0.002]	[0.782]	[0.000]	[0.743]	[0.001]	[0.702]
		1.021	1.241	1.005	1.209	1.007	1.201	1.079	1.414	1.027	1.343	1.027	1.306	1.133	1.507	1.085	1.464	1.040	1.425
	0.5	(0.196)	(0.520)	(0.104)	(0.477)	(0.122)	(0.454)	(0.401)	(0.684)	(0.200)	(0.601)	(0.210)	(0.571)	(0.523)	(0.721)	(0.603)	(0.723)	(0.237)	(0.663)
		[0.019]	[0.204]	[0.008]	[0.184]	[0.007]	[0.185]	[0.062]	[0.325]	[0.027]	[0.289]	[0.024]	[0.270]	[0.108]	[0.407]	[0.061]	[0.364]	[0.036]	[0.353]
		1.003	1.076	1.005	1.062	1.000	1.053	1.004	1.088	1.011	1.083	1.008	1.072	1.014	1.128	1.018	1.108	1.003	1.099
	1	(0.070)	(0.283)	(0.158)	(0.257)	(0.000)	(0.224)	(0.063)	(0.297)	(0.242)	(0.283)	(0.223)	(0.273)	(0.117)	(0.371)	(0.447)	(0.326)	(0.054)	(0.311)
		[0.002]	[0.071]	[0.001]	[0.058]	[0.000]	[0.053]	[0.004]	[0.084]	[0.003]	[0.081]	[0.002]	[0.068]	[0.014]	[0.117]	[0.005]	[0.103]	[0.003]	[0.095]
		1.003	1.011	1.005	1.008	1.000	1.006	1.016	1.015	1.001	1.016	1.006	1.019	1.004	1.024	1.000	1.013	1.000	1.017
	1.5	(0.094)	(0.104)	(0.158)	(0.089)	(0.000)	(0.077)	(0.444)	(0.121)	(0.031)	(0.125)	(0.189)	(0.136)	(0.099)	(0.153)	(0.000)	(0.113)	(0.000)	(0.129)
		[0.001]	[0.011]	[0.001]	[0.008]	[0.000]	[0.006]	[0.003]	[0.015]	[0.001]	[0.016]	[0.001]	[0.019]	[0.002]	[0.024]	[0.000]	[0.013]	[0.000]	[0.017]
		1.003	1.001	1.005	1.001	1.000	1.001	1.015	1.002	1.000	1.002	1.006	1.003	1.013	1.002	1.000	1.004	1.000	1.003
1	2	(0.094)	(0.031)	(0.158)	(0.031)	(0.000)	(0.031)	(0.443)	(0.044)	(0.000)	(0.044)	(0.189)	(0.054)	(0.330)	(0.044)	(0.000)	(0.063)	(0.000)	(0.054)
		[0.001]	[0.001]	[0.001]	[0.001]	[0.000]	[0.001]	[0.002]	[0.002]	[0.000]	[0.002]	[0.001]	[0.003]	[0.002]	[0.002]	[0.000]	[0.004]	[0.000]	[0.003]
		1.001	1.593	1.000	1.676	1.000	1.732	1.002	1.489	1.000	1.544	1.000	1.592	1.003	1.514	1.000	1.549	1.015	1.561
	∞	(0.031)	(1.140)	(0.000)	(1.261)	(0.000)	(1.413)	(0.063)	(0.837)	(0.000)	(0.927)	(0.000)	(1.060)	(0.070)	(0.795)	(0.000)	(0.884)	(0.474)	(0.886)
		[0.001]	[0.299]	[0.000]	[0.314]	[0.000]	[0.312]	[0.001]	[0.315]	[0.000]	[0.323]	[0.000]	[0.320]	[0.002]	[0.389]	[0.000]	[0.346]	[0.001]	[0.351]
		9.992	10.000	9.991	10.000	9.994	10.000	14.971	15.000	14.977	15.000	14.986	15.000	19.952	19.998	19.961	19.998	19.978	20.000
	0.5	(0.089)	(0.000)	(0.094)	(0.000)	(0.077)	(0.000)	(0.173)	(0.000)	(0.149)	(0.000)	(0.117)	(0.000)	(0.218)	(0.044)	(0.193)	(0.044)	(0.146)	(0.000)
		[0.008]	[0.000]	[0.009]	[0.000]	[0.006]	[0.000]	[0.028]	[0.000]	[0.023]	[0.000]	[0.014]	[0.000]	[0.047]	[0.002]	[0.039]	[0.002]	[0.022]	[0.000]
		9.998	10.000	9.998	10.000	9.999	10.000	14.996	15.000	14.997	15.000	14.997	15.000	19.995	20.000	19.991	20.000	19.997	20.000
