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Systematic Comovement in Threshold Group-Factor Models *

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Abstract

We study regime-specific systematic comovement between two large panels of variables that exhibit an approximate factor structure. Within each panel, we identify threshold-type regimes through shifts in the factor loadings. For the resulting regimes, and with regard to the relation between any two variables in different panels, we define as "systematic" the comovement that is generated by the common components of the variables. In our setup, changes in comovement are identified by regime shifts in the loadings. After constructing measures of systematic comovement between the two panels, we propose estimators for these measures and derive their asymptotic properties. We develop inferential procedures to formally test for changes in systematic comovement between regimes. The empirical analysis of two large panels of U.S. and international equity returns shows that their systematic comovement increases when U.S. macroeconomic uncertainty is high as determined by our estimation procedure.

Keywords: Comovement, Approximate Factor Model, Groups, Threshold, PCA

JEL Codes: C12, C24, C38, C55, F39

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1 Introduction

Comovement is paramount in economics and finance. The cross-sectional variation among macroeconomic variables is valuable to study business cycle fluctuations, see e.g. Forni and Reichlin (1998), Stock and Watson (2002b), and Cheng, Liao, and Schorfheide (2016). Comovement among asset returns has implications for portfolio diversification, see e.g. Ang and Timmermann (2012), and deepens the understanding of the global financial integration, see e.g. Miranda-Agrippino and Rey (2020). When a large number of variables is involved, these are likely to exhibit a dense structure that reflects common sources of variation, as argued in Giannone, Lenza, and Primiceri (2021). It is thus appealing to assume an underlying approximate factor structure, in which a small number of common factors drives the cross-sectional systematic variation among the variables in a panel through the common components, see Stock and Watson (2010), and references therein. Analysis of comovement in large dimensional factor models has mainly looked at a single panel of variables, which means that it has focused on comovement *within* the group. We depart from this scenario and consider a more general group-factor structure involving two distinct panels of variables, each of which admits a common factor representation, and study comovement *between* the groups. We further allow for time variation in comovement induced by discrete changes in the factor loadings. We thus study comovement subject to possible discrete time variation between two groups of variables, each of which allows for an approximate factor structure. Our paper makes three main contributions: it develops a suitable econometric model; it proposes valid measures of systematic comovement; it illustrates the usefulness of our methodological framework through an extensive empirical analysis.

Factor representations are widely used to model comovement within large panels of financial and economic data. Seminal contributions studying static or dynamic frameworks are Connor and Korajczyk (1986, 1988), Forni, Hallin, Lippi, and Reichlin (2000, 2004), Bai and Ng (2002), Stock and Watson (2002a,b), Bai (2003), and Forni, Hallin, Lippi, and Zaffaroni (2015, 2017). All these studies focus on factor models as applied to one group of variables and assume that both loadings and number of factors remain constant over time. These assumptions impose restrictions along both the cross-sectional and the time series dimensions of the underlying data generating process, and such restrictions may not always be accurate in empirically relevant scenarios. The one-group restriction implies that all factors are pervasive. However, as argued in Goyal, Pérignon, and Villa (2008), there may arise situations in which it is necessary to distinguish between *common* factors, which affect variables in all groups, and *group-specific* factors, which affect only variables within a given group. Factor loadings may also be time-varying, as argued for example in Bekaert, Hodrick, and Zhang (2009) in the context of analyzing international equity return comovement. An accurate analysis of comovement should consider these two features within a unifying framework, otherwise the empirical results may lead to misleading conclusions.

Multi-level factor models have been the focus of attention in a number of contributions. Goyal, Pérignon, and Villa (2008) study the factor structure between two groups of returns from stocks on NYSE and Nasdaq, and develop a procedure to determine the number of common factors. Breitung and Eickmeier (2016), and Choi, Kim, Kim, and Kwark (2018), propose a canonical correlation estimator for multi-level factor models. Ando and Bai (2017), and Han (2021), focus upon shrinkage-based estimation. Hallin and Liska (2011) extend the dynamic factor model to the case of a finite number of groups; their model is then applied in Barigozzi and Hallin (2016, 2017), and Barigozzi, Hallin, and Soccorsi (2019). Andreou, Gagliardini, Ghysels, and Rubin (2019) formally propose a test for the number of common factors between two groups of variables. All these contributions focus on linear factor representations and do not consider the possibility of potentially time-dependent loadings.

There now exists a vast literature studying time variation in the loadings in large dimensional factor models. Bates, Plagborg-Moller, Stock, and Watson (2013) study the robustness properties of the asymptotic principal components estimator as applied to a misspecified linear factor model when the true underlying data generating process exhibits time-varing loadings. Breitung and Eickmeier (2011), Corradi and Swanson (2014), Chen, Dolado, and Gonzalo (2014), Han and Inoue (2015), Yamamoto and Tanaka (2015), Massacci (2020), and Barigozzi and Trapani (2021), develop inferential procedures to detect discrete shifts in factor loadings. Cheng, Liao, and Schorfheide (2016), Baltagi, Kao, and Wang (2017, 2021), Su and Wang (2017), Massacci (2017), Ma and Su (2018), Pelger and Xiong (2018), Zaffaroni (2019), and Kelly, Pruitt, and Su (2020), propose model specifications that allow for either discrete or continuous shifts in the factor loadings. These contributions work under the one-group maintained assumption, which implies that all factors are pervasive for the observable variables.

We fill the existing gap in the literature by developing a group-factor model that allows for time variation in the factor loadings: this is the first contribution of our paper. In order to ease the exposition, we propose a two-group specification, although the model can be extended to a finite number of groups. Each group admits an underlying approximate static factor representation in which the loadings exhibit two discrete regimes. Within each group, the shift between states is modeled through the threshold principle of Pearson (1900): at a given point in time, the factor loadings depend on the relative position of an observable state variable with respect to the corresponding unknown threshold parameter. This general set up extends the existing literature by allowing for regime-specific group-factor structure. We propose to estimate our threshold group-factor model by least squares: following Bai and Ng (2002) and Stock and Watson (2002a), we implement a group-by-group estimator based on asymptotic principal components to estimate factors, loadings and threshold value. This approach is appealing since it allows to estimate each group independently and therefore does not require any restriction across the groups.

Given our threshold group-factor model, we propose regime-specific measures of systematic comovement between the groups: this is the second contribution of our paper. The building block is the *pairwise* common component, which is defined as the product between the common components of two cross-sectional units that belong to different groups. As a generalization of the common components analyzed in Bai (2003), the pairwise common component captures the instantaneous comovement between a pair of cross-sectional units. Based on the pairwise common components, we formally develop two regime-specific measures of comovement, namely average systematic *covariance* and *correlation*. The former is obtained as the within-regime weighted average of the pairwise common components; the latter is constructed in an analogous way by suitably standardizing the pairwise common components so that the absolute value of the resulting measure lies within the unit interval. Our measures of comovement require assigning *a priori* weights to the cross-sectional units within each group; the comovement between each pair is then obtained by specifying suitable weighting schemes. We motivate our measures of comovement by showing that the average systematic correlation has an upper bound that depends on the canonical correlations among the factors in the two groups and the pervasiveness of the factors within each group, as measured by the R-square of the factors. We propose valid estimators for the pairwise common components and for the measures of systematic comovement and we analytically derive their asymptotic distributions based on a set of empirically plausible assumptions. We further strengthen our methodology by advancing formal inferential procedures to detect changes in systematic comovement among the regimes: our test statistics are easily implementable and we prove their asymptotically normal distribution.[1] We

[1]We develop tests for changes in systematic comovement between the regimes identified and estimated by the threshold factor model. Nevertheless, it is easy to show that the same tests can be applied for any two regimes

corroborate the validity of our theoretical results through a comprehensive Monte Carlo analysis, which shows the excellent finite sample performance of estimators and test statistics.

Finally, we employ our theoretical results to study comovement in global equity market returns: this is the third contribution of our paper. We consider two large groups of equity portfolio returns, namely U.S. and international, over the sample period running between January 1991 and December 2019. As a common state variable driving the regimes we opt for the U.S. macroeconomic uncertainty index of Jurado, Ludvigson, and Ng (2015): our model thus allows to identify low and high uncertainty regimes within each group. Interestingly, our estimates show that the regimes are perfectly synchronized across the groups, with a sample split occurring at an estimated value equal to the 77th percentile of the empirical distribution of the U.S. macroeconomic uncertainty index. Notice that this first result of the two equity markets switching at the same time between low and high uncertainty regimes is achieved without imposing any restrictions on the estimation procedure and thus is a genuine feature of the data. We further show that the first estimated factor within each group is highly correlated with the first factor of Miranda-Agrippino and Rey (2020): therefore, the dynamics of equity returns within each group follow those of the global financial cycle discussed in Rey (2018). Our empirical analysis thus investigates the dynamics of systematic comovement in global equity markets over the course of the global financial cycle. In particular, we show that both pairwise and average systematic comovement between the two groups is significantly higher during periods of high U.S. macroeconomic uncertainty as compared to times of low uncertainty. To the best of our knowledge, this result has not been previously documented.

The rest of the paper is organized as follows. Section 2 introduces the econometric model, the measures of systematic comovement, and the related hypotheses about their changes across regimes. Section 3 discusses estimation of the model and measures of systematic comovement. Section 4 collects all asymptotic results. Section 5 presents the main findings of an extensive Monte Carlo study. Section 6 covers the empirical analysis, and Section 7 concludes. Appendix A states all Assumptions. Appendix B provides the estimators for the asymptotic variances appearing in Theorems 1 through 5 in Section 4. The Online Appendix C includes the proofs of Proposition 1 and of all Theorems, together with technical Lemmas. The Online Appendix D collects the tables of results for the Monte Carlo experiments. The Supplementary Material (henceforth SM) collects additional Monte Carlo and empirical results.

identified exogenously by the econometrician.

2 Threshold group-factor model and systematic comovement

2.1 Model

We are interested in the threshold group-factor model

$$\begin{aligned} x_{1i_1t} &= \mathbb{I}\left(z_{1t} \le \theta_1\right) \boldsymbol{\lambda}'_{1Li_1} \boldsymbol{f}_{1Lt} + \mathbb{I}\left(z_{1t} > \theta_1\right) \boldsymbol{\lambda}'_{1Hi_1} \boldsymbol{f}_{1Ht} + e_{1i_1t} \ , \\ x_{2i_2t} &= \mathbb{I}\left(z_{2t} \le \theta_2\right) \boldsymbol{\lambda}'_{2Li_2} \boldsymbol{f}_{2Lt} + \mathbb{I}\left(z_{2t} > \theta_2\right) \boldsymbol{\lambda}'_{2Hi_2} \boldsymbol{f}_{2Ht} + e_{2i_2t} \ , \end{aligned} \tag{1}$$

where $\mathbb{I}(\cdot)$ denotes the indicator function. For $g = 1, 2$, $i_g = 1, \ldots, N_g$, and $t = 1, \ldots, T$, x_{gi_gt} is the observable dependent variable. The threshold variable z_{gt} is observable, with corresponding unknown threshold value θ_g: z_{gt} and θ_g are further discussed in Section 2.1.1 below. For $j_g = L, H$, the $K_{gj_g} \times 1$ vector $\boldsymbol{f}_{gj_gt} = \left[f_{gj_g1t}, ..., f_{gj_gK_{gj_g}t}\right]'$ collects the latent factors, with corresponding $K_{gj_g} \times 1$ vector of loadings $\boldsymbol{\lambda}_{gj_gi_g} = \left[\lambda_{gj_gi_g1}, \ldots, \lambda_{gj_gi_gK_{gjg}}\right]'$. Finally, e_{gi_gt} is the idiosyncratic component, with features given in Assumption A.4.

2.1.1 Regimes

The model in (1) generally allows for group-specific threshold variable and value z_{gt} and θ_g, respectively, for $g = 1, 2$. Four regimes arise within this general framework, namely: (i) $\{z_{1t} \le \theta_1\} \cap \{z_{2t} \le \theta_2\}$; (ii) $\{z_{1t} \le \theta_1\} \cap \{z_{2t} > \theta_2\}$; (iii) $\{z_{1t} > \theta_1\} \cap \{z_{2t} \le \theta_2\}$; (iv) $\{z_{1t} > \theta_1\} \cap \{z_{2t} > \theta_2\}$. Our set up also allows for a more restrictive scenario in which the threshold variable $z_{1t} = z_{2t} = z_t$ is common across the groups. In this case, the model generates three regimes, that is: (i) $z_t \le \min\{\theta_1, \theta_2\}$; (ii) $\min\{\theta_1, \theta_2\} < z_t \le \max\{\theta_1, \theta_2\}$; (iii) $z_t > \max\{\theta_1, \theta_2\}$. In the latter scenario, if $\theta_1 = \theta_2 = \theta$, than the model has only two regimes: (i) $z_t \le \theta$ and (ii) $z_t > \theta$.

In what follows, we deal with the general case in which the threshold variables are group-specific: the case $z_{1t} = z_{2t} = z_t$ is nested within this scenario. The threshold values θ_1 and θ_2 are generally unknown: we thus do not impose any restriction on them.

2.1.2 Common and group-specific factors

Following Andreou, Gagliardini, Ghysels, and Rubin (2019), we allow for both common and group specific factors within each regime. Formally, we let the pervasive factors $\boldsymbol{f}_{gj_gt}$ be defined as $\boldsymbol{f}_{gj_gt} = [\boldsymbol{f}^{c\prime}_{j_1j_2t}, \boldsymbol{f}^{s\prime}_{gj_gt}]'$, where $\boldsymbol{f}^{c}_{j_1j_2t}$ and $\boldsymbol{f}^{s}_{gj_gt}$ are $K^c_{j_1j_2} \times 1$ and $K^s_{gj_g} \times 1$ vectors of common and group-specific factors, respectively, such that either $K^c_{j_1j_2} > 0$ or $K^s_{gj_g} > 0$ (or both), and $K^c_{j_1j_2} + K^s_{gj_g} = K_{gj_g}$:

the model only has group-specific factors within regime j_g if $K^c_{j_1 j_2} = 0$; the common factors are the only drivers of comovement both within and across groups if $K^s_{gj_g} = 0$. Importantly, we do not impose ex-ante the pervasive factors to be common in the two groups, in either regime.

2.1.3 Further notation

In what follows, $\boldsymbol{x}_{gt} = \left[x_{g1t}, ..., x_{gN_gt}\right]'$ is the $N_g \times 1$ vector of observable dependent variables within each group. The $N_g \times K_{gj_g}$ matrix $\boldsymbol{\Lambda}_{gj_g} = \left[\boldsymbol{\lambda}_{gj_g1}, \ldots, \boldsymbol{\lambda}_{gj_gN_g}\right]'$ collects the loadings. The $N_g \times 1$ vector of idiosyncratic components is $\boldsymbol{e}_{gt} = \left[e_{g1t}, ..., e_{gN_gt}\right]'$. We define $\mathbb{I}_{gLt}(\theta_g) = \mathbb{I}(z_{gt} \leq \theta_g)$ and $\mathbb{I}_{gHt}(\theta_g) = \mathbb{I}(z_{gt} > \theta_g)$. We let θ^0_g, $\boldsymbol{f}^0_{gj_gt}$, and $\boldsymbol{\Lambda}^0_{gj_g}$, be the true values of θ_g, $\boldsymbol{f}_{gj_gt}$, and $\boldsymbol{\Lambda}_{gj_g}$, respectively.

2.2 Systematic comovement

2.2.1 Pairwise common components

Let $c^0_{gj_gi_gt}$ be the common component within group $g = 1, 2$, and regime $j_g = L, H$, for cross-sectional unit $i_g = 1, \ldots, N_g$, at time period $t = 1, \ldots, T$. Following Bai and Ng (2002), $c^0_{gj_gi_gt}$ is defined as

$$c^0_{gj_gi_gt} \;:=\; \boldsymbol{\lambda}^{0\prime}_{gj_gi_g}\boldsymbol{f}^0_{gj_gt}. \tag{2}$$

The common component in (2) naturally extends the common component defined in linear large dimensional factor models to allow for groups and regimes: see Bai (2003) for an analysis of common components in linear large dimensional factor models. Based on (2), the common component within group $g = 1, 2$, for $i_g = 1, \ldots, N_g$, and $t = 1, \ldots, T$, may be written as

$$c^0_{gi_gt} \;=\; \mathbb{I}_{gLt}\left(\theta^0_g\right) c^0_{gLi_gt} + \mathbb{I}_{gHt}\left(\theta^0_g\right) c^0_{gHi_gt} \;=\; \mathbb{I}_{gLt}\left(\theta^0_g\right) \boldsymbol{\lambda}^{0\prime}_{gLi_g}\boldsymbol{f}^0_{gLt} + \mathbb{I}_{gHt}\left(\theta^0_g\right) \boldsymbol{\lambda}^{0\prime}_{gHi_g}\boldsymbol{f}^0_{gHt}\;. \tag{3}$$

The common component in (3) reduces to the one in (2) within each regime $j_g = L, H$. The common components in (2) and (3) apply to an individual cross-sectional unit i_g within group g. They also allow to construct *pairwise common components* between the two groups. Formally, from (2) we define the pairwise common component between cross-sectional units i_1 and i_2 within regimes j_1 and j_2 as

$$c^0_{j_1j_2i_1i_2t} \;:=\; c^0_{1j_1i_1t} \cdot c^0_{2j_2i_2t} \;=\; \boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\boldsymbol{f}^0_{1j_1t}\boldsymbol{f}^{0\prime}_{2j_2t}\boldsymbol{\lambda}^0_{2j_2i_2}. \tag{4}$$

Similarly, from (3) we define the pairwise common component between cross-sectional units i_1 and i_2 unconditional upon the regime as

$$c^0_{i_1 i_2 t} \;=\; c^0_{1 i_1 t} \cdot c^0_{2 i_2 t} \;=\; \sum_{j_1 = L,H} \; \sum_{j_2 = L,H} \mathbb{I}_{1 j_1 t}\left(\theta^0_1\right) \mathbb{I}_{2 j_2 t}\left(\theta^0_2\right) \boldsymbol{\lambda}^{0\prime}_{1 j_1 i_1} \boldsymbol{f}^0_{1 j_1 t} \boldsymbol{f}^{0\prime}_{2 j_2 t} \boldsymbol{\lambda}^0_{2 j_2 i_2} \,. \tag{5}$$

2.2.2 Measuring systematic comovement

From the pairwise common component in (5) we define measures of *systematic comovement* between two cross-sectional units conditional upon the regimes. Formally, for $g = 1, 2$, $j_g = L, H$, and $i_g = 1, \ldots, N_g$, we define the *systematic covariance* between the i_1-th element of $\boldsymbol{x}_{1t}$ and the i_2-th element of $\boldsymbol{x}_{2t}$ within regimes j_1 and j_2 as

$$c^0_{j_1 j_2 i_1 i_2} \;:=\; \mathrm{E}\left[c^0_{i_1 i_2 t} | \mathbb{I}_{1 j_1 t}\left(\theta^0_1\right) = \mathbb{I}_{2 j_2 t}\left(\theta^0_2\right) = 1\right] \;=\; \boldsymbol{\lambda}^{0\prime}_{1 j_1 i_1} \mathrm{E}\left[\boldsymbol{f}^0_{1 j_1 t} \boldsymbol{f}^{0\prime}_{2 j_2 t}\right] \boldsymbol{\lambda}^0_{2 j_2 i_2} : \tag{6}$$

$c^0_{j_1 j_2 i_1 i_2}$ measures the degree of comovement between $x_{1 i_1 t}$ and $x_{2 i_2 t}$ in regime j_1 and j_2 as induced by the pervasive factors $\boldsymbol{f}^0_{1 j_1 t}$ and $\boldsymbol{f}^0_{2 j_2 t}$; as such, it is a measure of systematic comovement.

The systematic covariance in (6) measures comovement between two individual cross-sectional units. In some instances, it may be useful to quantify the average comovement between the two groups: for example, if $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$ are returns from assets that belong to two separate portfolios, this is informative about the comovement between the average returns from the two portfolios. To this purpose, for $g = 1, 2$, consider the $N_g \times 1$ vector of weights $\boldsymbol{w}'_g = \left[w_{g1}, \ldots, w_{g N_g}\right]$ such that $\boldsymbol{\iota}'_{N_g} \boldsymbol{w}_g = 1$, where $\boldsymbol{\iota}_{N_g}$ is the $N_g \times 1$ vector of ones. Given the unconditional pairwise common component $c^0_{i_1 i_2 t}$ in (5), we define the average unconditional pairwise common component as

$$c^0_{\boldsymbol{w}_1 \boldsymbol{w}_2 t} = \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} w_{1 i_1} w_{2 i_2} c^0_{i_1 i_2 t}. \tag{7}$$

We then define the *average systematic covariance* between the groups in regime j_1 and j_2 as

$$c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} \;=\; \mathrm{E}\left[c^0_{\boldsymbol{w}_1 \boldsymbol{w}_2 t} | \mathbb{I}_{1 j_1 t}\left(\theta^0_1\right) = \mathbb{I}_{2 j_2 t}\left(\theta^0_2\right) = 1\right] \;=\; \boldsymbol{\lambda}^{0\prime}_{1 j_1}(\boldsymbol{w}_1) \cdot \mathrm{E}\left[\boldsymbol{f}^0_{1 j_1 t} \boldsymbol{f}^{0\prime}_{2 j_2 t}\right] \cdot \boldsymbol{\lambda}^0_{2 j_2}(\boldsymbol{w}_2) \,, \tag{8}$$

with $\boldsymbol{\lambda}^0_{g j_g}(\boldsymbol{w}_g) = \sum_{i_g=1}^{N_g} w_{g i_g} \boldsymbol{\lambda}^0_{g j_g i_g}$ with $g = 1, 2$. If the i_g-th element of $\boldsymbol{w}_g$ is equal to one and all other elements are equal to zero, for $g = 1, 2$, equation (8) reduces to equation (6); $c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ thus is a general measure of systematic covariance.

The quantity $c^0_{j_1 j_2 i_1 i_2}$ defined in (6) is not standardized. Let

$$\sigma^0_{\boldsymbol{x} g j_1 j_2 i_g} = \sqrt{\text{Var}\left[x_{g i_g t} | \mathbb{I}_{1 j_1 t}\left(\theta^0_1\right) = \mathbb{I}_{2 j_2 t}\left(\theta^0_2\right) = 1\right]}$$

be the standard deviation of $x_{g i_g t}$ conditional upon j_1 and j_2. We define the *systematic correlation* between $x_{1 i_1 t}$ and $x_{2 i_2 t}$ in regime j_1 and j_2 as the standardized version of $c^0_{j_1 j_2 i_1 i_2}$ in (6), namely

$$R^0_{j_1 j_2 i_1 i_2} = \frac{c^0_{j_1 j_2 i_1 i_2}}{\sigma^0_{\boldsymbol{x} 1 j_1 j_2 i_1} \sigma^0_{\boldsymbol{x} 2 j_1 j_2 i_2}} \quad : \tag{9}$$

by construction, $-1 < R^0_{j_1 j_2 i_1 i_2} < 1$. We can then measure the *average systematic correlation* between the groups as

$$R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} = \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} w_{1 i_1} w_{2 i_2} R^0_{j_1 j_2 i_1 i_2} \ . \tag{10}$$

When the i_g-th element of $\boldsymbol{w}_g$ is equal to one and all other elements are equal to zero, for $g = 1, 2$, then $R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ becomes equal to $R^0_{j_1 j_2 i_1 i_2}$.

The vectors of weights $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ in (8) and (10) are *a priori* chosen. For example, if $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$ contain returns from financial assets, a natural choice is the equal weighting scheme, namely $\boldsymbol{w}_g = \boldsymbol{\iota}_{N_g}/N_g$, for $g = 1, 2$: in portfolio choice, this is advocated in DeMiguel, Garlappi, and Uppal (2009). Alternatively, one could follow Bekaert, Hodrick, and Zhang (2009), and make the weights depend upon the market capitalization of the underlying assets.

2.2.3 Testing for changes in systematic comovement

The quantities $c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ and $R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ defined in (8) and (10), respectively, allow to construct tests for changes in systematic comovement between regimes. In the case of the systematic covariance $c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$, for $g = 1, 2$, and $(j_1, j_2) \neq (j^*_1, j^*_2)$, this requires testing the null hypothesis

$$\mathcal{H}^c_0 : c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} = c^0_{j^*_1 j^*_2 \boldsymbol{w}_1 \boldsymbol{w}_2} \tag{11}$$

against the alternative

$$\mathcal{H}^c_1 : c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} \neq c^0_{j^*_1 j^*_2 \boldsymbol{w}_1 \boldsymbol{w}_2}. \tag{12}$$

Similarly, we can construct a test for changes in systematic correlation $R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ by testing the null hypothesis $\mathcal{H}^R_0 : R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} = R^0_{j^*_1 j^*_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ against the alternative $\mathcal{H}^R_1 : R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} \neq R^0_{j^*_1 j^*_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$.

When the i_1-th element of $\boldsymbol{w}_1$ and the i_2-th element of $\boldsymbol{w}_2$ are equal to one and all other elements are equal to zero, the tests for $c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ and $R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ become tests for $c^0_{j_1 j_2 i_1 i_2}$ and $R^0_{j_1 j_2 i_1 i_2}$, respectively. In the case of $c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$, the null and alternative hypotheses in (11) and (12) become $\mathcal{H}^c_0 : c^0_{j_1 j_2 i_1 i_2} = c^0_{j_1^* j_2^* i_1 i_2}$ and $\mathcal{H}^c_1 : c^0_{j_1 j_2 i_1 i_2} \neq c^0_{j_1^* j_2^* i_1 i_2}$, respectively. Analogous considerations hold for $R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$.

2.2.4 Interpreting systematic comovement

We now show that the systematic correlation is related to both the canonical correlations among the factors in the two panels and the pervasiveness of the factors within each panel. Denote by $\mathcal{R}^2_{g j_1 j_2 i_g}$ the R-square of the factors $\boldsymbol{f}_{g j_g t}$ for the cross-sectional unit i_g in group g, regime (j_1, j_2):

$$\mathcal{R}^2_{g j_1 j_2 i_g} \ := \ \frac{\text{Var}\left[\boldsymbol{\lambda}'_{g j_g i_g} \boldsymbol{f}_{g j_g t} \ \mid \ \mathbb{I}_{1 j_1 t}\left(\theta^0_1\right) = \mathbb{I}_{2 j_2 t}\left(\theta^0_2\right) = 1\right]}{\text{Var}\left[x_{g i_g t} | \mathbb{I}_{1 j_1 t}\left(\theta^0_1\right) = \mathbb{I}_{2 j_2 t}\left(\theta^0_2\right) = 1\right]}. \tag{13}$$

Let $\phi_{j_1 j_2, 1}$ be the largest canonical correlation between $\boldsymbol{f}_{1 j_1 t}$ and $\boldsymbol{f}_{2 j_2 t}$, that is the maximum correlation achievable among a linear combination of $\boldsymbol{f}_{1 j_1 t}$ and another linear combination of $\boldsymbol{f}_{2 j_2 t}$ in the generic regime (j_1, j_2). More generally, let $\phi_{j_1 j_2, \ell}$ be the $\ell - th$ largest canonical correlation between $\boldsymbol{f}_{1 j_1 t}$ and $\boldsymbol{f}_{2 j_2 t}$. By definition $1 \geq \phi_{j_1 j_2, 1} \geq \phi_{j_1 j_2, 2} \geq ... \geq \phi_{j_1 j_2, min(K_{1 j_1}, K_{2 j_2})} \geq 0$, while $\phi_{j_1 j_2, \ell} = 0$ for $\ell > min(K_{1 j_1}, K_{2 j_2})$. The next proposition determines the relationship among our measure of systematic correlation $R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$, the largest canonical correlation among the factors $\phi_{j_1 j_2, 1}$, and the cross-sectional averages of the square roots of $\mathcal{R}^2_{g j_1 j_2 i_g}$, with $g = 1, 2$.

PROPOSITION 1. *Let all the weights* $w_{g i_g} \geq 0$ *be such that* $\sum_{i_g=1}^{N_g} w_{g i_g} = 1$, *for* $i_g = 1, ..., N_g$ *and* $g = 1, 2$. *Under Assumptions A.1 - A.3, A.10 and further assuming that the factors and the errors are uncorrelated within and across groups we have:*
$\left| R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} \right| \ \leq \ \phi_{j_1 j_2, 1} \cdot \left(\sum_{i_1=1}^{N_1} w_{1 i_1} \sqrt{\mathcal{R}^2_{1 j_1 j_2 i_1}} \right) \cdot \left(\sum_{i_2=1}^{N_2} w_{2 i_2} \sqrt{\mathcal{R}^2_{2 j_1 j_2 i_2}} \right).$

Proof. See Section C.1 in Appendix C.

The average of the square root of the R-squares in each group, namely $\sum_{i_g=1}^{N_g} w_{g i_g} \sqrt{\mathcal{R}^2_{g j_1 j_2 i_g}}$, is a measure of pervasiveness of the factors $\boldsymbol{f}_{g j_g t}$ within group g only, for $g = 1, 2$. Proposition 1 shows that neither $\sum_{i_g=1}^{N_g} w_{g i_g} \sqrt{\mathcal{R}^2_{g j_1 j_2 i_g}}$, nor their products across groups, are sufficient to determine the level of systematic correlation between the two groups $g = 1, 2$. The same consideration holds for the largest canonical correlation among the pervasive factors in the two groups, $\phi_{j_1 j_2, 1}$. Using the definition of canonical correlation, Proposition 1 also implies $\left| R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} \right| \ \leq \ \left(\sum_{\ell=1}^{min(K_{1 j_1}, K_{2 j_2})} \phi_{j_1 j_2, \ell} \right)$.

$\left(\sum_{i_1=1}^{N_1} w_{1i_1}\sqrt{\mathcal{R}^2_{1j_1j_2i_1}}\right)\cdot\left(\sum_{i_2=1}^{N_2} w_{2i_2}\sqrt{\mathcal{R}^2_{2j_1j_2i_2}}\right)$. Therefore, neither $\phi_{j_1j_2,1}$, nor $\sum_{\ell=1}^{min(K_{1j_1},K_{2j_2})}\phi_{j_1j_2,\ell}$, are sufficient to characterize changes in systematic correlation across two large groups of observations when taken as stand-alone measures.[2] On the other hand, our measure $R^0_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2}$ efficiently summarizes in one number the information from both the pervasiveness of the factors within each group and the factor correlation across groups.

2.3 Comovement within group

We conclude this section by noting that although the definitions $c^0_{\boldsymbol{w}_1\boldsymbol{w}_2t}$ and $R^0_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2}$ of systematic comovement involve (averages of) individual observations in different groups, these measures can be easily adapted to include only individuals within the same group. The same consideration extends to the theorems and tests for change in systematic comovement derived in the following sections.

3 Estimation

3.1 Least squares estimation of the model

Following Massacci (2017), we estimate model (1) by group-by-group least squares. For $g = 1, 2$ and $j_g = L, H$, we assume the true number of factors $K^0_{gj_g}$ is known. Otherwise θ^0_g can be estimated by choosing a number of factors $\bar{K}_{gjg}$ such that $\bar{K}_{gjg} \geq K^0_{gj_g}$. Given the estimate of θ^0_g, $K^0_{gj_g}$ can be consistently determined using standard model selection criteria for static factor models, such as those proposed in Bai and Ng (2002), Alessi, Barigozzi, and Capasso (2010), Ahn and Horenstein (2013), and Caner and Han (2014): these require the convergence rate $C_{N_gT} := min(\sqrt{N_g}, \sqrt{T})$, which holds for the principal components estimators for factors and loadings.[3]

[2] For instance, finding that the average explanatory power of the factors in their own panel ($\sum_{i_g=1}^{N_g} w_{gi_g}\sqrt{\mathcal{R}^2_{gj_1j_2i_g}}$) is high does not imply a high systematic correlation if all the factors are not much correlated, i.e when their maximum canonical correlation is small, or when the sum of all the canonical correlations $\sum_{\ell=1}^{min(K_{1j_1},K_{2j_2})}\phi_{j_1j_2,\ell}$ is small. Symmetrically, either finding $K^c_{j_1j_2} > 0$ common factors - that is $K^c_{j_1j_2}$ canonical correlation equal to 1 between the pervasive factors of the two panels in regime (j_1, j_2), or finding that there is a high degree of comovement between (some of) the factors as measured by $\sum_{\ell=1}^{min(K_{1j_1},K_{2j_2})}\phi_{j_1j_2,\ell}$, is not enough to generate an increase in the systematic across-panel correlation $R^0_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2}$ with respect to another regime with fewer common factors, or less comovement among the factors. For example, if the $K^c_{j_1j_2}$ common factors have very small loadings (relatively to the other factors) in one of the two panel, their contribution to the overall R-squared in that panel is small, and the average systematic correlation between variables $x_{1j_1i_1}$ and $x_{2j_2i_2}$ could be low. Therefore, relatively high values of the pervasiveness of the factors are also needed to generate non-negligible systematic correlation between two individuals belonging to two groups.

[3] We conjecture that other procedures to determine the number of factors in large dimensional static linear factor models, such as Kapetanios (2010), Onatski (2010), and Trapani (2018), may be suitably generalized to become applicable to the model in (1).

Let $\hat{\theta}_g$, $\hat{\mathbf{\Lambda}}_{gj_g}$ and $\hat{\boldsymbol{f}}_{gj_gt}$ be the estimators for θ_g^0, $\mathbf{\Lambda}_{gj_g}^0$ and $\boldsymbol{f}_{gj_gt}^0$, respectively. For a given value of θ_g, define

$$\hat{\mathbf{\Sigma}}_{\boldsymbol{x}gj_g}(\theta_g) = (N_gT)^{-1}\sum_{t=1}^{T}\mathbb{I}_{gj_gt}(\theta_g)\,\boldsymbol{x}_{gt}\boldsymbol{x}_{gt}', \quad g=1,2, \quad j_g = L,H. \tag{14}$$

Under the identification assumption $N_g^{-1}\left(\mathbf{\Lambda}_{gj_g}'\mathbf{\Lambda}_{gj_g}\right) = \mathbf{I}_{K_{gj_g}^0}$, the estimator $\hat{\theta}_g$ for θ_g^0 is

$$\hat{\theta}_g = \arg\min_{\theta_g}(N_gT)^{-1}\sum_{t=1}^{T}\boldsymbol{x}_{gt}'\left\{\mathbf{I}_{N_g} - N_g^{-1}\left[\begin{array}{l}\mathbb{I}_{gLt}(\theta_g)\,\hat{\mathbf{\Lambda}}_{gL}(\theta_g)\,\hat{\mathbf{\Lambda}}_{gL}(\theta_g)' \\ +\mathbb{I}_{gHt}(\theta_g)\,\hat{\mathbf{\Lambda}}_{gH}(\theta_g)\,\hat{\mathbf{\Lambda}}_{gH}(\theta_g)'\end{array}\right]\right\}\boldsymbol{x}_{gt}, \quad g=1,2,$$

where $\hat{\mathbf{\Lambda}}_{gj_g}(\theta_g)$ is the estimator for $\mathbf{\Lambda}_{gj_g}^0$ for given θ_g: $\hat{\mathbf{\Lambda}}_{gj_g}(\theta_g)$ is equal to $\sqrt{N_g}$ times the $N_g \times K_{gj_g}^0$ matrix of eigenvectors of $\hat{\mathbf{\Sigma}}_{\boldsymbol{x}gj_g}(\theta_g)$ in (14) corresponding to its $K_{gj_g}^0$ largest eigenvalues. Given $\hat{\theta}_g$, the estimator for $\mathbf{\Lambda}_{gj_g}^0$ is $\hat{\mathbf{\Lambda}}_{gj_g} = \hat{\mathbf{\Lambda}}_{gj_g}\left(\hat{\theta}_g\right)$. The estimator for $\boldsymbol{f}_{gj_gt}^0$ is then obtained as $\hat{\boldsymbol{f}}_{gj_gt} = N_g^{-1}\hat{\mathbf{\Lambda}}_{gj_g}\left(\hat{\theta}_g\right)'\boldsymbol{x}_{gt}$, for $g=1,2$, $t=1,\ldots,T$, and $\mathbb{I}_{gj_gt}\left(\hat{\theta}_g\right)=1$.

3.2 Estimating systematic comovement

Given $\hat{\theta}_g$, $\hat{\mathbf{\Lambda}}_{gj_g}$ and $\hat{\boldsymbol{f}}_{gj_gt}$, the common components $c_{i_1i_2t}^0$ in (5) may be estimated as

$$\hat{c}_{i_1i_2t} = \sum_{j_1=L,H}\sum_{j_2=L,H}\mathbb{I}_{1j_1t}\left(\hat{\theta}_1\right)\mathbb{I}_{2j_2t}\left(\hat{\theta}_2\right)\hat{\boldsymbol{\lambda}}_{1j_1i_1}'\hat{\boldsymbol{f}}_{1j_1t}\hat{\boldsymbol{f}}_{2j_2t}'\hat{\boldsymbol{\lambda}}_{2j_2i_2}^0 \,. \tag{15}$$

For $g=1,2$, $j_g = L,H$, and $i_g = 1,\ldots,N_g$, given $\hat{c}_{i_1i_2t}$ in (15) we estimate $c_{j_1j_2i_1i_2}^0$ in (6) as

$$\hat{c}_{j_1j_2i_1i_2} = T_{j_1j_2}\left(\hat{\theta}_1,\hat{\theta}_2\right)^{-1}\left[\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\hat{\theta}_1\right)\mathbb{I}_{2j_2t}\left(\hat{\theta}_2\right)\hat{c}_{i_1i_2t}\right], \tag{16}$$

where $T_{j_1j_2}(\theta_1,\theta_2) := \sum_{t=1}^{T}\mathbb{I}_{1j_1t}(\theta_1)\,\mathbb{I}_{2j_2t}(\theta_2)$. It follows that $R_{j_1j_2i_1i_2}^0$ in (9) may be estimated through its empirical counterpart as

$$\hat{R}_{j_1j_2i_1i_2} = \frac{\hat{c}_{j_1j_2i_1i_2}}{\hat{\sigma}_{\boldsymbol{x}1j_1j_2i_1}\hat{\sigma}_{\boldsymbol{x}2j_1j_2i_2}}, \tag{17}$$

where

$$\hat{\sigma}_{\boldsymbol{x}gj_1j_2i_g} = \sqrt{\hat{\sigma}_{\boldsymbol{x}gj_1j_2i_g}^2\left(\hat{\theta}_1,\hat{\theta}_2\right)}, \tag{18}$$

and

$$\hat{\sigma}^2_{\boldsymbol{x}gj_1j_2i_g}(\theta_1,\theta_2) \ := \ T_{j_1j_2}(\theta_1,\theta_2)^{-1}\left[\sum_{t=1}^{T}\mathbb{I}_{1j_1t}(\theta_1)\,\mathbb{I}_{2j_2t}(\theta_2)\,x^2_{gi_gt}\right].$$

Finally, the average systematic covariance between the groups in (8) may be estimated as

$$\hat{c}_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2} = \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{1i_1}w_{2i_2}\hat{c}_{j_1j_2i_1i_2}; \tag{19}$$

similarly, the estimator for the average systematic correlation in (10) is

$$\hat{R}_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2} = \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{1i_1}w_{2i_2}\hat{R}_{j_1j_2i_1i_2}. \tag{20}$$

4 Asymptotic results

In what follows we define $C_{NT} := min(\sqrt{N},\sqrt{T})$, with $N := min\,(N_1, N_2)$.

4.1 Limiting distribution of estimator for pairwise common component

The following theorem provides the asymptotic distribution of the estimator $\hat{c}_{i_1i_2t}$ for the pairwise systematic component $c^0_{i_1i_2t}$ defined in (15) and (5), respectively.

THEOREM 1. *Let Assumptions A.1–A.9 hold . Then as* $C_{NT}\to\infty$

$$C_{NT}\left(\hat{c}_{i_1i_2t} - c^0_{i_1i_2t}\right) \xrightarrow{d} \mathcal{N}\left(0, \boldsymbol{Q}^0_{i_1i_2t}\right),$$

for $g = 1, 2$, $i_g = 1,\ldots,N_g$, *and* $t = 1,\ldots,T$, *with asymptotic variance* $\boldsymbol{Q}^0_{i_1i_2t}$ *defined as*

$$\boldsymbol{Q}^0_{i_1i_2t} \ = \ \sum_{j_1=L,H}\ \sum_{j_2=L,H}\mathbb{I}_{1j_1t}(\theta^0_1)\mathbb{I}_{2j_2t}(\theta^0_2)\boldsymbol{Q}^0_{j_1j_2i_1i_2t}\ ,$$

where $\boldsymbol{Q}^0_{j_1j_2i_1i_2t} = \left(c^0_{1j_1i_1t}\right)^2\left(\mu^2_{N_2}\boldsymbol{V}^0_{2j_2i_2t} + \mu^2_T\boldsymbol{W}^0_{2j_2i_2t}\right) + \left(c^0_{2j_2i_2t}\right)^2\left(\mu^2_{N_1}\boldsymbol{V}^0_{1j_1i_1t} + \mu^2_T\boldsymbol{W}^0_{1j_1i_1t}\right)$, *for* $j_g = L, H$, *with* $\mu_{N_g} = \lim_{N,T\to\infty}\dfrac{C_{NT}}{\sqrt{N_g}}$, $\mu_T = \lim_{N,T\to\infty}\dfrac{C_{NT}}{\sqrt{T}}$,

$$\boldsymbol{V}^0_{gj_gi_gt} = \boldsymbol{\lambda}^{0\prime}_{gj_gi_g}\left(\boldsymbol{D}^0_{\boldsymbol{\Lambda}gj_g}\right)^{-1}\boldsymbol{\Gamma}^0_{gj_gt}\left(\boldsymbol{D}^0_{\boldsymbol{\Lambda}gj_g}\right)^{-1}\boldsymbol{\lambda}^0_{gj_gi_g},\quad \boldsymbol{W}^0_{gj_gi_gt} = \boldsymbol{f}^{0\prime}_{gj_gt}\left(\boldsymbol{\Sigma}^0_{\boldsymbol{f}gj_g}\right)^{-1}\boldsymbol{\Omega}^0_{gj_gi_g}\left(\boldsymbol{\Sigma}^0_{\boldsymbol{f}gj_g}\right)^{-1}\boldsymbol{f}^0_{gj_gt},$$

and where $\boldsymbol{\Sigma}^0_{\boldsymbol{f}gj_g} = \boldsymbol{\Sigma}^0_{\boldsymbol{f}gj_g}\left(\theta^0_g\right)$. *Matrices* $\boldsymbol{\Sigma}^0_{\boldsymbol{f}gj_g}\left(\theta^0_g\right)$, $\boldsymbol{D}^0_{\boldsymbol{\Lambda}gj_g}$, $\boldsymbol{\Gamma}^0_{gj_gt}$ *and* $\boldsymbol{\Omega}^0_{gj_gi_g}$ *are defined in Assumptions A.1, A.2, A.7 and A.8, respectively.*

Proof. See Section C.2 in Appendix C.

Theorem 1 shows that the estimator for the pairwise common components is asymptotically normal, with convergence rate C_{NT} accounting for the cross-sectional dimensions of the two groups of variables $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$. No restriction on the relationship between N and T is required to achieve asymptotic normality. The variance $\boldsymbol{Q}^0_{i_1 i_2 t}$ is made of the four mutually exclusive terms $\boldsymbol{Q}^0_{j_1 j_2 i_1 i_2 t}$, each corresponding to one of the four regimes described in Section 2.1.1. Each term $\boldsymbol{Q}^0_{j_1 j_2 i_1 i_2 t}$ consists of two additive parts, which are proportional to the asymptotic variance of the estimators for the unit-specific common components $c^0_{1 j_1 i_1 t}$ and $c^0_{2 j_2 i_2 t}$: these asymptotic variances are equal to $\mu^2_{N_2} \boldsymbol{V}^0_{2 j_2 i_2 t} + \mu^2_T \boldsymbol{W}^0_{2 j_2 i_2 t}$ and $\mu^2_{N_1} \boldsymbol{V}^0_{1 j_1 i_1 t} + \mu^2_T \boldsymbol{W}^0_{1 j_1 i_1 t}$, respectively, and are analogous to the asymptotic variance of the common components estimator in linear factor models as derived in Theorem 3 in Bai (2003). The following corollary describes two special cases.

COROLLARY 1. *Let the assumptions of Theorem 1 hold and recall $\boldsymbol{Q}^0_{i_1 i_2 t}$ from the same theorem. Then, for $g = 1, 2$, $j_g = L, H$, $i_g = 1, \ldots, N_g$, and $t = 1, \ldots, T$:*

(a) if $N/T \to 0$, then $\sqrt{N}\left(\hat{c}_{i_1 i_2 t} - c^0_{i_1 i_2 t}\right) \xrightarrow{d} \mathcal{N}\left(0, \boldsymbol{Q}^0_{i_1 i_2 t}\right)$, with $\boldsymbol{Q}^0_{j_1 j_2 i_1 i_2 t} = \left(c^0_{1 j_1 i_1 t}\right)^2 \dfrac{N}{N_2} \boldsymbol{V}^0_{2 j_2 i_2 t} + \left(c^0_{2 j_2 i_2 t}\right)^2 \dfrac{N}{N_1} \boldsymbol{V}^0_{1 j_1 i_1 t}$;

(b) if $T/N \to 0$, then $\sqrt{T}\left(\hat{c}_{i_1 i_2 t} - c^0_{i_1 i_2 t}\right) \xrightarrow{d} \mathcal{N}\left(0, \boldsymbol{Q}^0_{i_1 i_2 t}\right)$, with $\boldsymbol{Q}^0_{j_1 j_2 i_1 i_2 t} = \left(c^0_{1 j_1 i_1 t}\right)^2 \boldsymbol{W}^0_{2 j_2 i_2 t} + \left(c^0_{2 j_2 i_2 t}\right)^2 \boldsymbol{W}^0_{1 j_1 i_1 t}$.