	1	(0.044)	(0.000)	(0.044)	(0.000)	(0.031)	(0.000)	(0.063)	(0.000)	(0.054)	(0.000)	(0.054)	(0.000)	(0.070)	(0.000)	(0.104)	(0.000)	(0.054)	(0.000)
		[0.002]	[0.000]	[0.002]	[0.000]	[0.001]	[0.000]	[0.004]	[0.000]	[0.003]	[0.000]	[0.003]	[0.000]	[0.005]	[0.000]	[0.008]	[0.000]	[0.003]	[0.000]
		10.000	10.000	10.000	10.000	10.000	10.000	14.998	15.000	14.998	15.000	15.000	15.000	19.995	20.000	19.999	20.000	19.999	20.000
	1.5	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.044)	(0.000)	(0.044)	(0.000)	(0.000)	(0.000)	(0.070)	(0.000)	(0.031)	(0.000)	(0.031)	(0.000)
		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.002]	[0.000]	[0.002]	[0.000]	[0.000]	[0.000]	[0.005]	[0.000]	[0.001]	[0.000]	[0.001]	[0.000]
		10.000	10.000	10.000	10.000	10.000	10.000	15.000	15.000	15.000	15.000	15.000	15.000	20.000	20.000	19.999	20.000	20.000	20.000
n	2	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.031)	(0.000)	(0.000)	(0.000)
		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.001]	[0.000]	[0.000]	[0.000]
		10.000	10.000	10.000	10.000	10.000	10.000	15.000	15.000	15.000	15.000	15.000	15.000	20.000	20.000	20.000	20.000	20.000	20.000
	∞	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]

The table XXX

TABLE 11

Comparison between BCT and BIC - large N, positive autocorrelation

$\theta = 0.5$	n	10						15						20					
	T	1000		1500		2000		1000		1500		2000		1000		1500		2000	
		BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC
r	η																		
		0.009	0.382	0.010	0.346	0.001	0.304	0.037	0.596	0.009	0.520	0.012	0.507	0.051	0.836	0.016	0.708	0.005	0.623
	0.5	(0.144)	(0.602)	(0.199)	(0.612)	(0.031)	(0.545)	(0.497)	(0.785)	(0.094)	(0.719)	(0.232)	(0.696)	(0.413)	(0.893)	(0.125)	(0.838)	(0.070)	(0.753)
		[0.006]	[0.324]	[0.005]	[0.282]	[0.001]	[0.262]	[0.022]	[0.444]	[0.009]	[0.401]	[0.006]	[0.406]	[0.038]	[0.574]	[0.016]	[0.506]	[0.005]	[0.481]
		0.004	0.118	0.007	0.109	0.000	0.076	0.018	0.169	0.004	0.289	0.009	0.126	0.017	0.224	0.001	0.019	0.001	0.166
	1	(0.126)	(0.349)	(0.192)	(0.327)	(0.000)	(0.268)	(0.479)	(0.432)	(0.063)	(0.530)	(0.225)	(0.355)	(0.372)	(0.462)	(0.031)	(0.429)	(0.031)	(0.403)
		[0.001]	[0.109]	[0.002]	[0.104]	[0.000]	[0.075]	[0.003]	[0.146]	[0.004]	[0.253]	[0.003]	[0.118]	[0.004]	[0.204]	[0.001]	[0.175]	[0.001]	[0.154]
		0.004	0.025	0.007	0.026	0.000	0.012	0.017	0.036	0.001	0.135	0.008	0.027	0.015	0.038	0.000	0.033	0.000	0.037
	1.5	(0.126)	(0.156)	(0.192)	(0.159)	(0.000)	(0.108)	(0.478)	(0.186)	(0.031)	(0.359)	(0.223)	(0.168)	(0.370)	(0.191)	(0.000)	(0.178)	(0.000)	(0.194)
		[0.004]	[0.025]	[0.002]	[0.026]	[0.000]	[0.012]	[0.002]	[0.036]	[0.001]	[0.129]	[0.002]	[0.026]	[0.002]	[0.038]	[0.000]	[0.033]	[0.000]	[0.036]