Corollary 1 states the asymptotic distribution of $\hat{c}_{i_1 i_2 t}$ when either $N/T \to 0$ or $T/N \to 0$. However, the general result in Theorem 1 does not require any restriction on the limits of N/T and T/N.

4.2 Limiting distribution of estimators for systematic comovement

The theorem below states the asymptotic distribution of $\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$, defined in equation (19), as an estimator for the systematic covariance $c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ defined in equation (8).

THEOREM 2. *Let Assumptions A.1 - A.3 and A.5 - A.11 hold, with $C_{NT} \to \infty$ and $\sqrt{T}/N \to 0$. Then for $g = 1, 2$, and $j_g = L, H$,*

$$\sqrt{T}\left(\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}\right) \xrightarrow{d} \mathcal{N}\left(\ 0\ ,\ \boldsymbol{Q}^0_{j_1 j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2)\ \right),$$

with

$$\boldsymbol{Q}^0_{j_1j_2}(\boldsymbol{w}_1,\boldsymbol{w}_2) = \frac{T^2}{T_{j_1j_2}(\theta^0_1,\theta^0_2)\cdot T_{j_1^*j_2^*}(\theta^0_1,\theta^0_2)}\cdot\Bigg[\ \boldsymbol{\Psi}^0_{j_1j_2,j_1j_2}(\boldsymbol{w}_1,\boldsymbol{w}_2) + \boldsymbol{Q}^0_{12j_1j_2,j_1j_2}(\boldsymbol{w}_1,\boldsymbol{w}_2;\boldsymbol{w}_1,\boldsymbol{w}_2) + \boldsymbol{Q}^0_{21j_1j_2,j_1j_2}(\boldsymbol{w}_1,\boldsymbol{w}_2;\boldsymbol{w}_1,\boldsymbol{w}_2)\ \Bigg],$$

where

$$\boldsymbol{\Psi}^0_{j_1j_2,j_1^*j_2^*}(\boldsymbol{w}_1,\boldsymbol{w}_2) = \lim_{N_1,N_2,T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\sum_{v=1}^{T}\mathrm{Cov}\left[\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)c^0_{\boldsymbol{w}_1\boldsymbol{w}_2t}\ ,\ \mathbb{I}_{1j_1^*v}\left(\theta^0_1\right)\mathbb{I}_{2j_2^*v}\left(\theta^0_2\right)c^0_{\boldsymbol{w}_1\boldsymbol{w}_2v}\right]$$

is computed for $(j_1^*,j_2^*) = (j_1,j_2)$. *Moreover, for a vector of weights* $\boldsymbol{w}_g^*$ *potentially different from* $\boldsymbol{w}_g$, *let*

$$\boldsymbol{Q}^0_{12j_1j_2,j_1^*j_2^*}(\boldsymbol{w}_1,\boldsymbol{w}_2;\boldsymbol{w}_1^*,\boldsymbol{w}_2^*) = \lim_{N_1,N_2,T\to\infty}\boldsymbol{\lambda}^{0\prime}_{1j_1}(\boldsymbol{w}_1)\boldsymbol{\Sigma}^0_{\mathbf{f}12j_1j_2}\left(\boldsymbol{\Sigma}^0_{\mathbf{f}2j_2}\right)^{-1}\boldsymbol{\Omega}^0_{2j_2j_2^*}(\boldsymbol{w}_2,\boldsymbol{w}_2^*)\left(\boldsymbol{\Sigma}^0_{\mathbf{f}2j_2^*}\right)^{-1}\boldsymbol{\Sigma}^{0\prime}_{\mathbf{f}12j_1^*j_2^*}\boldsymbol{\lambda}^0_{1j_1^*}(\boldsymbol{w}_1^*),$$

$$\boldsymbol{Q}^0_{21j_1j_2,j_1^*j_2^*}(\boldsymbol{w}_1,\boldsymbol{w}_2;\boldsymbol{w}_1^*,\boldsymbol{w}_2^*) = \lim_{N_1,N_2,T\to\infty}\boldsymbol{\lambda}^{0\prime}_{2j_2}(\boldsymbol{w}_2)\boldsymbol{\Sigma}^0_{\mathbf{f}21j_1j_2}\left(\boldsymbol{\Sigma}^0_{\mathbf{f}1j_1}\right)^{-1}\boldsymbol{\Omega}^0_{1j_1j_1^*}(\boldsymbol{w}_1,\boldsymbol{w}_1^*)\left(\boldsymbol{\Sigma}^0_{\mathbf{f}1j_1^*}\right)^{-1}\boldsymbol{\Sigma}^{0\prime}_{\mathbf{f}21j_1^*j_2^*}\boldsymbol{\lambda}^0_{2j_2^*}(\boldsymbol{w}_2^*),$$

where $c^0_{\boldsymbol{w}_1\boldsymbol{w}_2t}$ *is defined in equation* (7), *and*

$$\boldsymbol{\Omega}^0_{gj_gj_g^*}(\boldsymbol{w}_g,\boldsymbol{w}_g^*) = \sum_{i_g=1}^{N_g}\sum_{l_g=1}^{N_g}w_{gi_g}w^*_{gl_g}\left\{\frac{1}{T}\sum_{t=1}^{T}\sum_{v=1}^{T}\mathrm{E}\left[\mathbb{I}_{gj_gt}\left(\theta^0_g\right)\mathbb{I}_{gj_g^*v}\left(\theta^0_g\right)\mathbf{f}^0_{gj_gt}\mathbf{f}^{0\prime}_{gj_g^*v}e_{gi_gt}e_{gl_gv}\right]\right\},$$

with $\boldsymbol{\Sigma}_{\boldsymbol{f}gj_g} = \boldsymbol{\Sigma}_{\boldsymbol{f}gj_g}(\theta_g)$. *Matrices* $\boldsymbol{\Sigma}_{\boldsymbol{f}gj_g}(\theta_g)$ *and* $\boldsymbol{\Sigma}_{\boldsymbol{f}12j_1j_2}$ *are defined in Assumptions A.2 and A.10, respectively.*

Proof. See Section C.3 in Appendix C.

The asymptotic variance $\boldsymbol{Q}^0_{j_1j_2}(\boldsymbol{w}_1,\boldsymbol{w}_2)$ of $\hat{c}_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2}$ is made of three terms. The first term $\boldsymbol{\Psi}^0_{j_1j_2,j_1j_2}(\cdot)$ depends on the fourth cross-moments of factors $\boldsymbol{f}^0_{1j_1t}$ and $\boldsymbol{f}^0_{2j_2v}$ and is due to the estimation of the true cross-covariance matrix of the factors, $\boldsymbol{\Sigma}^0_{\boldsymbol{f}12j_1j_2}$. It would be present even if the true loadings $\boldsymbol{\lambda}^0_{gj_g\boldsymbol{w}_g}$ and the time series of the true factors $\boldsymbol{f}_{gj_gt}$, $t=1,...,T$ were known. The second and third terms, $\boldsymbol{Q}^0_{12j_1j_2,j_1j_2}(\cdot)$ and $\boldsymbol{Q}^0_{21j_1j_2,j_1j_2}(\cdot)$, originate from the estimation of the loadings $\boldsymbol{\lambda}^0_{1j_2i_1}$ and $\boldsymbol{\lambda}^0_{2j_2i_2}$, respectively. These two terms are exactly zero when all the factors $\boldsymbol{f}^0_{1j_1t}$ of the first group are uncorrelated with all the factors of the second group $\boldsymbol{f}^0_{2j_2t}$, that is when $\boldsymbol{\Sigma}^0_{\boldsymbol{f}12j_1j_2} = 0$.[4] The rate of convergence of $\hat{c}_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2}$ is $\sqrt{T}$, since it is estimated as the time average of the product of the (estimated) common components $\hat{c}_{1j_1i_1t}$ and $\hat{c}_{2j_2i_2t}$, and we assume $N,T\to\infty$ such that

[4] This is a special case of interest in group-factor models, as it corresponds to two groups not sharing any common factor and whose group-specific factors are also uncorrelated across groups.

$\sqrt{T}/N \to 0$ as customary in approximate factor models, see e.g. Bai and Ng (2006). Notice that Assumption A.11 only requires the absolute summability of the weights. This ensures that the asymptotic variance of $\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$, which is defined as a generic average across all individuals in groups 1 and 2 with generic weights w_{1i_1} and w_{2i_2} converges as $N_1, N_2 \to \infty$. This assumption accommodates the two most interesting cases (i) $w_{gi_g} = 1/N_g$, $\forall i_g$, and (ii) $w_{gi_g} = 1$ when $i_g = i_g^*$, while $w_{gi_g} = 0$ $\forall i_g \neq i_g^*$, with $g = 1, 2$.

Theorem 3 provides the asymptotic distribution of $\hat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$, the estimator of the systematic correlation $R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$.

THEOREM 3. *Let Assumptions A.1 - A.3 and A.5 - A.11 hold, and $C_{NT} \to \infty$ with $\sqrt{T}/N \to 0$. Define the $N_g \times 1$ vector of rescaled weights $\boldsymbol{w}_{\boldsymbol{\sigma},gj_1j_2} = [w_{\boldsymbol{\sigma},gj_1j_2i_g}, ..., w_{\boldsymbol{\sigma},gj_1j_2i_g}]'$, with $w_{\boldsymbol{\sigma},gj_1j_2i_g} = w_{gi_g}/\sigma^0_{\boldsymbol{x}gj_1j_2i_g}$, $i_g = 1, ..., N_g$. Then for $g = 1, 2$, and $j_g = L, H$,*

$$\sqrt{T}\left(\widehat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}\right) \xrightarrow{d} \mathcal{N}\left(\ 0\ ,\ \boldsymbol{Q}^0_{R,j_1j_2}\ \right),$$

where $\boldsymbol{Q}^0_{R,j_1j_2} = \boldsymbol{Q}^0_{R,j_1j_2,j_1j_2}(\boldsymbol{w}_{\sigma,1j_1j_2}, \boldsymbol{w}_{\sigma,2j_1j_2})$, and

$$\begin{aligned}
&\boldsymbol{Q}^0_{R,j_1j_2,j_1^*j_2^*}(\boldsymbol{w}_{\sigma,1j_1j_2}, \boldsymbol{w}_{\sigma,2j_1j_2}) \\
&= \frac{T^2}{T_{j_1j_2}(\theta_1^0,\theta_2^0)\cdot T_{j_1^*j_2^*}(\theta_1^0,\theta_2^0)} \cdot \Big[\boldsymbol{\Psi}^0_{R,j_1j_2,j_1^*j_2^*}(\boldsymbol{w}_{\sigma,1j_1j_2}, \boldsymbol{w}_{\sigma,2j_1j_2}; \boldsymbol{w}_{\sigma,1j_1j_2}, \boldsymbol{w}_{\sigma,2j_1j_2}) \\
&+ \boldsymbol{Q}^0_{12j_1j_2,j_1j_2}(\boldsymbol{w}_{\sigma,1j_1j_2}, \boldsymbol{w}_{\sigma,2j_1j_2}; \boldsymbol{w}_{\sigma,1j_1j_2}, \boldsymbol{w}_{\sigma,2j_1j_2}) \\
&+ \boldsymbol{Q}^0_{21j_1j_2,j_1j_2}(\boldsymbol{w}_{\sigma,1j_1j_2}, \boldsymbol{w}_{\sigma,2j_1j_2}; \boldsymbol{w}_{\sigma,1j_1j_2}, \boldsymbol{w}_{\sigma,2j_1j_2}) + 2\cdot\boldsymbol{\Xi}^0_{1,j_1j_2,j_1j_2} + 2\cdot\boldsymbol{\Xi}^0_{2,j_1j_2,j_1j_2}\Big],
\end{aligned}$$

with

$$\begin{aligned}
&\boldsymbol{\Psi}^0_{R,j_1j_2,j_1^*j_2^*}(\boldsymbol{w}_{\sigma,1j_1j_2}, \boldsymbol{w}_{\sigma,2j_1j_2}; \boldsymbol{w}_{\sigma,1j_1^*j_2^*}, \boldsymbol{w}_{\sigma,2j_1^*j_2^*}) \\
&= \lim_{N_1,N_2,T\to\infty} \frac{1}{T}\sum_{t=1}^{T}\sum_{v=1}^{T} \mathrm{Cov}\left[\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right) c^0_{R,j_1j_2t}(\boldsymbol{w}_{\sigma,1j_1j_2}, \boldsymbol{w}_{\sigma,2j_1j_2})\ ,\right. \\
&\left.\mathbb{I}_{1j_1^*v}\left(\theta_1^0\right)\mathbb{I}_{2j_2^*v}\left(\theta_2^0\right) c^0_{R,j_1^*j_2^*v}(\boldsymbol{w}_{\sigma,1j_1^*j_2^*}, \boldsymbol{w}_{\sigma,2j_1^*j_2^*})\right],
\end{aligned}$$

being computed for $(j_1^*, j_2^*) = (j_1, j_2)$. Moreover,

$$c^0_{R,j_1j_2t}(\boldsymbol{w}_{\sigma,1j_1j_2}, \boldsymbol{w}_{\sigma,2j_1j_2}) = \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{\boldsymbol{\sigma},1j_1j_2i_1} w_{\boldsymbol{\sigma},2j_1j_2i_2} c^0_{j_1j_2i_1i_2}\left[\frac{c^0_{j_1j_2i_1i_2t}}{c^0_{j_1j_2i_1i_2}} - \frac{x^2_{1i_1t}}{2(\sigma^0_{\boldsymbol{x}1j_1j_2i_1})^2} - \frac{x^2_{2i_2t}}{2(\sigma^0_{\boldsymbol{x}2j_1j_2i_2})^2}\right],$$

the quantities $\boldsymbol{Q}^0_{12,j_1j_2,j_1^*j_2^*}(\cdot,\cdot)$ *and* $\boldsymbol{Q}^0_{21,j_1j_2,j_1^*j_2^*}(\cdot,\cdot)$ *are defined in Theorem 2, and*

$$\boldsymbol{\Xi}^0_{1,j_1j_2,j_1^*j_2^*} = \lim_{N_1,N_2,T\to\infty} \frac{1}{T}\sum_{t=1}^{T}\sum_{v=1}^{T}\text{Cov}\Bigg(\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{\boldsymbol{\sigma},1j_1j_2i_1}w_{\boldsymbol{\sigma},2j_1j_2i_2}\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}$$
$$\times\boldsymbol{\Sigma}^{0\prime}_{\mathbf{f}12j_1j_2}\left(\boldsymbol{\Sigma}^0_{\mathbf{f}1j_1}\right)^{-1}\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\ ,\ \mathbb{I}_{1j_1^*v}\left(\theta_1^0\right)\mathbb{I}_{2j_2^*v}\left(\theta_2^0\right)\sum_{i_1'=1}^{N_1}\sum_{i_2'=1}^{N_2} w_{\boldsymbol{\sigma},1j_1^*j_2^*i_1'}w_{\boldsymbol{\sigma},2j_1^*j_2^*i_2'}\frac{c^0_{j_1^*j_2^*i_1'i_2'}}{(\sigma^0_{\boldsymbol{x}1j_1^*j_2^*i_1'})^2}\boldsymbol{\lambda}'_{1j_1^*i_1'}\boldsymbol{f}_{1j_1^*v}e_{1,i_1'v}\Bigg)\ ,$$

$$\boldsymbol{\Xi}^0_{2,j_1j_2,j_1^*j_2^*} = \lim_{N_1,N_2,T\to\infty} \frac{1}{T}\sum_{t=1}^{T}\sum_{v=1}^{T}\text{Cov}\Bigg(\mathbb{I}_{2j_1t}\left(\theta_1^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{\boldsymbol{\sigma},1j_1j_2i_1}w_{\boldsymbol{\sigma},2j_1j_2i_2}\boldsymbol{\lambda}^{0\prime}_{1j_2i_2}$$
$$\times\boldsymbol{\Sigma}^{0}_{\mathbf{f}12j_1j_2}\left(\boldsymbol{\Sigma}^0_{\mathbf{f}2j_2}\right)^{-1}\boldsymbol{f}^0_{2j_2t}e_{2i_2t}\ ,\ \mathbb{I}_{1j_1^*v}\left(\theta_1^0\right)\mathbb{I}_{2j_2^*v}\left(\theta_2^0\right)\sum_{i_1'=1}^{N_1}\sum_{i_2'=1}^{N_2} w_{\boldsymbol{\sigma},1j_1^*j_2^*i_1'}w_{\boldsymbol{\sigma},2j_1^*j_2^*i_2'}\frac{c^0_{j_1^*j_2^*i_1'i_2'}}{(\sigma^0_{\boldsymbol{x}2j_1^*j_2^*i_2'})^2}\boldsymbol{\lambda}'_{2j_2^*i_2'}\boldsymbol{f}_{2j_2^*v}e_{2,i_2'v}\Bigg)\ ,$$

are also computed for $(j_1^*,j_2^*)=(j_1,j_2)$.

Proof. See Section C.4 in Appendix C.

Compared to Theorem 1, we derive Theorems 2 and 3, as well as Theorems 4 and 5 below, by replacing Assumption A.4, which only imposes within-group weak dependence between factors and idiosyncratic components, with the stronger Assumption A.12, which requires independence between factors and idiosyncratic components both within and between groups. Assumption A.12 may be generalized by imposing some worm of weak dependence between factors and idiosyncratic components, as in Andreou, Gagliardini, Ghysels, and Rubin (2019), at the expense of a higher final degree of complexity induced by the presence of cross-moments of order three or higher in the asymptotic variances of those theorems.

4.3 Limiting distributions of tests for changes in systematic comovement

The following theorems allow to construct tests for the null of no change in the comovement defined as a change either in the systematic covariance (Theorem 4), or in the systematic correlation (Theorem 5).

THEOREM 4. *For* $g=1,2$, *and* $j_g, j_g^* = L,H$, *with either* $j_1\neq j_1^*$, *or* $j_2\neq j_2^*$ *(or both), consider the test statistic*

$$\widehat{\mathcal{T}}^c_{j_1j_2j_1^*j_2^*\boldsymbol{w}_1\boldsymbol{w}_2} = \sqrt{T}\ \frac{\left(\hat{c}_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2}-\hat{c}_{j_1^*j_2^*\boldsymbol{w}_1\boldsymbol{w}_2}\right)}{\sqrt{\widehat{\boldsymbol{Q}}^{\Delta}_{j_1j_2j_1^*j_2^*}}},$$

where $\widehat{\boldsymbol{Q}}^{\Delta}_{j_1 j_2 j_1^* j_2^*}$*, defined in equation (B.6) in Appendix B, is a consistent estimator for*

$$\begin{aligned}\boldsymbol{Q}^{\Delta,0}_{j_1 j_2 j_1^* j_2^*} &= \boldsymbol{Q}^{0}_{j_1 j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2) + \boldsymbol{Q}^{0}_{j_1^* j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2) - \frac{2 \cdot T^2}{T_{j_1 j_2}(\theta_1^0, \theta_2^0) \cdot T_{j_1^* j_2^*}(\theta_1^0, \theta_2^0)} \cdot \Big[\boldsymbol{\Psi}^{0}_{j_1 j_2, j_1^* j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2) \\ &\quad + \boldsymbol{Q}^{0}_{12 j_1 j_2, j_1^* j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2) + \boldsymbol{Q}^{0}_{21 j_1 j_2, j_1^* j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2) \Big]\end{aligned}$$

with $(j_g, j_g^*) = (H, L)$ *or* (L, H)*, and all quantities in the last equation are defined in Theorem 2. Let the assumptions of Theorem 2 hold, then* $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ *is such that as* $N_1, N_2, T \to \infty$*: (i)* $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2} \xrightarrow{d} \mathcal{N}(0,1)$ *under* $\mathcal{H}_0^c : c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} = c^0_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$*; and (ii)* $P\left(|\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}| > \kappa \right) \to 1$ *for any constant* $\kappa \in \mathbb{R}$ *under* $\mathcal{H}_1^c : c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} \neq c^0_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$.

Proof. See Section C.5 in Appendix C.

THEOREM 5. *For* $g = 1, 2$*, and* $j_g, j_g^* = L, H$*, with either* $j_1 \neq j_1^*$*, or* $j_2 \neq j_2^*$ *(or both), consider the test statistic*

$$\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2} = \sqrt{T} \, \frac{\left(\hat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - \hat{R}_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2} \right)}{\sqrt{\widehat{\boldsymbol{Q}}^{\Delta}_{R, j_1 j_2 j_1^* j_2^*}}},$$

where $\widehat{\boldsymbol{Q}}^{\Delta}_{R, j_1 j_2 j_1^* j_2^*}$ *defined in equation (B.7) in Appendix B is a consistent estimator for*

$$\begin{aligned}\boldsymbol{Q}^{\Delta,0}_{R, j_1 j_2 j_1^* j_2^*} &= \boldsymbol{Q}^{0}_{R, j_1 j_2} + \boldsymbol{Q}^{0}_{R, j_1^* j_2^*} \\ &\quad - \frac{2 \cdot T^2}{T_{j_1 j_2}(\theta_1^0, \theta_2^0) \cdot T_{j_1^* j_2^*}(\theta_1^0, \theta_2^0)} \times \Big[\boldsymbol{\Psi}^{0}_{R, j_1 j_2, j_1^* j_2^*}(\boldsymbol{w}_{\sigma,1 j_1 j_2}, \boldsymbol{w}_{\sigma,2 j_1 j_2}; \boldsymbol{w}_{\sigma,1 j_1^* j_2^*}, \boldsymbol{w}_{\sigma,2 j_1^* j_2^*}) \\ &\quad + \boldsymbol{Q}^{0}_{12 j_1 j_2, j_1^* j_2^*}(\boldsymbol{w}_{\sigma,1 j_1 j_2}, \boldsymbol{w}_{\sigma,2 j_1 j_2}; \boldsymbol{w}_{\sigma,1 j_1^* j_2^*}, \boldsymbol{w}_{\sigma,2 j_1^* j_2^*}) \\ &\quad + \boldsymbol{Q}^{0}_{21 j_1 j_2, j_1^* j_2^*}(\boldsymbol{w}_{\sigma,1 j_1 j_2}, \boldsymbol{w}_{\sigma,2 j_1 j_2}; \boldsymbol{w}_{\sigma,1 j_1^* j_2^*}, \boldsymbol{w}_{\sigma,2 j_1^* j_2^*}) \\ &\quad + \boldsymbol{\Xi}^{0}_{1, j_1 j_2, j_1^* j_2^*} + \boldsymbol{\Xi}^{0}_{2, j_1 j_2, j_1^* j_2^*} + \boldsymbol{\Xi}^{0}_{1, j_1^* j_2^*, j_1 j_2} + \boldsymbol{\Xi}^{0}_{2, j_1^* j_2^*, j_1 j_2} \Big],\end{aligned}$$

with $(j_g, j_g^*) = (H, L)$ *or* (L, H)*, and where* $\boldsymbol{Q}^{0}_{R, j_1 j_2}$*,* $\boldsymbol{Q}^{0}_{R, j_1^* j_2^*}$ $\boldsymbol{\Psi}^{0}_{R, j_1 j_2, j_1^* j_2^*}(\cdot)$*,* $\boldsymbol{\Xi}^{0}_{1, j_1 j_2, j_1^* j_2^*}$*, and* $\boldsymbol{\Xi}^{0}_{2, j_1 j_2, j_1^* j_2^*}$ *are defined in Theorem 3, while* $\boldsymbol{Q}^{0}_{12 j_1 j_2, j_1^* j_2^*}(\cdot)$ *and* $\boldsymbol{Q}^{0}_{21 j_1 j_2, j_1^* j_2^*}(\cdot)$ *are defined in Theorem 2. Let the assumptions of Theorem 2 hold, then* $\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ *is such that as* $N_1, N_2, T \to \infty$*: (i)* $\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2} \xrightarrow{d} \mathcal{N}(0,1)$ *under* $\mathcal{H}_0^R : R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} = R^0_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$*; and (ii)* $P\left(|\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}| > \kappa \right) \to 1$ *for any constant* $\kappa \in \mathbb{R}$ *under* $\mathcal{H}_1^c : R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} \neq R^0_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$.