		0.004	0.007	0.007	0.004	0.000	0.001	0.017	0.006	0.000	0.031	0.008	0.005	0.015	0.006	0.000	0.004	0.000	0.002
0	2	(0.126)	(0.083)	(0.192)	(0.063)	(0.000)	(0.031)	(0.478)	(0.077)	(0.000)	(0.184)	(0.223)	(0.070)	(0.370)	(0.077)	(0.000)	(0.063)	(0.000)	(0.044)
		[0.004]	[0.007]	[0.002]	[0.004]	[0.000]	[0.001]	[0.002]	[0.006]	[0.000]	[0.029]	[0.002]	[0.005]	[0.002]	[0.005]	[0.000]	[0.004]	[0.000]	[0.002]
		0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.005	0.000	0.000	0.000	0.016	0.000
	∞	(0.044)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.113)	(0.000)	(0.000)	(0.000)	(0.505)	(0.000)
		[0.002]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.002]	[0.000]	[0.000]	[0.000]	[0.001]	[0.000]
		1.000	1.342	1.001	1.278	1.000	1.259	1.026	1.544	1.024	1.485	1.008	1.455	1.046	1.765	1.046	1.676	1.008	1.598
	0.5	(0.118)	(0.587)	(0.114)	(0.548)	(0.195)	(0.508)	(0.339)	(0.754)	(0.342)	(0.712)	(0.209)	(0.688)	(0.374)	(0.874)	(0.407)	(0.820)	(0.264)	(0.748)
		[0.011]	[0.291]	[0.010]	[0.234]	[0.015]	[0.233]	[0.026]	[0.416]	[0.012]	[0.385]	[0.022]	[0.378]	[0.042]	[0.556]	[0.028]	[0.507]	[0.020]	[0.473]
		1.000	1.098	0.999	1.086	0.991	1.067	1.012	1.154	0.996	1.256	1.000	1.117	1.010	1.209	0.995	1.173	0.995	1.153
	1	(0.109)	(0.320)	(0.176)	(0.291)	(0.094)	(0.250)	(0.447)	(0.405)	(0.063)	(0.492)	(0.205)	(0.339)	(0.334)	(0.437)	(0.070)	(0.404)	(0.070)	(0.384)
		[0.004]	[0.091]	[0.007]	[0.083]	[0.009]	[0.067]	[0.005]	[0.137]	[0.004]	[0.231]	[0.007]	[0.111]	[0.005]	[0.196]	[0.005]	[0.163]	[0.005]	[0.144]
		0.998	1.018	0.997	1.018	0.990	1.011	1.011	1.029	0.996	1.121	1.002	1.022	1.011	1.035	0.995	1.032	0.995	1.003
	1.5	(0.118)	(0.133)	(0.181)	(0.133)	(0.099)	(0.104)	(0.448)	(0.167)	(0.063)	(0.338)	(0.200)	(0.153)	(0.033)	(0.183)	(0.070)	(0.176)	(0.070)	(0.176)
		[0.006]	[0.018]	[0.009]	[0.018]	[0.010]	[0.011]	[0.006]	[0.029]	[0.004]	[0.117]	[0.005]	[0.021]	[0.004]	[0.035]	[0.005]	[0.032]	[0.005]	[0.029]
		1.000	1.004	0.998	1.002	0.988	1.001	1.014	1.003	0.994	1.021	1.003	1.005	1.010	1.005	0.996	1.004	0.993	1.002
1	2	(0.109)	(0.063)	(0.178)	(0.044)	(0.108)	(0.031)	(0.444)	(0.054)	(0.077)	(0.150)	(0.197)	(0.070)	(0.334)	(0.077)	(0.063)	(0.063)	(0.083)	(0.044)
		[0.006]	[0.004]	[0.008]	[0.002]	[0.012]	[0.001]	[0.003]	[0.003]	[0.006]	[0.020]	[0.004]	[0.005]	[0.005]	[0.005]	[0.004]	[0.004]	[0.007]	[0.002]
		0.998	1.000	0.998	1.000	0.989	1.000	0.998	1.000	0.992	1.000	0.990	1.000	1.000	1.000	1.000	1.000	1.009	1.000
	∞	(0.063)	(0.000)	(0.063)	(0.000)	(0.104)	(0.000)	(0.044)	(0.000)	(0.089)	(0.000)	(0.099)	(0.000)	(0.085)	(0.000)	(0.000)	(0.000)	(0.480)	(0.000)