Proof. See Section C.6 in Appendix C.

As noted in the Introduction, all Theorems 1 - 5 have been derived in the context of the threshold-group factor model (1) where the Thresholds θ_1 and θ_2 need to be estimated. It turns out that the estimation error of the thresholds does not affect the asymptotic distribution of the comovement measures. Therefore, the results of all our Theorems 1 - 5 can be applied also in the case of threshold values exogenously chosen by the econometrician. For instance, if $z_t = t/T$, the values of the threshold could be set a priori in order to test for a change in comovement across groups between two distinct periods of time. This can be seen as a formalization of the approach in Bekaert, Hodrick, and Zhang (2009), who study changes in systematic comovement across non-overlapping time widows.

5 Monte Carlo

We conduct Monte Carlo experiments to study the finite sample properties of the estimators and testing procedures proposed in the previous sections. Section 5.1 describes the simulations designs while Section 5.2 presents the results. We denote with $0_{A\times B}$ the $A \times B$ dimensional matrix of zeros, and with 1_A the $A \times 1$ vector of ones.

5.1 Simulation Designs

We simulate data from the following Data Generating Process (DGP):

$$\begin{aligned} x_{1i_1t} &= \mathbb{I}(z_t \le \theta) \cdot \left[\boldsymbol{\lambda}_{1Li_1}^{c\prime} \boldsymbol{f}_{Lt}^{c} + \boldsymbol{\lambda}_{1Li_1}^{s\prime} \boldsymbol{f}_{1,Lt}^{s} \right] + \mathbb{I}(z_t > \theta) \cdot \left[\boldsymbol{\lambda}_{1Hi_1}^{c\prime} \boldsymbol{f}_{Ht}^{c} + \boldsymbol{\lambda}_{1Hi_1}^{s\prime} \boldsymbol{f}_{1,Ht}^{s} \right] + e_{1i_1t} \ , \\ x_{2i_2t} &= \mathbb{I}(z_t \le \theta) \cdot \left[\boldsymbol{\lambda}_{2Li_2}^{c\prime} \boldsymbol{f}_{Lt}^{c} + \boldsymbol{\lambda}_{2Li_2}^{s\prime} \boldsymbol{f}_{2,Lt}^{s} \right] + \mathbb{I}(z_t > \theta) \cdot \left[\boldsymbol{\lambda}_{2Hi_2}^{c\prime} \boldsymbol{f}_{Ht}^{c} + \boldsymbol{\lambda}_{2Hi_2}^{s\prime} \boldsymbol{f}_{2,Ht}^{s} \right] + e_{2i_2t} \ , \end{aligned}$$

for $i_1 = 1, ..., N_1, i_2 = 1, ..., N_2,\ t = 1, ..., T$. This is a constrained version of model (1) where, compatibly with the findings of the empirical analysis, the threshold variable z_t and the threshold value are the same across the two groups, that is $z_t = z_{1t} = z_{2t}$ and $\theta = \theta_1 = \theta_2$. All dates when $z_t \le \theta$ are denoted as regime L, while all dates when $z_t > \theta$ are denoted as regime H. Table 1 (resp. Table 2) displays the number of factors and the values of the other DGP parameters characterizing each MC design used with respect to Theorems 1, 2 and 3 (resp. Theorems 4 and 5). The number of MC replications used for each design is 4000.

The vectors of factors $\boldsymbol{f}_{L,t}^{c}$, $\boldsymbol{f}_{H,t}^{c}$, $\boldsymbol{f}_{L,1t}^{s}$, $\boldsymbol{f}_{L,2t}^{s}$, $\boldsymbol{f}_{H,1t}^{s}$ and $\boldsymbol{f}_{H,2t}^{s}$ have dimensions K_L^c, K_H^c, K_L^s, K_L^s, K_H^s, and K_H^s respectively. Since $K_L^s := K_{1L}^s = K_{2L}^s$ (resp $K_H^s := K_{1H}^s = K_{2H}^s$), we are imposing that the number of group-specific factors in regime L (resp. regime H) is the same for group 1 and group 2. Furthermore, in all simulation designs we assume that the number of common factors (resp. group-specific factors) is the same in both regimes, that is $K_L^c = K_H^c$ (resp. $K_L^s = K_H^s$). Let

Table 1 – Parameters of Monte Carlo simulation designs for Theorems 1, 2 and 3

Design / Param.	$K_L^C = K_H^C$	$K_L^s = K_H^s$	$\delta_{i_g}^c = \delta_{i_g}^s$	c_{gj}^*	β	a_F	a_z	π_0
Design 1	0	3	0.25	0.5	0	0	0	0.75
Design 2	0	3	1.00	0.5	0	0	0	0.75
Design 3	**1**	3	0.25	0.5	0	0	0	0.75
Design 4	**1**	3	1.00	0.5	0	0	0	0.75
Design 5	0	3	0.25	0.5	0	**0.5**	0	0.75
Design 6	0	3	1.00	0.5	0	**0.5**	0	0.75
Design 7	1	3	0.25	0.5	0	**0.5**	0	0.75
Design 8	1	3	1.00	0.5	0	**0.5**	0	0.75

Table 1 provides values of the parameters in the DGP described in Section 5 for each of the MC simulation designs used to assess the properties of the statistics in Theorems 1, 2 and 3. In all simulation designs we also set $\sigma_{gj}^c = \sigma_{gj}^s = 1$, $\alpha = 1$, $\Phi_L^s = diag(0.4,\ 0.2,\ 0.1)$ and $\Phi_H^s = diag(0.8,\ 0.4,\ 0.2)$.

Table 2 – Parameters of Monte Carlo simulation designs for Theorems 4 and 5

Design / Param.	$K_L^C = K_H^C$	$K_L^s = K_H^s$	$\delta_{i_g}^c = \delta_{i_g}^s$	c_{gj}^*	β	a_F	a_z	π_0
Design 1 H_0	0	3	0.25	0.5	0	0	0	0.75 or 0.5
Design 1 H_1	0	3	0.25	0.5	0	0	0	0.75 or 0.5
Design 2 H_0	0	3	1.00	0.5	0	0	0	0.75 or 0.5
Design 2 H_1	0	3	1.00	0.5	0	0	0	0.75 or 0.5
Design 3 H_0	0	3	0.25	0.5	0	**0.5**	0	0.75 or 0.5
Design 3 H_1	0	3	0.25	0.5	0	**0.5**	0	0.75 or 0.5
Design 4 H_0	0	3	1.00	0.5	0	**0.5**	0	0.75 or 0.5
Design 4 H_1	0	3	1.00	0.5	0	**0.5**	0	0.75 or 0.5

Table 2 provides values of the parameters in the DGP described in Section 5 for each of the MC simulation designs used with respect to Theorems 4 and 5. Designs 1 H_0 to 4 H_0, where data is simulated under the null hypothesis of no change in comovement across regimes by setting $\Phi_L^s = \Phi_H^s = 0_{3\times3}$, are used to assess the properties of the statistics and the size of the tests in Theorems 4 and 5. Designs 1 H_1 to 4 H_1, where data is simulated under the alternative hypothesis of change in comovement across regimes by setting $\Phi_L^s = 0_{3\times3}$ and $\Phi_H^s = diag(0.8,\ 0.4,\ 0.2)$, are used to assess the power of the tests in Theorems 4 and 5. In all simulation designs we also set $\sigma_{gj}^c = \sigma_{gj}^s = 1$ and $\alpha = 1$. For all designs we consider $\pi_0 = 0.5$ only for Theorem 5.

$K^* := K_L^c + K_H^c + 2\cdot K_L^s + 2\cdot K_H^s$, the K^*-dimensional vector $\boldsymbol{f}_t := [\boldsymbol{f}_{L,t}^{c\,\prime},\ \boldsymbol{f}_{H,t}^{c\,\prime},\ \boldsymbol{f}_{L,1t}^{s\,\prime},\ \boldsymbol{f}_{L,2t}^{s\,\prime},\ \boldsymbol{f}_{H,1t}^{s\,\prime},\ \boldsymbol{f}_{H,2t}^{s\,\prime}]'$ follows an autoregressive process: $\boldsymbol{f}_t = a_F \boldsymbol{f}_{t-1} + \sqrt{1 - a_F^2}\ \eta_t$, where the scalar a_F is an AR(1) coefficient common to all factors. The innovation vector $\eta_t := [\eta_{L,t}^{c\,\prime},\ \eta_{H,t}^{c\,\prime},\ \eta_{L,1t}^{s\,\prime},\ \eta_{L,2t}^{s\,\prime},\ \eta_{H,1t}^{s\,\prime},\ \eta_{H,2t}^{s\,\prime}]'$ is simulated such that $\eta_t \sim i.i.N(0, \Sigma_\eta)$, with

$$\Sigma_\eta = \begin{bmatrix} I_{K_L^c} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{K_H^c} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{K_L^s} & \Phi_L^s & 0 & 0 \\ 0 & 0 & \Phi_L^s & I_{K_L^s} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{K_H^s} & \Phi_H^s \\ 0 & 0 & 0 & 0 & \Phi_H^s & I_{K_H^s} \end{bmatrix},$$

where $\Phi_L^s = diag(\phi_{L,1}^s, ..., \phi_{L,K_L^s})$ and $\Phi_H^s = diag(\phi_{H,1}^s, ..., \phi_{H,K_H^s})$ are $(K_L^s \times K_L^s)$ and $(K_H^s \times K_H^s)$ diagonal matrices, respectively. The scalar parameters in the main diagonal of Φ_L^s (resp. Φ_H^s) are such that $1 \geq \phi_{L,1}^s \geq \phi_{L,2}^s \geq ... \geq \phi_{L,K_L^s} \geq 0$ (resp. $1 \geq \phi_{H,1}^s \geq \phi_{H,2}^s \geq ... \geq \phi_{H,K_H^s} \geq 0$), and generate correlation between the first specific factor in group 1 and the first specific factor in group 2 in regime L (resp. H), the second specific factor in group 1 and the second specific factor in group 2 in regime L (resp. H), and so on. By definition, they are the ordered non-zero canonical correlations between the group specific factors in group 1 and 2 in regime L (resp. H). In Designs 1 - 8 of Table 1, we let $\Phi_L^s = diag(0.4,\ 0.2,\ 0.1)$ and $\Phi_H^s = diag(0.8,\ 0.4,\ 0.2)$. In Designs 1 H_0 - 4 H_0 of Table 2, which refer to data simulated under the null hypothesis of no change in comovement across regimes, $\Phi_L^s = \Phi_H^s = 0_{3\times 3}$. In Designs 1 H_1 - 4 H_1 of Table 2, which refer to data simulated under the the alternative hypothesis of change in comovement across regimes, $\Phi_L^s = 0_{3\times 3}$ and $\Phi_H^s = diag(0.8,\ 0.4,\ 0.2)$. The initial values of the factors are drawn from their stationary distributions, and their paths are re-sampled in each MC simulation.

For each group $g = 1, 2$, the first $K_{min}^c = \min(K_L^c,\ K_H^c)$ common factors loadings are drawn for the L regime as $\boldsymbol{\lambda}_{gLi_g}^c = \sigma_g^c \tilde{\boldsymbol{\lambda}}_{gLi_g}^c$, with $\tilde{\boldsymbol{\lambda}}_{gLi_g}^c \sim i.i.N\left(\ 1_{K_{min}^c},\ I_{K_{min}^c}\ \right)$; the first $K_{min}^s = \min(K_L^s, K_H^s)$ group-specific factors loadings are drawn for the L regime as $\boldsymbol{\lambda}_{gLi_g}^s = \sigma_g^s \tilde{\boldsymbol{\lambda}}_{gLi_g}^s$, $\tilde{\boldsymbol{\lambda}}_{gLi_g}^s \sim i.i.N\left(\ 1_{K_{min}^s},\ I_{K_{min}^s}\ \right)$; the first K_{min}^c common factors loadings are generated for the H regime as $\boldsymbol{\lambda}_{1Hi_1}^c = \boldsymbol{\lambda}_{1Li_1}^c + \delta_{i_1}^c 1_{K_{min}^c}$; the first K_{min}^s group-specific factor loadings are generated for the H regime as $\boldsymbol{\lambda}_{gHi_g}^s = \boldsymbol{\lambda}_{gLi_g}^s + \delta_{i_g}^s 1_{K_{min}^s}$. If $K_H^c > K_L^c$, the additional $\Delta K^c := K_H^c - K_L^c$ common factor loadings in regime H are drawn from $\boldsymbol{\lambda}_{1Li_1}^c = \sigma_1^c \tilde{\boldsymbol{\lambda}}_{1Li_1}^c$, with $\tilde{\boldsymbol{\lambda}}_{1Li_1}^c \sim i.i.N\left(\ 1_{\Delta K^c}(1 + \delta_{i_1}^c),\ I_{\Delta K^c}\ \right)$. If $K_H^c < K_L^c$, the additional $\Delta K^c := K_L^c - K_H^c$ common factor loadings in regime L are drawn from $\boldsymbol{\lambda}_{1Li_1}^c = \sigma_1^c \tilde{\boldsymbol{\lambda}}_{1Li_1}^c$, with $\tilde{\boldsymbol{\lambda}}_{1Li_1}^c \sim i.i.N\left(\ 1_{\Delta K^c},\ I_{\Delta K^c}\ \right)$. If $K_H^s > K_L^s$, the additional $\Delta K^s := K_H^s - K_L^s$ group-specific factor loadings in regime H are drawn from $\boldsymbol{\lambda}_{1Hi_1}^s = \sigma_1^s \tilde{\boldsymbol{\lambda}}_{1Hi_1}^s$, with $\tilde{\boldsymbol{\lambda}}_{1Hi_1}^s \sim i.i.N\left(\ 1_{\Delta K^s}(1 + \delta_{i_1}^s),\ I_{\Delta K^s}\ \right)$. Finally, if $K_H^s < K_L^s$, the additional $\Delta K^s := K_L^s - K_H^s$ group-specific factor loadings in regime L are drawn from $\boldsymbol{\lambda}_{gLi_g}^s = \sigma_g^s \tilde{\boldsymbol{\lambda}}_{gLi_g}^s$, with $\tilde{\boldsymbol{\lambda}}_{gLi_g}^s \sim i.i.N\left(\ 1_{\Delta K^s},\ I_{\Delta K^s}\ \right)$ with:

$$\delta_{i_g}^c > 0 \quad \text{for} \quad i_1 = 1, ..., [N_g^\alpha], \qquad \delta_{i_g}^c = 0 \quad \text{for} \quad i_g = [N_g^\alpha] + 1, ..., N_g,$$
$$\delta_{i_g}^s > 0 \quad \text{for} \quad i_1 = 1, ..., [N_g^\alpha], \qquad \delta_{i_g}^s = 0 \quad \text{for} \quad i_g = [N_g^\alpha] + 1, ..., N_g,$$

where $[\cdot]$ denotes the integer part of the argument, and the scalars σ_1^c, σ_2^c, σ_1^s, and σ_2^s determine the contribution of each factor to the overall variability of the "common component", that is the variance of the observables due to all the pervasive factors for each group $g = 1, 2$, and regime $j = L, H$. Finally, we set $\sigma_1^c = \sigma_2^c = 0$, if $K_L^c = 0$, and $\sigma_1^s = \sigma_2^s = 0$, if $K_L^s = 0$. For each design and sample sizes combination the loading matrices are the same across all MC simulations.

The idiosyncratic innovations vector $e_{gt} = [e_{gi_g t}, ..., e_{gN_g t}]'$ has dimension N_g, with $g = 1, 2$. We define the $(N_1 + N_2)$-dimensional vector $e_t := [e'_{1t}, e'_{2t}]'$, and assume the following AR(1) process:

$$e_{i_g t} = a_e e_{i_g t-1} + \left(c^*_{gj} \cdot c_{e,i_g t} \cdot \sqrt{1 - a_e^2}\right) v_{i_g t} \ ,$$

where a_e is a common AR(1) scalar coefficient for groups $g = 1, 2$, and

$$c^2_{e,i_g t} = \begin{cases} 2K_L^c \sigma_g^{c2} + 2K_L^s \sigma_g^{s2} & \text{if} \quad z_t \leq \theta_g \\ K_H^c \left[(\sigma_g^c + \delta_{i_g}^c)^2 + (\sigma_g^c)^2\right] + K_H^s \left[(\sigma_g^s + \delta_{i_g}^s)^2 + (\sigma_g^s)^2\right] & \text{if} \quad z_t > \theta_g \end{cases} .$$

Let $v_{gt} := [v_{1t}, ..., v_{i_g t}, ..., v_{N_g t}]'$, then the $(N_1 + N_2)$-dimensional vector $v_t := [v'_{1t}, v'_{2t}]'$ is simulated as $v_t \sim i.i.N(0, \Sigma_v)$, where $\Sigma_v = \{\beta^{|i-j|}\}_{ij}$, for $i, j = 1, ..., N_1 + N_2$, The scalar β in $[0, 1)$ induces cross-sectional dependence among the idiosyncratic innovations, similarly to Bates, Plagborg-Moller, Stock, and Watson (2013). The variance of the idiosyncratic part is c^*_{gj} times the variance of the common component in group g and state j.[5] The initial values of the idiosyncratic innovations are drawn from their stationary distributions, and all the innovations paths are re-sampled in each MC simulation.

The DGP for the threshold variable is an AR(1) process: $z_t = a_z z_{t-1} + \sqrt{1 - a_z^2}\, v_{z,t}$, where $v_{z,t} \sim i.i.N(0, 1)$. Let $\Phi(\theta)$ be the cumulative distribution function of the Standard Normal computed in $\theta \in \mathbb{R}$, and $\pi^0 = P(z_t \leq \theta^0) = \Phi(\theta^0)$ be the unconditional probability of observing a value of $z_t \leq \theta^0$, then $\theta^0 = \Phi^{-1}(\pi^0)$. We report simulation results for all Theorems setting $a_z = 0$ and $\pi^0 = 0.75$ (corresponding to $\theta^0 \approx 0.6745$). Only for the test in Theorem 5, we also report the simulated size and power for the combination of parameters $a_z = 0.0$ and $\pi^0 = 0.5$ (corresponding to $\theta^0 = 0$). Additionally, in the SM we report results for all Theorems for $a_z = 0.5$ and/or $\pi^0 = 0.50$. The initial values of z_t are drawn from its stationary distribution, and its sample paths are re-sampled in each MC simulation. The innovations of factors, errors, threshold variable, and loadings are drawn as mutually independent.

[5] When $c^*_{gj} = 1$ for $g = 1, 2$ and $j = L, H$, then the scalar $c_{e,i_g t}$ is defined so that the variance of the idiosyncratic part is equal to the variance of the common component within each group $g = 1, 2$ and state H, L.

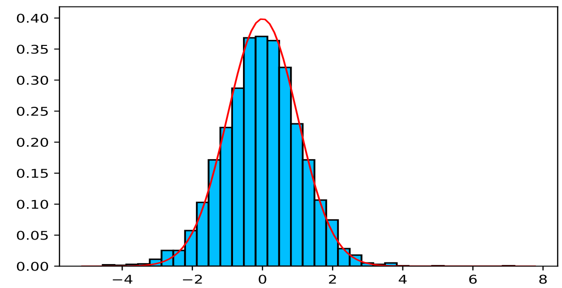
5.2 Monte Carlo results

In order to save space, in Appendix D we only display Tables D.1 and D.2 with the first two moments, median and interquantile range of the simulated distributions for Theorems 1, 2, respectively, and Table D.3 in the same appendix with the empirical size and power of the test of Theorem 5. Analogously, Figure 1 reports the histograms of the simulated distributions for Theorems 1 and 2, and the distribution under the null of no change in systematic correlation for the test is Theorem 5, only for two combination of sample sizes, namely: $N_1 = N_2 = 50$ (resp. 500), and $T = 200$ (resp. 1000). In Section E of the SM we report analogous tables and figures for all the remaining Theorems. As it can be seen from the moments and the quantiles reported in Tables D.1 - D.3, from the histograms in Figure 1, and from the results in Section E of the SM, for all the Theorems 1 - 5 and also for relatively small sample sizes (with the smallest ones being $N_1 = N_2 = 30$ and $T = 100$) the simulated distributions approximate well a Gaussian distribution with zero mean and variance equal to the estimated asymptotic variance form each theorem.

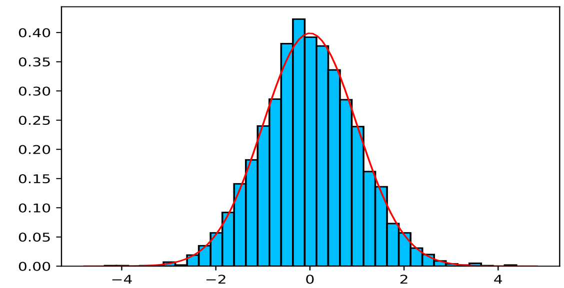
Table D.3 displays the empirical size of the tests for the null hypotheses of no change in systematic correlation across all individuals in the two groups corresponding to nominal sizes of 1%, 5%, and 10%. It also reports the empirical power of the same test performed on a DGP corresponding to the alternative hypothesis, with a significance level of 5%. The null hypothesis is imposed in the two regimes by simulating group-specific factors with zero correlations across groups, which implies that the systematic covariance and correlation across groups are zero in both regimes. The alternative hypothesis is imposed by simulating group-specific factors with correlation structure changing from one regime to the other, as described above. We observe that the asymptotic Gaussian distribution provides an overall very good approximation for the tails of the feasible test statistics of Theorems 4 and 5 under the null. For the vast majority of sample sizes and simulation designs, the size distortions range from 0.1% to 5%. The largest size distortions and the lowest values of the power (that is around 70%) are observed for the smallest value of T, that is $T = 100$, and for the designs where $\pi^0 = 0.75$. As expected, when the sample sizes increase the size distortions monotonically disappear, and the power approaches 1. By comparing Tables D.3 (a) and (b), we notice that decreasing π^0 from 0.75 to 0.5 (that is reducing the threshold value θ^0 from 0.6745 to 0) also substantially improves the size and the power of the test in Theorem 5. Analogous considerations can be made by looking at the results in Section E of the SM for all the other Theorems and simulation designs.

Figure 1 – Finite sample distribution of statistics in Theorems 1, 2 and 5, with $\pi_0 = 0.75$

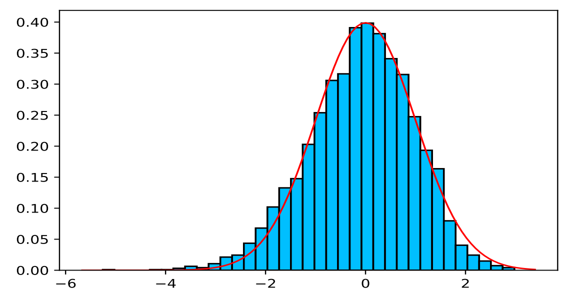
(a) Design 2, $N_1 = N_2 = 50$, $T = 200$



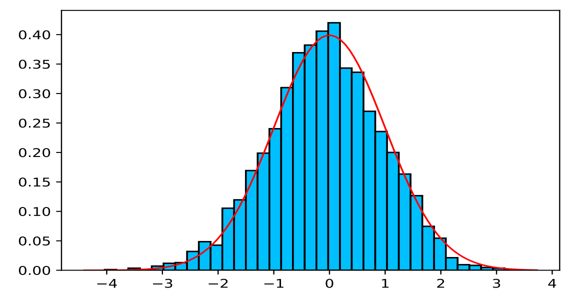
(b) Design 2, $N_1 = N_2 = 500$, $T = 1000$



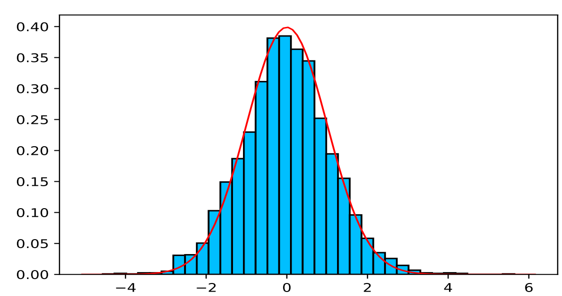
(c) Design 2, $N_1 = N_2 = 50$, $T = 200$



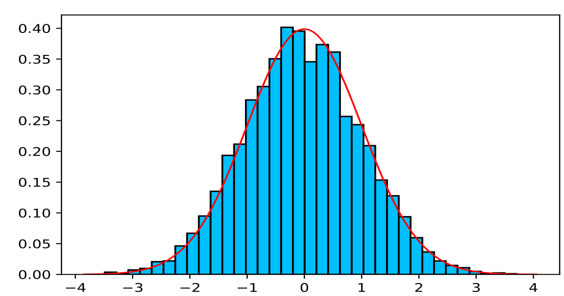
(d) Design 2, $N_1 = N_2 = 500$, $T = 1000$



(e) Design 2, $N_1 = N_2 = 50$, $T = 200$



(f) Design 2, $N_1 = N_2 = 500$, $T = 1000$



This figure shows the simulated empirical distribution of relevant statistics obtained with 4000 Monte Carlo simulations and computed for different sample sizes (N_1, N_2, T) and for the values of the DGP parameters in Design 2 of Tables 1 and 2 with $\pi_0 = 0.75$. Panels (a) and (b) refer to the recentered and standardized statistic $\hat{c}_{i_1 i_2 t}$ in Theorem 1 defined as: $C_{NT}\left(\hat{c}_{i_1 i_2 t} - c^0_{i_1 i_2 t}\right) / \sqrt{\hat{\boldsymbol{Q}}_{i_1 i_2 t}}$. Panels (c) and (d) refer to the recentered and standardized statistic $\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 2 defined as: $\sqrt{T}\left(\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}\right) / \sqrt{\hat{\boldsymbol{Q}}_{j_1 j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2)}$, with $\boldsymbol{w}_1 = [1, 0, 0, ..., 0]$ and $\boldsymbol{w}_2 = [1, 0, 0, ..., 0]$. Panels (e) and (f) refer to the test statistic $\widehat{\mathcal{T}}^R_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 5, with $\boldsymbol{w}_1 = [1/N_1, ..., 1/N_1]$ and $\boldsymbol{w}_2 = [1/N_2, ..., 1/N_2]$. Under the Assumptions of Theorems 1, 2 and 5, the asymptotic distribution of the three statistics is standard Gaussian (solid red line).

6 Systematic Comovement over the Global Financial Cycle

There exists empirical evidence that increasing financial integration has led to the emergence of a global financial cycle: see Rey (2018). As documented in Miranda-Agrippino and Rey (2020), and further in Habib and Venditti (2019), one of the main drivers of the global financial cycle is U.S.

monetary policy. Given the misalignment between countries' specific macroeconomic conditions and the global financial cycle discussed in Rey (2018), it is unclear how the comovement of asset returns evolve over the latter when the former change. We thus use the tools developed in this paper to study systematic comovement in global equity markets over the financial cycle depending on the state of U.S. macroeconomic uncertainty. Section 6.1 describes the data and the empirical specification. Section 6.2 deals with estimation and model selection. Section 6.3 presents the empirical findings about systematic comovement. Section 6.4 discusses the results.

6.1 Data and Empirical Specification

We consider monthly data and study the period running between January 1991 and December 2019, a total of $T = 348$ time series observations. Financial data are obtained from Kenneth French website.[6] In the case of the U.S., we consider the following $N_1 = 100$ value-weighted portfolios: 25 portfolios sorted by size and book-to-market ratio; 25 portfolios sorted by size and operating profitability; 25 portfolios sorted by size and investment; 25 portfolios sorted by size and momentum. We then consider the $N_2 = 100$ homologous portfolios for international equity markets: these are a subset of the portfolios considered in Fama and French (2012) and ensure that U.S. and international portfolios are obtained through the same sorting schemes.[7] Notice that, unlike higher frequency returns such as daily or weakly, lower frequency monthly returns mitigate the effect on our comovement measure that may be induced by the fact that markets in different countries may be open at different times. All returns are in U.S. dollars and are computed in excess of the U.S. risk-free rate at the end of each month: this is defined as the 1-month U.S. treasury bill rate and reflects the cost of short term funding in the U.S. dollar market. Therefore, we conduct our analysis from the perspective of a U.S. investor. As a measure of U.S. macroeconomic uncertainty we take the one month ahead index developed in Jurado, Ludvigson, and Ng (2015) and available from Sydney Ludvigson website, which we denote by $\mathbb{U}_t^{\mathrm{M}}$.[8]

Given the theoretical model in (1), we let $g = 1$ and $g = 2$ denote the groups of U.S. and international portfolios, respectively: x_{1i_1t} is the excess return on the U.S. portfolio i_1 at time t;

[6]See `https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html` .

[7]International (ex-U.S.) portfolios are formed from the set of individual stock returns from the following 22 countries: Australia, Austria, Belgium, Canada, Switzerland, Germany, Denmark, Spain, Finland, France, Great Britain, Greece, Hong Kong, Ireland, Italy, Japan, Netherlands, Norway, New Zealand, Portugal, Sweden, Singapore.

[8]U.S. macroeconomic uncertainty data are kindly made available at `https://www.sydneyludvigson.com/data-and-appendixes` .

similarly, x_{2i_2t} is the excess return on the international portfolio i_2 at time t. The threshold variable is defined as $z_t = \mathbb{U}_t^{\mathrm{M}}$. Our empirical model thus allows us to measure changes in systematic comovement induced by contemporaneous values of U.S. macroeconomic uncertainty.

6.2 Estimation and Model Selection

We apply to each individual group the test proposed in Massacci (2020) to detect threshold-type regime changes in the loadings: the test is robust to factor heteroskedasticity, a key feature of factor models applied to financial returns as pointed out in Baele, Bekaert, and Inghelbrecht (2010).[9] In both U.S. and international portfolios, the null hypothesis is rejected at the 1% significance level: this strong rejection of the null provides evidence in favor of regime changes in the loadings.[10]

We estimate the model as detailed in Section 3.1. In order to span the true factor space, for $g = 1, 2$ we estimate θ_g^0 by imposing an upper bound $K_g^{\max}$ on the number of factors in each panel within each regime such that $K_g^{\max}$ is greater than or equal to the true number of factors. We set $K_1^{\max} = K_2^{\max} = 10$: following Fama and French (2018) and Fama and French (2012), this is greater than the number of factors expected to drive the cross-sectional variation in U.S. and international equity returns, respectively. The estimated threshold values are identical to each other and are equal to $\hat{\theta}_1 = \hat{\theta}_2 = 0.674$, which corresponds to the 77^{th}-percentile of the empirical distribution of $\mathbb{U}_t^{\mathrm{M}}$: this is illustrated in Figure 2, which displays $\hat{\theta}_1 = \hat{\theta}_2 = \hat{\theta}$, the time series of $\mathbb{U}_t^{\mathrm{M}}$, and gray bars denoting the event $\mathbb{U}_t^{\mathrm{M}} > \hat{\theta}$.

Notice that we obtain identical values for $\hat{\theta}_1$ and $\hat{\theta}_2$ without imposing any restriction on the estimation procedure, as we separately estimate the models for U.S. and international equity returns. This result implies that, according to our model, regimes changes in the cross-sectional variation of U.S. and international returns that are induced by U.S. macroeconomic uncertainty are perfectly synchronized. It also implies that our empirical model identifies two regimes as a whole: this is perfectly consistent with the description of the regimes provided in Section 2.1.1.[11] Given $\hat{\theta}_1$ and $\hat{\theta}_2$, we estimate the number of factors driving the cross-sectional variations of returns. Within each group our model lets the number of factors change between the regimes and we adopt the

[9] We conduct the test on demeaned variables. The test requires an estimate of the number of factors in the linear model under the null. We estimate this by applying the IC_{p2} information criterion of Bai and Ng (2002), for which we impose an upper bound equal to 10 factors in both panels. The results of the test are unaffected by the choice of the maximum number of factors. As for the auxiliary threshold regression needed to implement the test, we use an equal-weighting scheme to obtain cross-sectional averages of U.S. and international returns. Finally, we approximate the asymptotic distribution of the test statistic using the fixed regressor bootstrap of Hansen (1996) with 1000 replications.

[10] Additional details about inference on the number of regimes are available upon request.

[11] Following Theorem 3.4 in Massacci (2017), the estimator for θ_g^0 is T-consistent, for $g = 1, 2$, which prevents us from using standard inferential procedures to test the null hypothesis $\theta_1^0 = \theta_2^0$.

Figure 2 – Time series of Macroeconomic Uncertainty Index and estimated thresholds

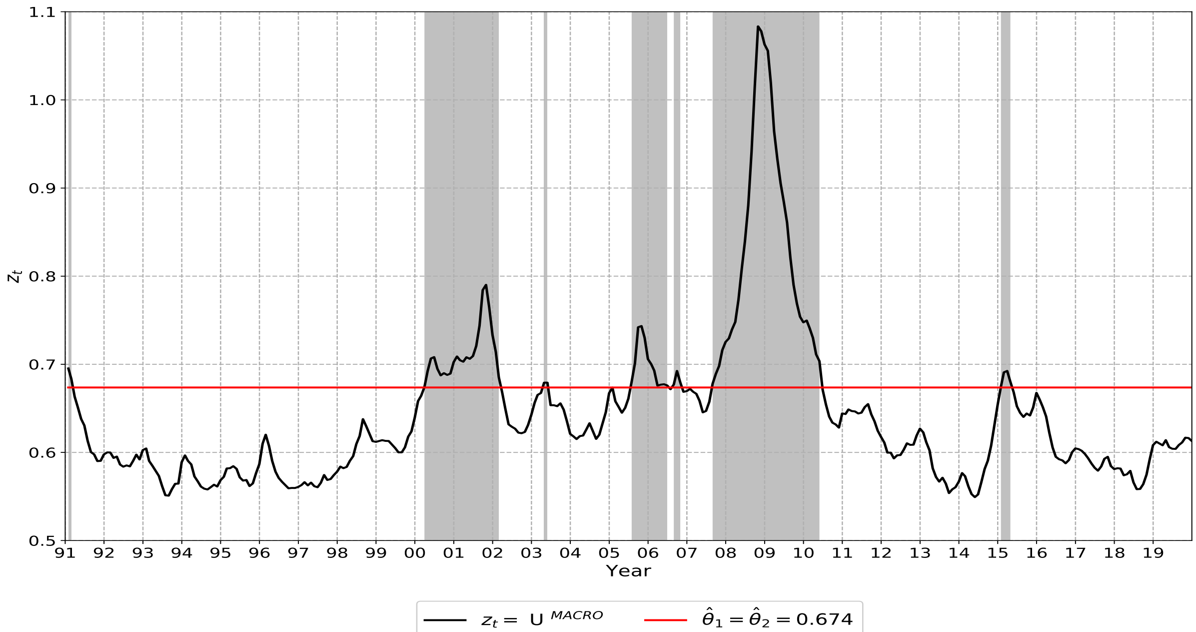


Figure 2 displays the end-of-month values of the macroeconomic uncertainty index ($\mathbb{U}^{\mathrm{M}}$) constructed by Jurado, Ludvigson, and Ng (2015) and used as threshold variable (z_t) in the threshold group-factor model. The sample period is January 1991 to December 2019. The red line corresponds to the estimated threshold value $\hat{\theta}_1 = \hat{\theta}_2 = 0.674$ for the panel of monthly U.S. and international equity excess returns.

following two-step approach. For $g = 1, 2$, we first estimate the factors in each regime using the $IC_{p2}\left(K_{gj_g}, K_{gj_g}\right)$ criterion of Massacci (2017). This suitably extends the IC_{p2} selection criterion of Bai and Ng (2002) by making it robust to the threshold effect in the factor loadings: it gives a consistent estimator for the number of factors if this does not change between the regimes; and it provides an upper bound to the number of factors when this varies between the regimes since the factor space needs to be fully spanned for the criterion to be minimized. In the second step we estimate the number of factors by applying the IC_{p2} selection criterion of Bai and Ng (2002) within each group and regime and taking as an upper bound the number factors estimated through the $IC_{p2}\left(K_{gj_g}, K_{gj_g}\right)$ criterion of Massacci (2017). Following this strategy we estimate 5 factors within each regime and group: formally, this means that $\hat{K}_{gj_g} = 5$, for $g = 1, 2$ and $j_g = L, H$. Therefore, the number of estimated factors does not change across regimes and groups; it is also important to stress that this result is a genuine feature of the data and it is not obtained by imposing any a priori restriction, since estimation of $\hat{K}_{gj_g}$ is run independently for each group and regime.

6.3 Systematic Comovement

We now study how global equity market comovement changes between the periods of low and high U.S. macroeconomic uncertainty identified by our model, namely $\mathbb{U}_t^{\mathrm{M}} \leq \hat{\theta}$ and $\mathbb{U}_t^{\mathrm{M}} > \hat{\theta}$, respectively. Given the interpretation of systematic comovement in Section 2.2.4, we focus on the average systematic correlation $R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ defined in (10): for specific combinations of assets i_1 and i_2 belonging to U.S. and international portfolios, respectively, this reduces to the systematic correlation $R^0_{j_1 j_2 i_1 i_2}$ defined in (9). Since regimes are synchronized across groups then $j_1 = j_2$.

We first look at the correlation matrix of excess returns between the two groups in low and high uncertainty regimes: these are displayed in the top-left and top-right panels of Figure 3, respectively. During periods of high uncertainty, the average sample correlation between U.S. and international returns is higher than in periods of low uncertainty and it is equal to 0.707 and 0.543, respectively.

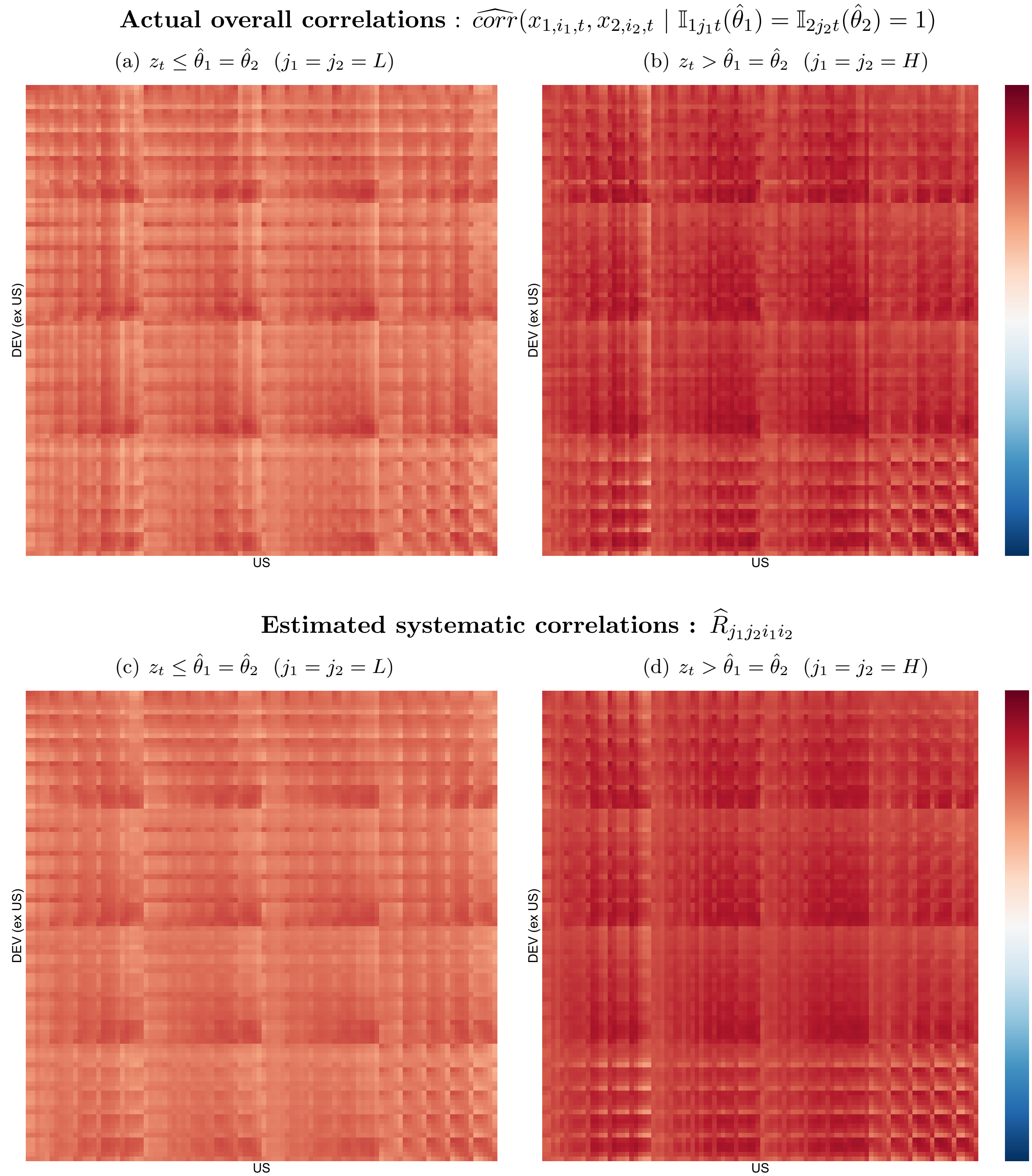
We then investigate to what extent this increase in correlation is due to systematic comovement. To this purpose, we estimate $R^0_{j_1 j_2 i_1 i_2}$ through $\hat{R}_{j_1 j_2 i_1 i_2}$ defined in (17): this is displayed in the bottom-left and bottom-right panels of Figure 3, respectively. A visual inspection shows that systematic comovement increases during periods of high uncertainty. This can be more clearly seen from the top panel of Figure 4, which displays the difference $(\hat{R}_{HHi_1i_2} - \hat{R}_{LLi_1i_2})$ for each pair of returns i_1 and i_2: all the $100 \times 100 = 10,000$ entries are positive, with the exception of only 10 of them. These results are confirmed by running the inferential procedure detailed in Theorem 5: in the bottom panel of Figure 4, 80.14% and 57.04% of the differences $(\hat{R}_{HHi_1i_2} - \hat{R}_{LLi_1i_2})$ are greater than zero in a one-sided test with significance level of 10% and 5%, respectively. We also apply the test in Theorem 5 to $\hat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ defined in (20): the equal-weighted average difference $\hat{R}_{HH\boldsymbol{w}_1\boldsymbol{w}_2} - \hat{R}_{LL\boldsymbol{w}_1\boldsymbol{w}_2} = 0.165$ results in a test statistic $\widehat{\mathcal{T}}^R_{HHLL\boldsymbol{w}_1\boldsymbol{w}_2} = 2.431$, and the one-sided test for the null hypothesis on no positive change in correlation is significant at 1% level.

We can thus conclude that our results clearly show that systematic comovement in global equity markets increases during periods of high U.S. macroeconomic uncertainty.

6.4 Discussion

Our empirical results may be interpreted through the lens of the global financial cycle (GFC) discussed in Rey (2018), and Miranda-Agrippino and Rey (2020). Figure 5 plots the cumulated values of the first factor, that is the first principal component (PC), in group 1, and of the first factor in group 2.