		[0.004]	[0.000]	[0.004]	[0.000]	[0.011]	[0.000]	[0.002]	[0.000]	[0.008]	[0.000]	[0.010]	[0.000]	[0.002]	[0.000]	[0.000]	[0.000]	[0.007]	[0.000]
		9,722	10.000	9,771	9.999	9,810	9.999	14.439	14.999	14.543	14.999	14.597	15.000	18.994	19.999	19.251	20.000	19.380	19.998
	0.5	(0.494)	(0.000)	(0.456)	(0.031)	(0.426)	(0.031)	(0.696)	(0.031)	(0.653)	(0.031)	(0.647)	(0.000)	(0.933)	(0.031)	(0.837)	(0.000)	(0.723)	(0.044)
		[0.257]	[0.000]	[0.261]	[0.001]	[0.177]	[0.001]	[0.457]	[0.001]	[0.378]	[0.001]	[0.328]	[0.000]	[0.659]	[0.001]	[0.540]	[0.000]	[0.487]	[0.002]
		9.711	10.000	9.728	10.000	9,750	10.000	14.352	15.000	14.506	15.000	14.544	15.000	18.875	20.000	19.178	20.000	19.305	20.000
	1	(0.515)	(0.000)	(0.510)	(0.000)	(0.497)	(0.000)	(0.735)	(0.000)	(0.681)	(0.000)	(0.646)	(0.000)	(0.996)	(0.000)	(0.880)	(0.000)	(0.780)	(0.000)
		[0.262]	[0.000]	[0.243]	[0.000]	[0.221]	[0.000]	[0.504]	[0.000]	[0.398]	[0.000]	[0.379]	[0.000]	[0.701]	[0.000]	[0.569]	[0.000]	[0.528]	[0.000]
		9.704	10.000	9.754	10.000	9.759	10.000	14.380	15.000	14.498	15.000	14.551	15.000	18.903	20.000	19.193	20.000	19.330	20.000
	1.5	(0.529)	(0.000)	(0.493)	(0.000)	(0.489)	(0.000)	(0.763)	(0.000)	(0.660)	(0.000)	(0.658)	(0.000)	(0.998)	(0.000)	(0.844)	(0.000)	(0.807)	(0.000)
		[0.263]	[0.000]	[0.223]	[0.000]	[0.217]	[0.000]	[0.475]	[0.000]	[0.420]	[0.000]	[0.368]	[0.000]	[0.685]	[0.000]	[0.579]	[0.000]	[0.484]	[0.000]
		9.709	10.000	9.769	10.000	9,770	10.000	14.436	15.000	14.504	15.000	14.607	15.000	18.952	20.000	19.208	20.000	19.348	20.000
n	2	(0.512)	(0.000)	(0.455)	(0.000)	(0.474)	(0.000)	(0.717)	(0.000)	(0.694)	(0.000)	(0.639)	(0.000)	(1.005)	(0.000)	(0.913)	(0.000)	(0.799)	(0.000)
		[0.264]	[0.000]	[0.216]	[0.000]	[0.209]	[0.000]	[0.449]	[0.000]	[0.392]	[0.000]	[0.325]	[0.000]	[0.652]	[0.000]	[0.527]	[0.000]	[0.474]	[0.000]
		9,747	10.000	9,768	10.000	9.804	10.000	14.412	15.000	14.552	15.000	14.623	15.000	18.991	20.000	19.227	20.000	19.381	20.000
	∞	(0.466)	(0.000)	(0.471)	(0.000)	(0.449)	(0.000)	(0.750)	(0.000)	(0.672)	(0.000)	(0.610)	(0.000)	(0.974)	(0.000)	(0.856)	(0.000)	(0.806)	(0.000)
		[0.239]	[0.000]	[0.211]	[0.000]	[0.175]	[0.000]	[0.442]	[0.000]	[0.358]	[0.000]	[0.312]	[0.000]	[0.641]	[0.000]	[0.547]	[0.000]	[0.454]	[0.000]

The table XXX

Table 12

Comparison between BCT and BIC - large N, negative autocorrelation

$\theta=-0.5$	n	10						15						20					
	T	1000		1500		2000		1000		1500		2000		1000		1500		2000	
		BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC	BCT	BIC
r	η																		
		0.190	1.325	0.102	1.281	0.051	1.224	0.617	1.699	0.281	1.690	0.176	1.693	1.392	1.915	0.641	2.038	0.357	2.046