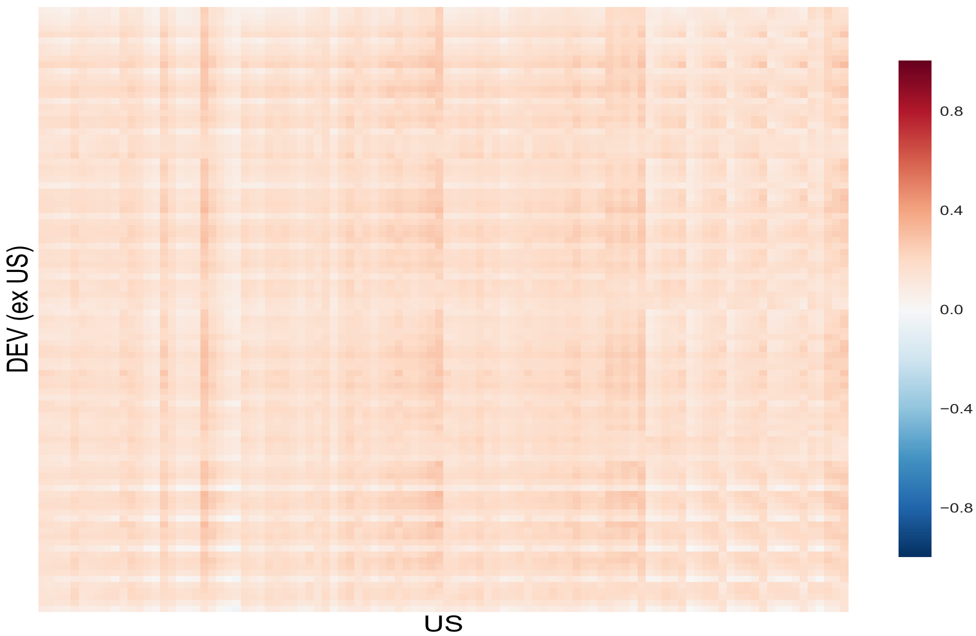
Figure 3 – Actual and systematic correlations across panels and in different regimes



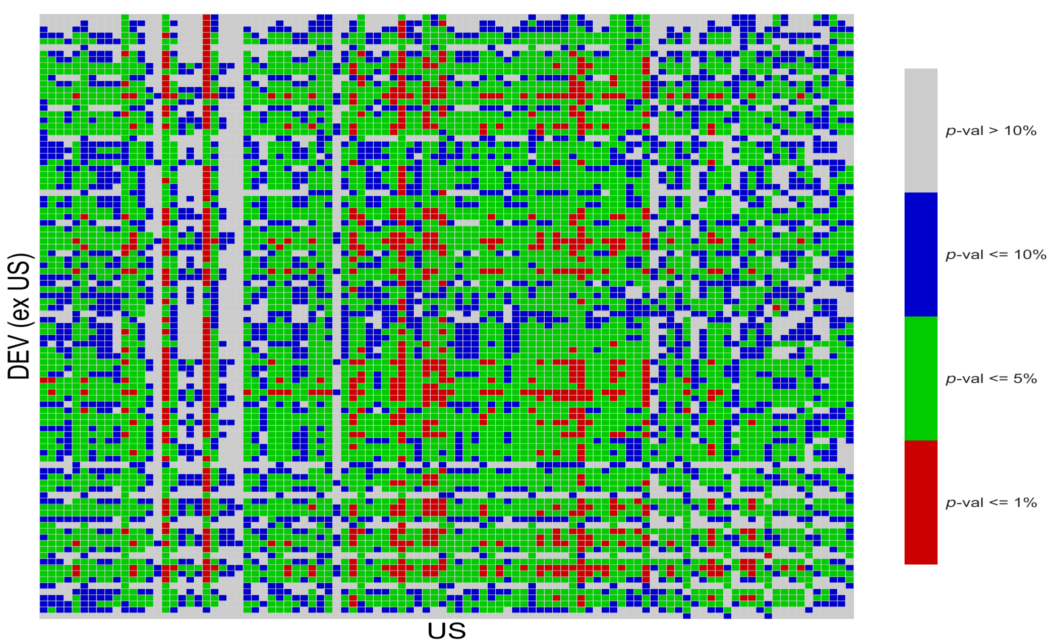
The figure displays the actual overall and estimated systematic correlations across panels of monthly excess returns on the U.S. (Group 1) and the international (Group 2) equity portfolios when $z_t \leq \hat{\theta}_1 = \hat{\theta}_2$, that is for all dates that $\mathbb{U}^{\mathrm{M}} \leq 0.674$, and when $z_t > \hat{\theta}_1 = \hat{\theta}_2$, that is for all dates that $\mathbb{U}^{\mathrm{M}} > 0.674$.

Figure 4 – Difference of systematic correlations across panels between the two regimes

(a) $\hat{R}_{HHi_1i_2} - \hat{R}_{LLi_1i_2}$



(b) $\hat{R}_{HHi_1i_2} - \hat{R}_{LLi_1i_2}$: p-value of signif. test



Panel (a) displays the difference, between high and low Macroeconomic Uncertainty regimes, of the estimated systematic correlations computed for each pair of returns i_1 and i_2, with i_1 from the panel of U.S. portfolios and i_2 from the panel of international (ex U.S.) portfolios. Panel (b) displays the p-value of the one-sided test of significance for each of the individual differences displayed in Panel (a). A gray square indicates a p-value $> 10\%$, a blue square indicates a p-value between 5% and 10%, a green square indicates a p-value between 1% and 5%, and a red square indicates a p-value $< 1\%$. In both panels, each column corresponds to a U.S. portfolio, while each row corresponds to an international (ex U.S.) portfolio.

As we have two regimes and all PCs are estimated separately for each group and regime, we need to choose a suitable standardization of either factors or loadings, and their signs, in order to represent them graphically.[12]

For all groups and regimes we impose the estimated loading matrices to be such that: $N_g^{-1}\left(\hat{\mathbf{\Lambda}}'_{gj_g}\hat{\mathbf{\Lambda}}_{gj_g}\right) = I_{K_g}$, therefore we standardize the scale of the loadings across all groups, regimes and factors. Moreover, for all dates in regime L, the sign of the first PC in group 1 (resp. group 2) is fixed such that the majority of the loadings of the returns in group 1 (resp. group 2) with respect to the PC are positive. The same procedure is used to compute the factors in each group for all dates in regime H.[13] From Figure E.5 in the SM we notice that all the loadings are positive in each regime and group, so the first factors can be interpreted as "Market factors" for the two geographical regions. For each group, we construct the unique time series of demeaned excess returns of the two factors, and we show the cumulated version of these two time series starting from the value 0 in December 1990. As the original non-cumulated factors are estimated from a panel of demeaned (within each regime) excess returns of individual assets, they are themselves (linear combinations of) demeaned excess returns, and thus have zero mean within each regime. Therefore, the cumulated factors can be interpreted as cyclical variations along long-run factor-specific trend, and no long-run linear trend appears when plotting them. To ease factor interpretation, Figure 5 also shows the dynamics of the Miranda-Agrippino and Rey (2020)'s GFC factor estimated from world-wide cross section of risky asset prices.[14]

The two groups of equity returns we consider follow a group-specific cycle, which is highly correlated, although not perfectly synchronized, with the global financial cycle.[15] This provides the empirical foundation for our analysis based on a group-factor model. Our empirical specification thus sheds light on the dynamics of systematic comovement as induced by changes in U.S. macroeconomics uncertainty for two sets of equity portfolio returns that follow group-specific cycles. Our

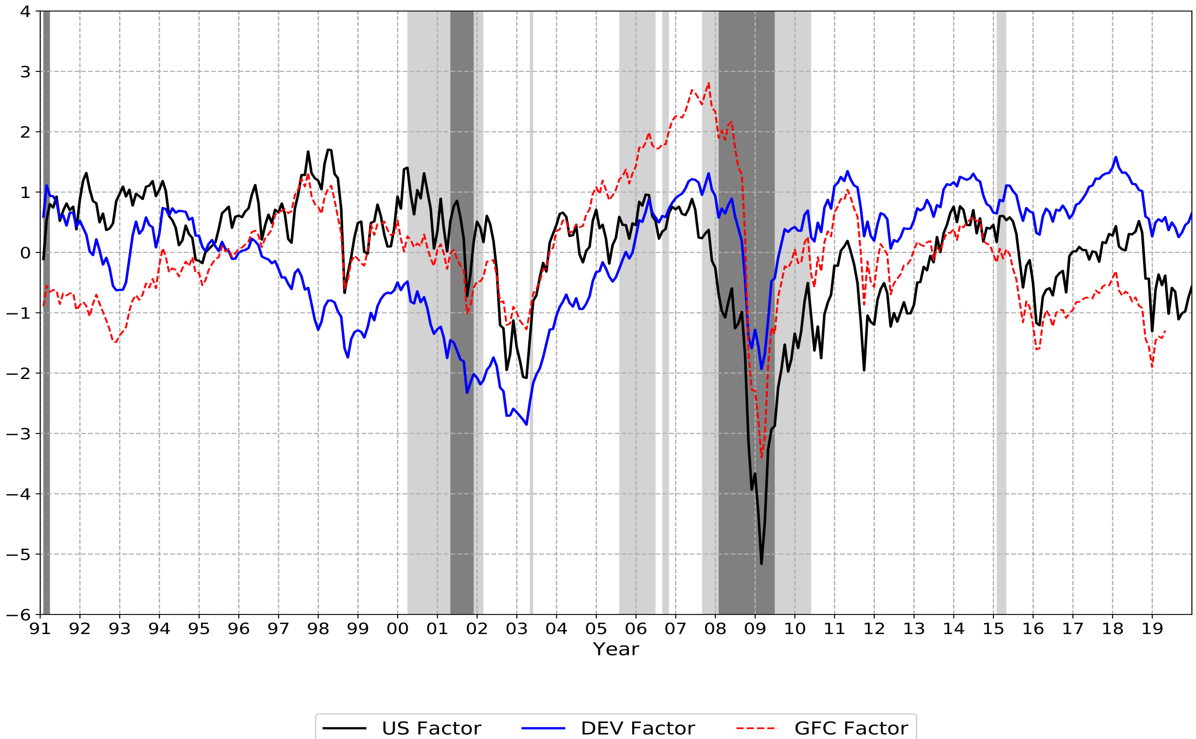
[12]These standardizations have no effect on the fit of the model, and on the values of the common components and of our measures of systematic comovement.

[13]Our standardization also implies that large (resp. small) changes in absolute value of the factor in a date correspond to average large changes (resp. small) across most of the excess returns of individual assets in each group. Moreover, the order of magnitude of these effects are comparable across regimes and groups simply by looking at trajectories of the factors themselves.

[14]More specifically, we show the global common factor of Miranda-Agrippino and Rey (2020) as extended by Miranda-Agrippino, Nenova, and Rey (2020) to cover the time period up to April 2019 and to reflect a larger and richer set of price series and compositional changes in global markets with the inclusion of Chinese stocks. Data have been downloaded from Silvia Miranda-Agrippino's website `http://silviamirandaagrippino.com/code-data/`.

[15]The correlation of the first U.S. (resp. International) not-cumulated factor with the non-cumulated global financial cycle of Miranda-Agrippino and Rey (2020) is 0.80 (resp. 0.86).

Figure 5 – Cumulative returns of First PCs in both panels



This figure plots the estimates of the cumulated global factor of Miranda-Agrippino and Rey (2020) together with those of the cumulated first factor for U.S. and international equity markets obtained according to the empirical specification detailed in Section 6.1. All cumulated factors have been standardized to have zero mean and unit variance to ease their comparison. Light gray areas correspond to the high macroeconomic uncertainty regimes, while the dark gray areas correspond to the NBER recession rates. All the NBER recessions coincide also with high macroeconomic uncertainty regimes, but the opposite is not true.

results show that, within those fluctuations, the two underlying factor models experience synchronized changed in the loadings that allow to identify variations in systematic comovement between the two groups: these are such that both pairwise and average systematic comovement are higher during periods of high macroeconomic uncertainty.

Our results have implications for investors and policy makers. From a portfolio choice perspective, our findings suggest that the benefits from global diversification are not constant over time and become weaker during periods of high U.S. macroeconomic uncertainty; they also imply that investors may have to rebalance their portfolios depending on how macroeconomic uncertainty affects factor loadings.[16] As shown in Ang and Chen (2002), correlations between asset returns tend to increase during market downturns: our results thus imply that periods of potential market instability are associated with high U.S. macroeconomic uncertainty. This suggests that regulators

[16]See Lehmann and Modest (2005) for a discussion on the link between factor loadings and portfolio weights in latent factor models.

may find it useful to track this indicator to monitor stability in global financial markets.

7 Conclusions

We develop measures of pairwise and average systematic comovement for high dimensional approximate group-factor models. We propose consistent estimators for these measures and analytically derive their asymptotic distributions. We further build formal procedures to test for changes in systematic comovement induced by threshold-type discrete shifts in the factor loadings. A comprehensive Monte Carlo analysis shows the good finite sample properties of our estimators and test statistics. An empirical analysis of U.S. and international large equity portfolios shows that measures of both pairwise and average comovement between the groups increase during periods of high U.S. macroeconomic uncertainty over the course of the global financial cycle.

Our work can be extended along several directions. On the methodological side, it would be interesting to have continuous variation in the factor loadings to allow for corresponding continuous dynamics in systematic comovement. Our framework is valid under the maintained assumption that the two groups are made of balanced panels: the case of unbalanced panels is definitely worth exploring. From an empirical perspective, our results may be useful to shed light on the systematic degree of, and comovement between, different classes of financial variables, which could provide valuable information for macroprudential policy implementation. These issues are high in our research agenda.

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Appendices

We use the following notation. Let $\|A\|=\sqrt{tr(A'A)}$ denote the Frobenius norm of matrix A.

A Assumptions

Assumption A.1. *For $g=1,2$ and $j_g=L,H$:* $\mathrm{E}[\mathbb{I}_{gj_gt}(\theta_g^0)\boldsymbol{f}_{gj_gt}^0]=0$*; for all θ_g and some $K_{gj_g}^0\times K_{gj_g}^0$ positive definite matrix $\boldsymbol{\Sigma}_{\boldsymbol{f}gj_g}^0(\theta_g)$, $T^{-1}\sum_{t=1}^{T}\mathbb{I}_{gj_gt}(\theta_g^0)\mathbb{I}_{gj_gt}(\theta_g)\boldsymbol{f}_{gj_gt}^0\boldsymbol{f}_{gj_gt}^{0\prime}\xrightarrow{p}\boldsymbol{\Sigma}_{\boldsymbol{f}gj_g}^0(\theta_g)$ as $T\to\infty$;* $\mathrm{E}\left\|\boldsymbol{f}_{gj_gt}^0\right\|^4<\infty$.

Assumption A.2. *For $g=1,2$, $j_g=L,H$ and $i_g=1,\ldots,N_g$, $\left\|\boldsymbol{\lambda}_{gj_gi_g}^0\right\|\leq\bar{\lambda}<\infty$ and $\left\|N_g^{-1}\boldsymbol{\Lambda}_{gj_g}^{0\prime}\boldsymbol{\Lambda}_{gj_g}^0-\boldsymbol{D}_{\boldsymbol{\Lambda}gj_g}^0\right\|\to 0$ as $N_g\to\infty$ for some $K_{gj_g}^0\times K_{gj_g}^0$ positive definite matrix $\boldsymbol{D}_{\boldsymbol{\Lambda}gj_g}^0$.*

Assumption A.3. *There exists a positive $M<\infty$ such that for $g=1,2$, $j_g=L,H$, for all θ_g and for all (N_g,T),*

1. $\mathrm{E}\left(e_{gi_gt}\right)=0$ *and* $\mathrm{E}\left|e_{gi_gt}\right|^8\leq M$*;*

2. $\mathrm{E}\left[\mathbb{I}_{gj_gt}(\theta_g)\mathbb{I}_{gj_gv}(\theta_g)e_{gi_gt}e_{gi_gv}\right]=\tau_{gj_gi_gtv}(\theta_g)$ *with $\left|\tau_{gj_gi_gtv}(\theta_g)\right|\leq\left|\tau_{gj_gi_gtv}\right|$ for some $\tau_{gj_gi_gtv}$ and for all i_g, and $T^{-1}\sum_{t=1}^{T}\sum_{v=1}^{T}\left|\tau_{gj_gi_gtv}\right|\leq M$.*

3. $\mathrm{E}\left[T^{-1}\sum_{t=1}^{T}\mathbb{I}_{gj_gt}(\theta_g)e_{gi_gt}e_{gl_gt}\right]=\sigma_{gj_gi_gl_g}(\theta_g)$, *$\left|\sigma_{gj_gl_gl_g}(\theta_g)\right|\leq M$ for all l_g, and* $N_g^{-1}\sum_{i_g=1}^{N_g}\sum_{l_g=1}^{N_g}\left|\sigma_{gj_gl_gl_g}(\theta_g)\right|\leq M$*;*

4. $\mathrm{E}\left|T^{-1/2}\sum_{t=1}^{T}\mathbb{I}_{gj_gt}(\theta_g)e_{gi_gt}e_{gl_gt}-\mathrm{E}\left[\mathbb{I}_{gj_gt}(\theta_g)e_{gi_gt}e_{gl_gt}\right]\right|^4\leq M$*, for every (i_g,l_g).*

Assumption A.4. *There exists some positive constant $M<\infty$ such that for all θ_g and for all (N_g,T),*

$$\mathrm{E}\left\{N_g^{-1}\sum_{i_g}^{N_g}\left\|T^{-1/2}\left[\sum_{t=1}^{T}\mathbb{I}_{gj_gt}(\theta_g)\boldsymbol{f}_{gj_gt}^0e_{gi_gt}\right]\right\|^2\right\}<M,\quad g=1,2,\quad j_g=L,H.$$

Assumptions A.1-A.4 are the natural extensions of Assumptions A-D in Bai and Ng (2002) and ensure consistency of the least squares estimator as applied to the model in (1).

Assumption A.5. *For $g=1,2$ and $i_g=1,\ldots,\lfloor N_g^{\alpha_g^0}\rfloor$ with $0.5<\alpha_g^0\leq 1$, $\boldsymbol{\lambda}_{gHi_g}^0\neq\boldsymbol{L}\boldsymbol{\lambda}_{gLi_g}^0$ for any $K_{gL}^0\times K_{gL}^0$ full rank matrix $\boldsymbol{L}$.*

For $g = 1, 2$, Assumption A.5 is sufficient to identify the threshold factor model from a linear specification. Following Massacci (2020), it requires that enough cross-sectional units experience a synchronized threshold effects within each group. If this condition fails to hold, the principal components estimator as applied to the misspecified linear model would achieve the convergence rate $min(\sqrt{N_g}, \sqrt{T})$ derived in Bai and Ng (2002) for correctly specified linear factor models and the threshold effect would not be identified. Assumption A.5 is trivially satisfied if $K^0_{gL} \neq K^0_{gH}$: in this case the change in the number of factors identifies the regime shift. If $K^0_{gL} = K^0_{gH}$ then Assumption A.5 holds provided that $\boldsymbol{\lambda}_{gHi_g}$ is not obtained as the rotation of $\boldsymbol{\lambda}_{gLi_g}$ induced by the same matrix $\boldsymbol{L}$, for $i_g = 1, \ldots, \lfloor N_g^{\alpha_g^0} \rfloor$: if this is not the case, the model becomes observationally equivalent to a linear model with a regime change in the covariance structure of the true factors, which is consistent with Assumption A in Bai and Ng (2002). Assumption A.5 orders the cross-sectional units for expositional purposes only: this condition can be relaxed without any consequence.

For $g = 1, 2$ and $j_g = L, H$, define $\delta^0_{\boldsymbol{f}gi_g}(\theta_g) = \mathrm{E}\left[\left(\boldsymbol{f}^{0\prime}_{gHt}\boldsymbol{\lambda}^0_{2Hi_g} - \boldsymbol{f}^{0\prime}_{gLt}\boldsymbol{\lambda}^0_{1Li_g}\right)^2 | z_{gt} = \theta_g\right]$ and let $f_{\mathcal{Z}g}(z_{gt})$ be the density function of z_{gt}.

Assumption A.6. *For $g = 1, 2$ and $j_g = L, H$:*

1. $\left\{\boldsymbol{f}^0_{gLt}, \boldsymbol{f}^0_{gHt}, z_{gt}, \boldsymbol{e}_{gt}\right\}_{t=1}^{T}$ *is strictly stationary, ergodic and $\rho-$mixing, with $\rho-$mixing coefficients satisfying* $\sum_{m=1}^{\infty} \rho_{gm}^{1/2} < \infty$;
2. *for all θ_g,* $\mathrm{E}\left(\left\|\boldsymbol{f}^0_{gj_gt}e_{i_gt}\right\|^4 | z_{gt} = \theta_g\right) \leq C$ *and* $\mathrm{E}\left(\left\|\boldsymbol{f}^0_{gj_gt}\right\|^4 | z_{gt} = \theta_g\right) \leq C$ *for some $C < \infty$ and for $i_g = 1, \ldots, N_g$, and* $f_{\mathcal{Z}g}(\theta_g) \leq \bar{f} < \infty$;
3. $\delta^0_{\boldsymbol{f}gi_g}(\theta_g)$ *and* $f_{\mathcal{Z}g}(z_{gt})$ *are continuous at* $\theta_g = \theta^0_g$;
4. $\delta^0_{\boldsymbol{f}gi_g}(\theta_g) > 0$, *for* $i_g = 1, \ldots, \lfloor N^{\alpha_g^0} \rfloor$ *and* $0.5 < \alpha^0_g \leq 1$; $f_{\mathcal{Z}g}(z_{gt}) > 0$ *for all* θ_g.

Assumption A.6 suitably extends Assumption 1 in Hansen (2000), to which we refer to for further comments: it is required to derive the convergence rates of the estimators for θ^0_1 and θ^0_2 and thus of the principal components estimators for factors and loadings.

Assumption A.7. *For $g = 1, 2$, $j_g = L, H$, and $t = 1, \ldots, T$, as $N_g \to \infty$,*

$$\frac{1}{\sqrt{N_g}} \sum_{i_g=1}^{N_g} \mathbb{I}_{gj_gt}(\theta^0_g)\boldsymbol{\lambda}_{gj_gi_g} e_{gi_gt} \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Gamma}^0_{gj_gt}\right),$$

where $\boldsymbol{\Gamma}^0_{gj_gt} = \lim_{N_g \to \infty} \frac{1}{N_g} \sum_{i_g=1}^{N_g} \sum_{l_g=1}^{N_g} \boldsymbol{\lambda}^0_{gj_gi_g} \boldsymbol{\lambda}^{0\prime}_{gj_gl_g} \mathrm{E}\left[\mathbb{I}_{gj_gt}(\theta^0_g) e_{gi_gt} e_{gl_gt}\right]$.

Assumption A.8. *For $g = 1, 2$, $j_g = L, H$, and $i = 1, \dots, N_g$, as $T \to \infty$,*

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \mathbb{I}_{gj_gt}(\theta_g^0) \boldsymbol{f}_{gj_gt} e_{gi_gt} \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Omega}_{gj_gi_g}^0\right),$$

where

$$\boldsymbol{\Omega}_{gj_gi_g}^0 = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{v=1}^{T} \mathrm{E}\left[\mathbb{I}_{gj_gt}(\theta_g^0) \mathbb{I}_{gj_gv}(\theta_g^0) \boldsymbol{f}_{gj_gt}^0 \boldsymbol{f}_{gj_gv}^{0\prime} e_{gi_gt} e_{gi_gv}\right].$$

Assumptions A.7 and A.8 are analogous to the central limit theorems in Assumption F in Bai (2003): they allow to obtain the asymptotic distribution of the principal components estimators for factors and loadings.

Assumption A.9. *For $g = 1, 2$, $i_g = 1, \dots, N_g$, and $t, v = 1, \dots, T$, the idiosyncratic components e_{1i_1t} and e_{2i_2v} are mutually independent.*

Assumption A.10. *For $g = 1, 2$, and $j_g = L, H$, $T^{-1} \sum_{t=1}^{T} \mathbb{I}_{1j_1t}(\theta_1^0) \mathbb{I}_{2j_2t}(\theta_2^0) \boldsymbol{f}_{1j_1t}^0 \boldsymbol{f}_{2j_2t}^{0\prime} \xrightarrow{p} \boldsymbol{\Sigma}_{\boldsymbol{f}12j_1j_2}^0$ as $T \to \infty$, for some $K_{1j_1}^0 \times K_{2j_2}^0$ matrix $\boldsymbol{\Sigma}_{\boldsymbol{f}12j_1j_2}^0$.*

Assumptions A.9 and A.10 impose conditions on the idiosyncratic components and the factors, respectively, that hold across the two groups.

Assumption A.11. *The sequence of weights $\{w_{gi_g}\}_{i_g=1,\dots,N_g}$ is such that $\lim_{N_g\to\infty} \sum_{i_g=1}^{N_g} |w_{gi_g}| \leq C$, where C is a strictly positive finite constant, for $g = 1, 2$.*

Assumption A.11 ensures that the asymptotic variances of the systematic covariance and correlations, which are defined as averages with generic weights w_{gi_g} across individuals in the two groups, converge as $N_1, N_2 \to +\infty$. For both $g = 1, 2$, this assumption accommodates the two most interesting cases (i) $w_{gi_g} = 1/N_g$, $\forall i_g$, and (ii) $w_{gi_g} = 1$ when $i_g = i_g^*$, while $w_{gi_g} = 0$ $\forall i_g \neq i_g^*$. Note that to ensure the convergence of the variances of the systematic covariance and correlations it is not necessary to assume that $\sum_{i_g=1}^{N_g} w_{gi_g} = 1$.

Assumption A.12. *Factors $\boldsymbol{f}_{gj_gt}^0$ and idiosyncratic components $e_{g^*i_{g^*}v}$ are mutually independent for all $t, v = 1, \dots, T$, $j_g = L, H$, $g, g^* = 1, 2$.*

B Covariances estimators

In this section we provide consistent estimators of the asymptotic variance-covariance matrices appearing in Theorems 1 - 5.

B.1 Estimators for Theorem 1

Similarly to Bai (2003), our Assumption A.9 implies cross-sectional independence of the errors within each group $g = 1, 2$ and in each regime j_g. In this case, matrix $\mathbf{\Gamma}_{gj_gt}$ defined in Assumption A.7 simplifies to: $\mathbf{\Gamma}^0_{gj_gt} = \lim_{N_g \to \infty} \frac{1}{N_g} \sum_{i_g=1}^{N_g} \boldsymbol{\lambda}^0_{gj_gi_g} \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \mathrm{E}\left[\mathbb{I}_{gj_gt}(\theta^0_g) e^2_{gi_gt}\right]$. Let $\hat{e}_{g,i_gt}$ be residual estimated at time t for individual i_g in group g as:

$$\hat{e}_{g,i_gt} \;=\; \mathbb{I}_{gj_gt}(\hat{\theta}_g)(x_{g,i_g,t} - \hat{c}_{g,i_gt}), \qquad t = 1, ..., T, \qquad i_g = 1, .., N_g, \qquad g = 1, 2.$$

The estimator of $\mathbf{\Gamma}^0_{gj_gt}$ is: $\hat{\mathbf{\Gamma}}_{gj_gt} \;=\; \frac{1}{N_g} \sum_{i_g=1}^{N_g} \hat{\boldsymbol{\lambda}}_{gj_gi_g} \hat{\boldsymbol{\lambda}}'_{gj_gi_g} \mathbb{I}_{gj_gt}(\hat{\theta}_g) \hat{e}^2_{gi_gt}$, and the estimator of $\boldsymbol{V}^0_{gj_gi_gt}$ is:

$$\hat{\boldsymbol{V}}_{gj_gi_gt} = \hat{\boldsymbol{\lambda}}'_{gj_gi_g} \left(\frac{1}{N_g} \sum_{i_g=1}^{N_g} \hat{\boldsymbol{\lambda}}_{gj_gi_g} \hat{\boldsymbol{\lambda}}'_{gj_gi_g} \right)^{-1} \hat{\mathbf{\Gamma}}_{gj_gt} \left(\frac{1}{N_g} \sum_{i_g=1}^{N_g} \hat{\boldsymbol{\lambda}}_{gj_gi_g} \hat{\boldsymbol{\lambda}}'_{gj_gi_g} \right)^{-1} \hat{\boldsymbol{\lambda}}_{gj_gi_g} \; .$$

The estimator of matrix $\mathbf{\Omega}_{gj_gi_g}$ is $\hat{\mathbf{\Omega}}_{gj_gi_g} = \hat{\boldsymbol{D}}_{j_gi_g,0} + \sum_{v=1}^{q} \left(1 - \frac{v}{q+1}\right) \left(\hat{\boldsymbol{D}}_{j_gi_g,v} + \hat{\boldsymbol{D}}'_{j_gi_g,v}\right)$, where:

$$\hat{\boldsymbol{D}}_{j_gi_g,v} \;=\; \frac{1}{T} \sum_{t=v+1}^{T} \mathbb{I}_{gj_gt}(\hat{\theta}_g) \mathbb{I}_{gj_gt-v}(\hat{\theta}_g) \hat{\boldsymbol{f}}_{g,t} \hat{\boldsymbol{f}}'_{g,t-v} \hat{e}_{g,i_gt} \hat{e}_{g,i_gt-v} \; .$$

This estimator is a Newey-West estimator computed only in the dates characterized by the regime j_g, and the role of the indicator function is to exclude from the computation all the dates which are not in regime j_g. Then, the estimator of $\boldsymbol{W}^0_{gj_gi_gt}$ is:

$$\hat{\boldsymbol{W}}_{gj_gi_gt} \;=\; \mathbb{I}_{gj_gt}(\hat{\theta}_g) \hat{\boldsymbol{f}}'_{gt} \left(\hat{\mathbf{\Sigma}}_{\mathbf{f}gj_1j_2}\right)^{-1} \hat{\mathbf{\Omega}}_{gj_gi_g} \left(\hat{\mathbf{\Sigma}}_{\mathbf{f}gj_1j_2}\right)^{-1} \hat{\boldsymbol{f}}_{gt} \; ,$$

where $\hat{\mathbf{\Sigma}}_{\mathbf{f}gj_g} := \frac{1}{T} \sum_{t=1}^{T} \mathbb{I}_{1j_gt}(\hat{\theta}_1) \mathbb{I}_{2j_2t}(\hat{\theta}_2) \widehat{\boldsymbol{f}}_{gt} \widehat{\boldsymbol{f}}'_{gt}$, with $g = 1,\ 2$, and the estimator of $\boldsymbol{Q}^0_{j_1j_2i_1i_2t}$ is:

$$\hat{\boldsymbol{Q}}_{j_1j_2i_1i_2t} = (\hat{c}_{1j_1i_1t})^2 \left(\hat{\mu}^2_{N_2} \hat{\boldsymbol{V}}_{2j_2i_2t} + \hat{\mu}^2_T \hat{\boldsymbol{W}}^0_{2j_2i_2t}\right) + (\hat{c}_{2j_2i_2t})^2 \left(\hat{\mu}^2_{N_1} \hat{\boldsymbol{V}}_{1j_1i_1t} + \mu^2_T \hat{\boldsymbol{W}}_{1j_1i_1t}\right) \; ,$$

with $\hat{\mu}_{N_g} = C_{NT}/\sqrt{N_g}$, and $\hat{\mu}_T = C_{NT}/\sqrt{T}$. Finally, the estimator of the asymptotic variance $\boldsymbol{Q}^0_{i_1 i_2 t}$ is: $\hat{\boldsymbol{Q}}_{i_1 i_2 t} = \sum_{j_1=L,H} \sum_{j_2=L,H} \mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\hat{\boldsymbol{Q}}_{j_1 j_2 i_1 i_2 t}$.

B.2 Estimators for Theorem 2

The estimator of $\boldsymbol{Q}^0_{j_1 j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2)$ is:

$$\begin{aligned}\hat{\boldsymbol{Q}}_{j_1 j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2) \;&=\; \frac{T^2}{T_{j_1 j_2}(\hat{\theta}_1, \hat{\theta}_2) \cdot T_{j_1^* j_2^*}(\hat{\theta}_1, \hat{\theta}_2)} \cdot \Big[\hat{\boldsymbol{\Psi}}_{j_1 j_2, j_1 j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2) \\ &\quad + \hat{\boldsymbol{Q}}_{12 j_1 j_2, j_1 j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2; \boldsymbol{w}_1, \boldsymbol{w}_2) + \hat{\boldsymbol{Q}}_{21 j_1 j_2, j_1 j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2; \boldsymbol{w}_1, \boldsymbol{w}_2) \Big] \;, \qquad \text{(B.1)}\end{aligned}$$

where the generic terms $\hat{\boldsymbol{\Psi}}_{j_1 j_2, j_1^* j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2)$, $\hat{\boldsymbol{Q}}_{12 j_1 j_2, j_1^* j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2; \boldsymbol{w}_1^*, \boldsymbol{w}_2^*)$, and $\hat{\boldsymbol{Q}}_{21 j_1 j_2, j_1^* j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2; \boldsymbol{w}_1^*, \boldsymbol{w}_2^*)$ are computed for $(j_1, j_2) = (j_1^*, j_2^*)$ and $\boldsymbol{w}_g = \boldsymbol{w}_g^*$, $g = 1, 2$. The more general definitions of these three terms, allowing for generic regimes $(j_1, j_2), (j_2^*, j_1^*)$ and potentially different weights $\boldsymbol{w}_g, \boldsymbol{w}_g^*$, will prove convenient to simplify the formulas for the estimators used in Theorem 5. The three terms in the square brackets of the last equations are:

$$\hat{\boldsymbol{\Psi}}_{j_1 j_2, j_1^* j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2) \;=\; \hat{\boldsymbol{D}}_{j_1 j_2, j_1^* j_2^*, 0}(\boldsymbol{w}_1, \boldsymbol{w}_2) + \sum_{v=1}^{q} 2 \left(1 - \frac{v}{q+1}\right) \hat{\boldsymbol{D}}_{j_1 j_2, j_1^* j_2^*, v}(\boldsymbol{w}_1, \boldsymbol{w}_2) \qquad \text{(B.2)}$$

with:

$$\begin{aligned}&\hat{\boldsymbol{D}}_{j_1 j_2, j_1^* j_2^*, v}(\boldsymbol{w}_1, \boldsymbol{w}_2) \\ &= \frac{1}{T} \sum_{t=v+1}^{T} \mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\mathbb{I}_{1j_1^*t-v}(\hat{\theta}_1)\mathbb{I}_{2j_2^*t-v}(\hat{\theta}_2) \left(\hat{c}_{j_1 j_2 \boldsymbol{w}_1, \boldsymbol{w}_2 t} - \hat{\bar{c}}_{j_1 j_2 \boldsymbol{w}_1, \boldsymbol{w}_2}\right) \left(\hat{c}_{j_1^* j_2^* \boldsymbol{w}_1, \boldsymbol{w}_2 t-v} - \hat{\bar{c}}_{j_1^* j_2^* \boldsymbol{w}_1, \boldsymbol{w}_2}\right),\end{aligned}$$

for $v = 0, 1, ..., q$, $\hat{c}_{j_1 j_2 \boldsymbol{w}_1, \boldsymbol{w}_2 t} = \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} w_{1,i_1} w_{2,i_2} \mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\widehat{c}_{i_1 i_2 t}$, and $\hat{\bar{c}}_{j_1 j_2 \boldsymbol{w}_1, \boldsymbol{w}_2} = \frac{1}{T_{j_1 j_2}(\hat{\theta}_1, \hat{\theta}_2)} \sum_{t=1}^{T} \hat{c}_{j_1 j_2 \boldsymbol{w}_1, \boldsymbol{w}_2 t}$. Quantity $\hat{\boldsymbol{D}}_{j_1 j_2, j_1 j_2, v}(\boldsymbol{w}_1, \boldsymbol{w}_2)$ is the estimator of the asymptotic auto-covariance of order v the product of (averages of) common components $\hat{c}_{i_1 i_2 t}$ computed only for the dates corresponding to the regime j_1 and j_2. Importantly, the sample average $\hat{\bar{c}}_{j_1 j_2 \boldsymbol{w}_1, \boldsymbol{w}_2}$ is estimated only in the dates corresponding to the regime j_1, j_2. Moreover,

$$\hat{\boldsymbol{Q}}_{12 j_1 j_2, j_1^* j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2; \boldsymbol{w}_1^*, \boldsymbol{w}_2^*) = \hat{\boldsymbol{\lambda}}'_{1j_1}(\boldsymbol{w}_1)\hat{\boldsymbol{\Sigma}}_{\mathbf{f}12 j_1 j_2} \left(\hat{\boldsymbol{\Sigma}}_{\mathbf{f}2 j_2}\right)^{-1} \hat{\boldsymbol{\Omega}}_{2 j_2 j_2^*}(\boldsymbol{w}_2, \boldsymbol{w}_2^*) \left(\hat{\boldsymbol{\Sigma}}_{\mathbf{f}2 j_2^*}\right)^{-1} \hat{\boldsymbol{\Sigma}}'_{\mathbf{f}12 j_1^* j_2^*} \hat{\boldsymbol{\lambda}}_{1j_1^*}(\boldsymbol{w}_1^*), \qquad \text{(B.3)}$$

$$\hat{\boldsymbol{Q}}_{21j_1j_2,j_1^*j_2^*}(\boldsymbol{w}_1,\boldsymbol{w}_2;\boldsymbol{w}_1^*,\boldsymbol{w}_2^*)=\hat{\boldsymbol{\lambda}}'_{2j_2}(\boldsymbol{w}_2)\hat{\boldsymbol{\Sigma}}_{\mathbf{f}21j_1j_2}\left(\hat{\boldsymbol{\Sigma}}_{\mathbf{f}1j_1}\right)^{-1}\hat{\boldsymbol{\Omega}}_{1j_1j_1^*}(\boldsymbol{w}_1,\boldsymbol{w}_1^*)\left(\hat{\boldsymbol{\Sigma}}_{\mathbf{f}1j_1^*}\right)^{-1}\hat{\boldsymbol{\Sigma}}'_{\mathbf{f}21j_1j_2^*}\hat{\boldsymbol{\lambda}}_{2j_2^*}(\boldsymbol{w}_2^*),\tag{B.4}$$

where for $g=1,2$, $\hat{\boldsymbol{\lambda}}_{gj_g}(\boldsymbol{w}_g,\boldsymbol{w}_g^*):=\sum_{i_g=1}^{N_g}w_{gi_g}\hat{\boldsymbol{\lambda}}_{gj_gi_g}$, $\hat{\boldsymbol{\Sigma}}_{\mathbf{f}12j_1j_2}\ :=\ \frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\widehat{\boldsymbol{f}}_{1t}\widehat{\boldsymbol{f}}'_{2t}$,

$$\hat{\boldsymbol{\Omega}}_{gj_gj_g^*}(\boldsymbol{w}_g,\boldsymbol{w}_g^*)=\hat{\boldsymbol{D}}_{j_gj_g^*,0}(\boldsymbol{w}_g,\boldsymbol{w}_g^*)+\sum_{v=1}^{q}\left(1-\frac{v}{q+1}\right)\left[\hat{\boldsymbol{D}}_{j_gj_g^*,v}(\boldsymbol{w}_g,\boldsymbol{w}_g^*)+\hat{\boldsymbol{D}}'_{j_gj_g^*,v}(\boldsymbol{w}_g,\boldsymbol{w}_g^*)\right],$$

and $\hat{\boldsymbol{D}}_{j_gj_g^*,v}(\boldsymbol{w}_g,\boldsymbol{w}_g^*)=\frac{1}{T}\sum_{t=v+1}^{T}\mathbb{I}_{gj_gt}(\hat{\theta}_g)\mathbb{I}_{gj_g^*t-v}(\hat{\theta}_g)\hat{\boldsymbol{f}}_{g,t}\hat{\boldsymbol{f}}'_{g,t-v}\left(\sum_{i_g=1}^{N_g}\sum_{l_g=1}^{N_g}w_{g,i_g}w^*_{g,l_g}\hat{e}_{g,i_gt}\hat{e}_{g,l_gt-v}\right)$.

B.3 Estimators for Theorem 3

Let $\hat{\boldsymbol{w}}_{\sigma,gj_1j_2i_g}=\dfrac{w_{gi_g}}{\hat{\sigma}_{\boldsymbol{x}gj_1j_2i_g}}$. The estimator of $\boldsymbol{Q}^0_{R,j_1j_2}$ is

$$\begin{aligned}\hat{\boldsymbol{Q}}_{R,j_1j_2}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2},\hat{\boldsymbol{w}}_{\sigma,2j_1j_2})\ &=\ \left(\frac{T}{T_{j_1j_2}(\hat{\theta}_1,\hat{\theta}_2)}\right)^2\cdot\Big[\hat{\boldsymbol{\Psi}}_{R,j_1j_2,j_1j_2}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2},\hat{\boldsymbol{w}}_{\sigma,2j_1j_2};\hat{\boldsymbol{w}}_{\sigma,1j_1j_2},\hat{\boldsymbol{w}}_{\sigma,2j_1j_2})\\&+\hat{\boldsymbol{Q}}_{12j_1j_2,j_1j_2}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2},\hat{\boldsymbol{w}}_{\sigma,2j_1j_2};\hat{\boldsymbol{w}}_{\sigma,1j_1j_2},\hat{\boldsymbol{w}}_{\sigma,2j_1j_2})\\&+\hat{\boldsymbol{Q}}_{21j_1j_2,j_1j_2}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2},\hat{\boldsymbol{w}}_{\sigma,2j_1j_2};\hat{\boldsymbol{w}}_{\sigma,1j_1j_2},\hat{\boldsymbol{w}}_{\sigma,2j_1j_2})+2\ \hat{\boldsymbol{\Xi}}_{1,j_1j_2,j_1j_2}+2\ \hat{\boldsymbol{\Xi}}_{2,j_1j_2,j_1j_2}\Big],\end{aligned}\tag{B.5}$$

where the variables inside the square brackets are defined below for the generic couple of regimes $(j_1,j_2),(j_1^*,j_2^*)$, as it will prove convenient to simplify the formulas for the estimators used in Theorem 5. The terms inside the square brackets are:

$$\hat{\boldsymbol{\Psi}}_{R,j_1j_2,j_1^*j_2^*}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2},\hat{\boldsymbol{w}}_{\sigma,2j_1j_2};\hat{\boldsymbol{w}}_{\sigma,1j_1^*j_2^*},\hat{\boldsymbol{w}}_{\sigma,2j_1^*j_2^*})=\hat{\boldsymbol{D}}^*_{R,j_1j_2,j_1^*j_2^*,0}+\sum_{v=1}^{q}2\left(1-\frac{v}{q+1}\right)\hat{\boldsymbol{D}}^*_{R,j_1j_2,j_1^*j_2^*,v},$$

with:

$$\begin{aligned}\hat{\boldsymbol{D}}^*_{R,j_1j_2,j_1^*j_2^*,v}\ &=\ \frac{1}{T}\sum_{t=v+1}^{T}\mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\mathbb{I}_{1j_1^*t-v}(\hat{\theta}_1)\mathbb{I}_{2j_2^*t-v}(\hat{\theta}_2)\\&\times\left[\widehat{c}_{R,j_1j_2t}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2},\hat{\boldsymbol{w}}_{\sigma,2j_1j_2})-\widehat{\overline{c}}_{R,j_1j_2}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2},\hat{\boldsymbol{w}}_{\sigma,2j_1j_2})\right]\\&\times\left[\widehat{c}_{R,j_1^*j_2^*t-v}(\hat{\boldsymbol{w}}_{\sigma,1j_1^*j_2^*},\hat{\boldsymbol{w}}_{\sigma,2j_1^*j_2^*})-\widehat{\overline{c}}_{R,j_1^*j_2^*}(\hat{\boldsymbol{w}}_{\sigma,1j_1^*j_2^*},\hat{\boldsymbol{w}}_{\sigma,2j_1^*j_2^*})\right],\end{aligned}$$

$$\widehat{c}_{R,j_1j_2t}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}, \hat{\boldsymbol{w}}_{\sigma,2j_1j_2}) = \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} \hat{\boldsymbol{w}}_{\sigma,1j_1j_2i_1}\hat{\boldsymbol{w}}_{\sigma,2j_1j_2i_2}\hat{c}_{j_1j_2i_1i_2}\left[\frac{\hat{c}_{j_1j_2i_1i_2t}}{\hat{c}_{j_1j_2i_1i_2}} - \frac{x_{1i_1t}^2}{2(\hat{\sigma}_{\boldsymbol{x}1j_1j_2i_1})^2} - \frac{x_{2i_2t}^2}{2(\hat{\sigma}_{\boldsymbol{x}2j_1j_2i_2})^2}\right],$$

and $\widehat{\widehat{c}}_{R,j_1j_2}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}, \hat{\boldsymbol{w}}_{\sigma,2j_1j_2}) = \frac{1}{T_{j_1j_2}(\hat{\theta}_1,\hat{\theta}_2)}\sum_{t=1}^{T}\widehat{c}_{R,j_1j_2t}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}, \hat{\boldsymbol{w}}_{\sigma,2j_1j_2})$. Moreover, $\hat{\boldsymbol{Q}}_{12j_1j_2,j_1^*j_2^*}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}, \hat{\boldsymbol{w}}_{\sigma,2j_1j_2}; \hat{\boldsymbol{w}}_{\sigma,1j_1^*j_2^*}, \hat{\boldsymbol{w}}_{\sigma,2j_1^*j_2^*})$, and $\hat{\boldsymbol{Q}}_{21j_1j_2,j_1^*j_2^*}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}, \hat{\boldsymbol{w}}_{\sigma,2j_1j_2}; \hat{\boldsymbol{w}}_{\sigma,1j_1^*j_2^*}, \hat{\boldsymbol{w}}_{\sigma,2j_1^*j_2^*})$ can be computed using substituting $\boldsymbol{w}_g$ with $\hat{\boldsymbol{w}}_{\sigma,gj_1j_2}$, and $\boldsymbol{w}_g^*$ with $\hat{\boldsymbol{w}}_{\sigma,gj_1^*j_2^*}$ in equations (B.3) and (B.4), with $g = 1, 2$. The estimator of $\boldsymbol{\Xi}^0_{g,j_1j_2,j_1^*j_2^*}$ is $\hat{\boldsymbol{\Xi}}_{g,j_1j_2,j_1^*j_2^*} = \hat{\boldsymbol{D}}_{\boldsymbol{\Xi},gj_1j_2,j_1^*j_2^*0} + \sum_{v=1}^{q} 2\left(1 - \frac{v}{q+1}\right)\hat{\boldsymbol{D}}_{\boldsymbol{\Xi},gj_1j_2,j_1^*j_2^*,v}$, where $\hat{\boldsymbol{D}}_{\boldsymbol{\Xi},gj_ggj_1j_2,v} = \dfrac{1}{T}\sum_{t=v+1}^{T}\mathbb{I}_{gj_gt}(\hat{\theta}_g)\mathbb{I}_{1j_1^*t-v}(\hat{\theta}_1)\mathbb{I}_{2j_2^*t-v}(\hat{\theta}_2)\widehat{G}^{I}_{gj_1j_2t}\widehat{G}^{II}_{gj_1^*j_2^*t-v}$, for $v = 0, 1, ..., q$, and

$$\begin{aligned}
\widehat{G}^{I}_{1j_1j_2t} &= \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}\hat{\boldsymbol{w}}_{\sigma,2j_1j_2}\hat{\boldsymbol{\lambda}}'_{2j_2i_2}\hat{\boldsymbol{\Sigma}}'_{\mathbf{f}12j_1j_2}\left(\hat{\boldsymbol{\Sigma}}_{\mathbf{f}1j_1}\right)^{-1}\hat{\boldsymbol{f}}_{1j_1t}\hat{e}_{1i_1t} ,\\
\widehat{G}^{I}_{2j_1j_2t} &= \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}\hat{\boldsymbol{w}}_{\sigma,2j_1j_2}\hat{\boldsymbol{\lambda}}'_{1j_1i_1}\hat{\boldsymbol{\Sigma}}_{\mathbf{f}12j_1j_2}\left(\hat{\boldsymbol{\Sigma}}_{\mathbf{f}2j_2}\right)^{-1}\hat{\boldsymbol{f}}_{2j_2t}\hat{e}_{2i_2t} ,\\
\widehat{G}^{II}_{gj_1^*j_2^*t} &= \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}\hat{\boldsymbol{w}}_{\sigma,1j_1^*j_2^*}\hat{\boldsymbol{w}}_{\sigma,2j_1^*j_2^*}\frac{\hat{c}_{j_1^*j_2^*i_1i_2}}{\hat{\sigma}^2_{\boldsymbol{x}1j_1^*j_2^*i_1}}\hat{\boldsymbol{\lambda}}'_{gj_g^*i_g}\hat{\boldsymbol{f}}_{gj_g^*t}\hat{e}_{g,i_gt} , \qquad g = 1, 2 .
\end{aligned}$$

B.4 Estimators for Theorems 4 and 5

The estimator of $\boldsymbol{Q}^{\Delta,0}_{j_1j_2j_1^*j_2^*}$ in Theorem 4 is :

$$\begin{aligned}
\hat{\boldsymbol{Q}}^{\Delta}_{j_1j_2j_1^*j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2) &= \hat{\boldsymbol{Q}}_{j_1j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2) + \hat{\boldsymbol{Q}}_{j_1^*j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2) - \frac{2\cdot T^2}{T_{j_1j_2}(\hat{\theta}_1,\hat{\theta}_2)\cdot T_{j_1^*j_2^*}(\hat{\theta}_1,\hat{\theta}_2)} \times \Big[\hat{\boldsymbol{\Psi}}_{j_1j_2j_1^*j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2) ,\\
&\quad + \hat{\boldsymbol{Q}}_{12j_1j_2,j_1^*j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2) + \hat{\boldsymbol{Q}}_{21j_1j_2,j_1^*j_2^*}(\boldsymbol{w}_1, \boldsymbol{w}_2)\Big], \qquad \text{(B.6)}
\end{aligned}$$

where all the terms in the RHS of the last equation are defined in Section B.2.