	0.5	(0.426)	(1.045)	(0.315)	(0.993)	(0.233)	(0.959)	(0.740)	(1.156)	(0.506)	(1.154)	(0.432)	(1.173)	(1.062)	(1.309)	(0.812)	(1.343)	(0.569)	(1.275)
		[0.177]	[0.771]	[0.098]	[0.768]	[0.048]	[0.758]	[0.480]	[0.854]	[0.255]	[0.255]	[0.159]	[0.847]	[0.812]	[0.877]	[0.487]	[0.890]	[0.313]	[0.899]
		0.054	1.090	0.030	0.987	0.004	0.965	0.266	1.192	0.072	1.238	0.039	1.179	0.832	1.190	0.205	1.286	0.087	1.310
	1	(0.243)	(0.888)	(0.321)	(0.850)	(0.063)	(0.835)	(0.481)	(0.974)	(0.266)	(0.975)	(0.282)	(0.952)	(0.796)	(0.989)	(0.543)	(0.998)	(0.292)	(0.976)
		[0.051]	[0.730]	[0.021]	[0.691]	[0.004]	[0.681]	[0.248]	[0.735]	[0.070]	[0.762]	[0.033]	[0.743]	[0.617]	[0.733]	[0.184]	[0.763]	[0.084]	[0.791]
		0.014	1.023	0.007	0.916	0.001	0.841	0.098	0.954	0.011	0.956	0.013	0.910	0.520	0.749	0.040	0.869	0.015	0.887
	1.5	(0.160)	(0.851)	(0.192)	(0.795)	(0.031)	(0.795)	(0.548)	(0.854)	(0.104)	(0.848)	(0.234)	(0.805)	(0.655)	(0.818)	(0.196)	(0.785)	(0.121)	(0.799)
		[0.011]	[0.715]	[0.002]	[0.673]	[0.001]	[0.626]	[0.083]	[0.669]	[0.011]	[0.673]	[0.007]	[0.662]	[0.438]	[0.551]	[0.040]	[0.645]	[0.015]	[0.652]
		0.004	0.960	0.006	0.909	0.000	0.784	0.055	0.821	0.002	0.875	0.009	0.820	0.446	0.598	0.010	0.683	0.003	0.752
0	2	(0.126)	(0.813)	(0.189)	(0.800)	(0.000)	(0.753)	(0.514)	(0.761)	(0.044)	(0.811)	(0.225)	(0.769)	(0.638)	(0.704)	(0.099)	(0.724)	(0.054)	(0.709)
		[0.001]	[0.690]	[0.001]	[0.664]	[0.000]	[0.605]	[0.040]	[0.628]	[0.002]	[0.639]	[0.003]	[0.619]	[0.380]	[0.481]	[0.010]	[0.543]	[0.003]	[0.602]
		0.003	0.761	0.000	0.686	0.000	0.631	0.017	0.642	0.000	0.677	0.000	0.590	0.302	0.398	0.002	0.546	0.023	0.539
	∞	(0.094)	(0.733)	(0.000)	(0.736)	(0.000)	(0.682)	(0.150)	(0.693)	(0.000)	(0.695)	(0.000)	(0.663)	(0.561)	(0.586)	(0.044)	(0.643)	(0.552)	(0.623)
		[0.001]	[0.601]	[0.000]	[0.534]	[0.000]	[0.524]	[0.015]	[0.521]	[0.000]	[0.553]	[0.000]	[0.499]	[0.266]	[0.349]	[0.002]	[0.465]	[0.002]	[0.470]
		1.126	2.156	1.063	2.114	1.040	2.053	1.488	2.552	1.231	2.533	1.142	2.533	2.137	2.783	1.511	2.897	1.300	2.931
	0.5	(0.355)	(0.945)	(0.288)	(0.947)	(0.220)	(0.920)	(0.669)	(1.149)	(0.646)	(1.107)	(0.399)	(1.099)	(0.979)	(1.278)	(0.728)	(1.259)	(0.792)	(1.259)
		[0.118]	[0.733]	[0.057]	[0.717]	[0.036]	[0.699]	[0.400]	[0.828]	[0.195]	[0.818]	[0.133]	[0.819]	[0.723]	[0.852]	[0.415]	[0.847]	[0.254]	[0.881]
		1.033	1.938	1.009	1.895	1.001	1.816	1.156	2.109	1.049	2.119	1.025	2.058	1.592	2.146	1.154	2.221	1.058	2.191
	1	(0.178)	(0.872)	(0.094)	(0.843)	(0.031)	(0.787)	(0.379)	(0.973)	(0.215)	(0.930)	(0.156)	(0.933)	(0.673)	(0.961)	(0.463)	(0.974)	(0.246)	(0.955)
		[0.033]	[0.646]	[0.009]	[0.638]	[0.001]	[0.610]	[0.150]	[0.706]	[0.049]	[0.724]	[0.025]	[0.694]	[0.495]	[0.736]	[0.140]	[0.750]	[0.055]	[0.747]
		1.005	1.924	1.005	1.840	1.000	1.741	1.062	1.880	1.013	1.897	1.020	1.870	1.347	1.742	1.026	1.821	1.008	1.878
	1.5	(0.070)	(0.826)	(0.158)	(0.771)	(0.000)	(0.748)	(0.297)	(0.811)	(0.243)	(0.832)	(0.343)	(0.801)	(0.546)	(0.745)	(0.159)	(0.789)	(0.089)	(0.813)
		[0.005]	[0.666]	[0.001]	[0.641]	[0.000]	[0.574]	[0.057]	[0.653]	[0.007]	[0.642]	[0.007]	[0.642]	[0.313]	[0.572]	[0.026]	[0.613]	[0.008]	[0.633]
		1.003	1.929	1.005	1.880	1.000	1.743	1.026	1.831	1.009	1.852	1.006	1.809	1.264	1.591	1.019	1.674	1.000	1.730
1	2	(0.070)	(0.792)	(0.158)	(0.783)	(0.000)	(0.716)	(0.339)	(0.766)	(0.225)	(0.782)	(0.161)	(0.770)	(0.590)	(0.678)	(0.508)	(0.719)	(0.000)	(0.706)
		[0.002]	[0.683]	[0.001]	[0.654]	[0.000]	[0.593]	[0.017]	[0.633]	[0.003]	[0.635]	[0.002]	[0.616]	[0.232]	[0.489]	[0.004]	[0.540]	[0.000]	[0.584]
		1.002	1.782	1.000	1.701	1.000	1.620	1.005	1.686	1.000	1.696	1.000	1.596	1.197	1.461	1.000	1.553	1.021	1.552
	∞	(0.063)	(0.747)	(0.000)	(0.708)	(0.000)	(0.705)	(0.070)	(0.699)	(0.000)	(0.707)	(0.000)	(0.667)	(0.508)	(0.623)	(0.000)	(0.630)	(0.510)	(0.629)
		[0.001]	[0.604]	[0.000]	[0.567]	[0.000]	[0.505]	[0.005]	[0.556]	[0.000]	[0.566]	[0.000]	[0.503]	[0.176]	[0.398]	[0.000]	[0.480]	[0.002]	[0.480]
		9,999	10.000	9.997	9.999	9.999	10.000	14.996	15.000	14.997	15.000	14.998	15.000	19.992	19.999	19.999	20.000	20.000	20.000
	0.5	(0.031)	(0.000)	(0.054)	(0.031)	(0.031)	(0.000)	(0.063)	(0.000)	(0.054)	(0.000)	(0.044)	(0.000)	(0.089)	(0.031)	(0.031)	(0.000)	(0.000)	(0.000)
		[0.001]	[0.000]	[0.003]	[0.001]	[0.001]	[0.000]	[0.004]	[0.000]	[0.003]	[0.000]	[0.002]	[0.000]	[0.008]	[0.001]	[0.001]	[0.000]	[0.000]	[0.000]
		10.000	10.000	10.000	10.000	10.000	10.000	15.000	15.000	15.000	15.000	15.000	15.000	19.999	20.000	20.000	20.000	20.000	20.000
	1	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.031)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.001]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]
		10.000	10.000	10.000	10.000	10.000	10.000	15.000	15.000	15.000	15.000	15.000	15.000	20.000	20.000	20.000	20.000	20.000	20.000
	1.5	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]
		10.000	10.000	10.000	10.000	10.000	10.000	15.000	15.000	15.000	15.000	15.000	15.000	20.000	20.000	20.000	20.000	20.000	20.000
n	2	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]
		10.000	10.000	10.000	10.000	10.000	10.000	15.000	15.000	15.000	15.000	15.000	15.000	20.000	20.000	20.000	20.000	20.000	20.000
	∞	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
		[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]	[0.000]

The table XXX