The estimator of $\boldsymbol{Q}^{\Delta,0}_{R,j_1j_2j_1^*j_2^*}$ in Theorem 5 is :

$$\begin{aligned}
\hat{\boldsymbol{Q}}^{\Delta}_{R,j_1j_2j_1^*j_2^*} &= \hat{\boldsymbol{Q}}_{R,j_1j_2}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}, \hat{\boldsymbol{w}}_{\sigma,2j_1j_2}) + \hat{\boldsymbol{Q}}_{R,j_1^*j_2^*}(\hat{\boldsymbol{w}}_{\sigma,1j_1^*j_2^*}, \hat{\boldsymbol{w}}_{\sigma,2j_1^*j_2^*})\\
&- \frac{2\cdot T^2}{T_{j_1j_2}(\hat{\theta}_1,\hat{\theta}_2)\cdot T_{j_1^*j_2^*}(\hat{\theta}_1,\hat{\theta}_2)} \times \Big[\hat{\boldsymbol{\Psi}}_{R,j_1j_2,j_1^*j_2^*}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}, \hat{\boldsymbol{w}}_{\sigma,2j_1j_2}; \hat{\boldsymbol{w}}_{\sigma,1j_1^*j_2^*}, \hat{\boldsymbol{w}}_{\sigma,2j_1^*j_2^*})\\
&+ \hat{\boldsymbol{Q}}_{12j_1j_2,j_1^*j_2^*}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}, \hat{\boldsymbol{w}}_{\sigma,2j_1j_2}; \hat{\boldsymbol{w}}_{\sigma,1j_1^*j_2^*}, \hat{\boldsymbol{w}}_{\sigma,2j_1^*j_2^*}) + \hat{\boldsymbol{Q}}_{21j_1j_2,j_1^*j_2^*}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}, \hat{\boldsymbol{w}}_{\sigma,2j_1j_2}; \hat{\boldsymbol{w}}_{\sigma,1j_1^*j_2^*}, \hat{\boldsymbol{w}}_{\sigma,2j_1^*j_2^*})\\
&+ \hat{\boldsymbol{\Xi}}_{1,j_1j_2,j_1^*j_2^*} + \hat{\boldsymbol{\Xi}}_{2,j_1j_2,j_1^*j_2^*} + \hat{\boldsymbol{\Xi}}_{1,j_1^*j_2^*,j_1j_2} + \hat{\boldsymbol{\Xi}}_{2,j_1^*j_2^*,j_1j_2}\Big],
\end{aligned}$$

where all the terms in the RHS of the last equation are defined in Section B.3.

Online Appendix

C Proofs

C.1 Proof of Proposition 1

To simplify the proof, we assume that both the threshold variables, and the threshold values are the same across the two groups, that is $z_{1t} = z_{2t} = z_t$, and $\theta_1 = \theta_2 = \theta$, respectively. Therefore we only have two regimes - synchronized across the two groups - denoted as $j = j_1 = j_2$, where $j = H, L$. Without loss of generality, we also assume that the number of factors in group 1 (resp. group 2) is K_1 (resp. K_2) in both regimes $j = L, H$, with $K_1 \leq K_2$, which implies $min(K_1, K_2) = K_1$. Under these simplifying assumptions, model (1) for the generic regime j can we written as:

$$\boldsymbol{x}_{1jt} = \Lambda_{1j} f_{1jt} + e_{1jt}, \tag{C.1}$$

$$\boldsymbol{x}_{2jt} = \Lambda_{2j} f_{2jt} + e_{2jt}, \tag{C.2}$$

where $f_{gjt} = \boldsymbol{f}_{gj_gt}$ and $\Lambda_{gj} = \left[\lambda_{gj1}, \ldots, \lambda_{gjN_g}\right]'$. The factors have zero mean within each regime, and the variance-covariance matrix is:

$$V \begin{bmatrix} f_{1jt} \\ f_{2jt} \end{bmatrix} = E \begin{bmatrix} f_{1jt} f'_{1jt} & f_{1jt} f'_{2jt} \\ f_{2jt} f'_{1jt} & f_{2jt} f'_{2jt} \end{bmatrix} = \begin{bmatrix} \Sigma_{fj11} & \Sigma_{fj12} \\ \Sigma'_{fj12} & \Sigma_{fj22} \end{bmatrix}, \qquad j = H, L \ ,$$

where Σ_{fj11} and Σ_{fj22} are full rank. Let D_{jg} be the $(K_g \times K_g)$ diagonal matrix collecting the eigenvalues of Σ_{fjgg}, and C_{jg} be the $(K_g \times K_g)$ matrix collecting the associated eigenvectors, that is:

$$\Sigma_{fjgg} C_{jg} = C_{jg} D_{jg} \ , \qquad C'_{jg} C_{jg} = C_{jg} C'_{jg} = I_{K_g} \ , \qquad g = 1, 2, \qquad j = H, L.$$

Define the full rank matrix $L_{jg} = D_{jg}^{-1/2} C'_{jg}$ and its inverse $L_{jg}^{-1} = C_{jg} D_{jg}^{1/2}$, for $g = 1, 2$ and $j = H, L$. Then model (C.1) - (C.2) is observationally equivalent to:

$$\boldsymbol{x}_{1jt} = \tilde{\Lambda}_{1j} \tilde{f}_{1jt} + e_{1jt}, \tag{C.3}$$

$$\boldsymbol{x}_{2jt} = \tilde{\Lambda}_{2j} \tilde{f}_{2jt} + e_{2jt}, \tag{C.4}$$

where $\tilde{\Lambda}_{gj} = \Lambda_{1jt} L_{jg}^{-1}$ and $\tilde{f}_{1jt} = L_{jg} f_{gjt}$. By definition, we have $\tilde{\Lambda}_{gj} = \left[\tilde{\lambda}_{gj1}, \ldots, \tilde{\lambda}_{gjN_g}\right]'$, where $\tilde{\lambda}_{gji_g} = (L_{jg}^{-1})' \lambda_{1ji_g}$. The factors in the new model (C.3)-(C.4) can be written as:

$$\begin{bmatrix} \tilde{f}_{1jt} \\ \tilde{f}_{2jt} \end{bmatrix} = \begin{bmatrix} L_{j1} & 0_{(K_1 \times K_2)} \\ 0_{(K_2 \times K_1)} & L_{j2} \end{bmatrix} \begin{bmatrix} f_{1jt} \\ f_{2jt} \end{bmatrix},$$

and their variance-covariance matrix is:

$$\begin{aligned} V \begin{bmatrix} \tilde{f}_{1jt} \\ \tilde{f}_{2jt} \end{bmatrix} &= \begin{bmatrix} L_{j1} & 0_{(K_1 \times K_2)} \\ 0_{(K_2 \times K_1)} & L_{j2} \end{bmatrix} \cdot \begin{bmatrix} \Sigma_{fj11} & \Sigma_{fj12} \\ \Sigma_{fj12}' & \Sigma_{fj22} \end{bmatrix} \cdot \begin{bmatrix} L_{j1}' & 0_{(K_2 \times K_1)} \\ 0_{(K_1 \times K_2)} & L_{j2}' \end{bmatrix} \\ &= \begin{bmatrix} L_{j1}\Sigma_{fj11}L_{j1}' & L_{j1}\Sigma_{fj12}L_{j2}' \\ L_{j2}'\Sigma_{fj12}'L_{j1}' & L_{j2}\Sigma_{fj22}L_{j2}' \end{bmatrix} = \begin{bmatrix} I_{K_1} & \tilde{\Phi}_j \\ \tilde{\Phi}_j' & I_{K_2} \end{bmatrix}, \end{aligned}$$

where the last equality follows by defining $\tilde{\Phi}_j = L_{j1}\Sigma_{fj12}L_{j2}'$, and from the fact that:

$$L_{jg}\Sigma_{fj11}L_{jg}' = D_{jg}^{-1/2} C_{jg}' \Sigma_{fjgg} C_{jg} D_{jg}^{-1/2} = D_{jg}^{-1/2} D_{jg} D_{jg}^{-1/2} = I_{K_g}, \quad \text{for } g = 1, 2, \quad j = H, L \, .$$

Therefore, we have just shown that by allowing the factor loadings to change in each regime and group - as we do in our model - we can always rewrite the original model for each regime as an equivalent factor model where $\mathrm{Var}[\tilde{f}_{gjt}] = I_{K_{g,j}}$ for $g = 1, 2$, and $E[\tilde{f}_{1jt}\tilde{f}_{2jt}'] = \tilde{\Phi}_j$ is the $K_1 \times K_2$ matrix of the correlations between the (rotated) pervasive factors in groups 1 and 2, in regime j. Therefore, the systematic correlation can be expressed as:

$$R_{12ji_1i_2}^0 = \frac{cov(\tilde{\lambda}_{1ji_1}'\tilde{f}_{1jt}, \tilde{\lambda}_{2ji_2}'\tilde{f}_{2jt})}{\sqrt{\mathrm{V}\left(x_{1ji_1}\right) \cdot \mathrm{V}\left(x_{2ji_2}\right)}} = \frac{\tilde{\lambda}_{1ji_1}'\tilde{\Phi}_j\tilde{\lambda}_{2ji_2}}{\sqrt{\mathrm{V}\left(x_{1ji_1}\right) \cdot \mathrm{V}\left(x_{2ji_2}\right)}}. \tag{C.5}$$

The zero correlation assumption between factors and idiosyncratic innovations implies:

$$V(x_{gji_gt}) = V(\tilde{\lambda}_{gji_g}'\tilde{f}_{gjt} + e_{gji_gt}) = \tilde{\lambda}_{gji_g}' V(\tilde{f}_{gjt})\tilde{\lambda}_{gji_g} + V(e_{gji_gt}) = \sum_{\ell=1}^{K_2} \tilde{\lambda}_{gji_g,\ell}^2 + \sigma_{gji_g}^2, \qquad g = 1, 2.$$

Let A_j and B_j be the matrices of *canonical directions* associated to the ordered *canonical correlations* between $\tilde{f}_{1jt}$, $\tilde{f}_{2jt}$, that is A_j and B_j are $(K_1 \times K_1)$ and $(K_2 \times K_2)$ matrices, respectively, such that:

$$A_j'A_j = A_jA_j' = I_{K_1}, \qquad B_j'B_j = B_jB_j' = I_{K_2}, \tag{C.6}$$

and

$$\tilde{\Phi}_j = A_j \left[\Phi_j \;\vdots\; 0_{K_1 \times (K_2 - K_1)} \right] B_j' \ , \tag{C.7}$$

where $\tilde{\Phi}_j$ is the $(K_1 \times K_1)$ diagonal matrix of all the K_1 canonical correlations between $\tilde{f}_{1jt}$ and $\tilde{f}_{2jt}$:

$$\Phi_j = \begin{bmatrix} \phi_{j,1} & 0 & ... & 0 \\ 0 & \phi_{j,2} & ... & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & ... & \phi_{j,K_1} \end{bmatrix},$$

with $1 \geq \phi_{j,1} \geq \phi_{j,2} \geq ... \geq \phi_{j,K_1} > 0$. As discussed at the end of Chapter 12.2 of Anderson (2003), equation (C.7), corresponds to the Singular Value Decomposition (SVD) of matrix $\tilde{\Phi}_j$. It can be shown that the columns of A_j are the eigenvectors of $\tilde{\Phi}_j \tilde{\Phi}_j'$, while the corresponding eigenvalues are equal to the squared elements of matrix Φ_j, that is:

$$\tilde{\Phi}_j \tilde{\Phi}_j' A_j = A_j \Phi_j^2 \ . \tag{C.8}$$

It can also be shown that the columns of B_j are the eigenvectors of $\tilde{\Phi}_j' \tilde{\Phi}_j$, while the corresponding non-zero eigenvalues are equal to the squared elements of matrix Φ_j, that is:

$$\tilde{\Phi}_j' \tilde{\Phi}_j B_j = B_j \begin{bmatrix} \Phi_j^2 & 0_{(K_1 \times (K_2 - K_1))} \\ 0_{((K_2 - K_1) \times K_1)} & 0_{((K_2 - K_1) \times (K_2 - K_1))} \end{bmatrix} .$$

Equations (C.6) and (C.7) imply:

$$A_j' \tilde{\Phi}_j B_j = \left[\Phi_j \;\vdots\; 0_{K_1 \times (K_2 - K_1)} \right].$$

Therefore, factors $\tilde{f}_{1jt}$ and $\tilde{f}_{2jt}$ can be rotated by means of matrices A_j and B_j respectively, in order to generate two new set of factors $\breve{f}_{1jt} = A_j' \tilde{f}_{1jt}$ and $\breve{f}_{2jt} = B_j' \tilde{f}_{2jt}$ such that:

$$V(\breve{f}_{1jt}) = A_j' V(\tilde{f}_{1jt}) A_j = A_j' A_j = I_{K_1} \ , \qquad V(\breve{f}_{2jt}) = B_j' V(\tilde{f}_{2jt}) B_j = B_j' B_j = I_{K_2} \ ,$$

and

$$
\begin{aligned}
cov(\breve{f}_{1jt} \ , \ \breve{f}_{2jt}) &= cov(A_j' \tilde{f}_{1jt} \ , \ B_j' \tilde{f}_{2jt}) \ = \ A_j' cov(\tilde{f}_{1jt} \ , \ \tilde{f}_{2jt}) B_j \ = \ A_j' \tilde{\Phi} B_j \\
&= \begin{bmatrix} \Phi_j & \vdots & 0_{K_1 \times (K_2 - K_1)} \end{bmatrix} . \qquad \text{(C.9)}
\end{aligned}
$$

The new factors $\breve{f}_{1jt} = [\breve{f}_{1jt,1}, ..., \breve{f}_{1jt,K_1}]'$ and $\breve{f}_{2jt} = [\breve{f}_{2jt,1}, ..., \breve{f}_{1jt,K_2}]'$ are commonly referred to as *canonical variates*. By construction the $\breve{f}_{1jt,1}$ and $\breve{f}_{2jt,1}$ are the linear combinations of $\breve{f}_{1jt}$ and $\breve{f}_{2jt}$ with the maximum correlation, that is $\phi_{j,1}$. Moreover, $\breve{f}_{1jt,2}$ and $\breve{f}_{2jt,2}$ are the linear combinations of $\breve{f}_{1jt}$ and $\breve{f}_{2jt}$ uncorrelated with $\breve{f}_{1jt,1}$ and $\breve{f}_{2jt,1}$, respectively, which have the maximum correlation, that is $\phi_{j,2}$. Analogously, $\breve{f}_{1jt,3}$ and $\breve{f}_{2jt,3}$ are the linear combinations of $\breve{f}_{1jt}$ and $\breve{f}_{2jt}$ uncorrelated with $[\breve{f}_{1jt,1}, \breve{f}_{1jt,2}]'$ and $[\breve{f}_{2jt,1}, \breve{f}_{2jt,2}]'$, respectively, which have the maximum correlation, that is $\phi_{j,3}$, and so on.[17]

By using the orthonormality of the the canonical direction matrices A_j and B_j, that is the equations in (C.6), we have:

$$
\begin{aligned}
\tilde{\lambda}_{1ji_1}' \tilde{f}_{1jt} &= \tilde{\lambda}_{1ji_1}' A_j A_j' \tilde{f}_{1jt} \ = \ \breve{\lambda}_{1ji_1}' \breve{f}_{1jt}, \qquad \text{(C.10)} \\
\tilde{\lambda}_{2ji_2}' \tilde{f}_{2jt} &= \tilde{\lambda}_{2ji_2}' B_j B_j' \tilde{f}_{2jt} \ = \ \breve{\lambda}_{2ji_2}' \breve{f}_{2jt}, \qquad \text{(C.11)}
\end{aligned}
$$

where $\breve{\lambda}_{1ji_1} = \tilde{\lambda}_{1ji_1}' A_j$ and $\breve{\lambda}_{2ji_2} = \tilde{\lambda}_{2ji_2}' B_j$. By using equations (C.10) and (C.11) can rewrite model (C.1)-(C.2) as an observationally equivalent model with new factors $\breve{f}_{1jt}$ and $\breve{f}_{2jt}$:

$$
\begin{aligned}
x_{1ji_1t} &= \tilde{\lambda}_{1ji_1}' \tilde{f}_{1jt} + e_{1ji_1t} \ = \ \breve{\lambda}_{1ji_1}' \breve{f}_{1jt} + e_{1ji_1t}, \qquad \text{(C.12)} \\
x_{2ji_2t} &= \tilde{\lambda}_{2ji_2}' \tilde{f}_{2jt} + e_{2ji_1t} \ = \ \breve{\lambda}_{2ji_2}' \breve{f}_{2jt} + e_{2ji_2t}, \qquad \text{(C.13)}
\end{aligned}
$$

Equations (C.10) - (C.13) and the assumption that the factors and the errors are uncorrelated imply that, for $g = 1, 2$ we have:

$$
V(\tilde{\lambda}_{gji_g}' \tilde{f}_{gjt}) \ = \ V(\breve{\lambda}_{gji_g}' \breve{f}_{gjt}) \ = \ \breve{\lambda}_{gji_g}' V(\breve{f}_{gjt}) \breve{\lambda}_{gji_g} \ = \ \sum_{\ell=1}^{K_g} \breve{\lambda}_{gji_g,\ell}^2 \ = \ \sum_{\ell=1}^{K_g} \breve{\lambda}_{1ji_g,\ell}^2 \ ,
$$

[17] For a thorough introduction to canonical correlation, canonical directions and variables see Chapter 12.2 of Anderson (2003) and Chapter 17.16 in Magnus and Neudecker (2007).

and

$$V(x_{gji_gt}) = V(\breve{\lambda}'_{gji_g}\breve{f}_{gjt} + e_{gji_gt}) = \sum_{\ell=1}^{K_g} \breve{\lambda}^2_{gji_g,\ell} + \sigma^2_{gji_g}, \qquad \text{(C.14)}$$

The last four equations and the definition of the population R-square given in equation (13) imply that

$$\begin{aligned} \mathcal{R}^2_{gji_g} &= \frac{V(\tilde{\lambda}'_{gji_g}\tilde{f}_{gjt})}{V(x_{gji_gt})} = \frac{\sum_{\ell=1}^{K_1}\tilde{\lambda}^2_{gji_1,\ell}}{\sum_{\ell=1}^{K_1}\tilde{\lambda}^2_{gji_g,\ell} + \sigma^2_{gji_1}} \\ &= \frac{V(\breve{\lambda}'_{gji_g}\breve{f}_{gjt})}{V(x_{gji_gt})} = \frac{\sum_{\ell=1}^{K_2}\breve{\lambda}^2_{gji_g,\ell}}{\sum_{\ell=1}^{K_2}\breve{\lambda}^2_{gji_g,\ell} + \sigma^2_{gji_1}}, \quad g = 1, 2 . \end{aligned} \qquad \text{(C.15)}$$

Moreover,

$$\begin{aligned} cov(\breve{\lambda}'_{1ji_1}\breve{f}_{1jt},\ \breve{\lambda}'_{2ji_2}\breve{f}_{2jt}) &= \breve{\lambda}'_{1ji_1}cov(\breve{f}_{1jt}\ ,\ \breve{f}_{2jt})\breve{\lambda}_{2ji_2} = \breve{\lambda}'_{1ji_1}\left[\Phi_j \ \vdots \ 0_{K_1\times(K_2-K_1)}\right]\breve{\lambda}_{2ji_2} \\ &= \sum_{\ell=1}^{K_1}\breve{\lambda}_{1ji_1,\ell}\breve{\lambda}_{2ji_2,\ell}\phi_{j,\ell} + \sum_{\ell=K_1+1}^{K_2-K_1} 0\cdot\breve{\lambda}_{2ji_2,\ell} . \end{aligned} \qquad \text{(C.16)}$$

By substituting (C.14) and (C.16) into equation (C.5) we get

$$\begin{aligned} \left|R^0_{12ji_1i_2}\right| &= \frac{\left|\sum_{\ell=1}^{K_1}\breve{\lambda}_{1ji_1,\ell}\breve{\lambda}_{2ji_2,\ell}\phi_\ell\right|}{\sqrt{\sum_{\ell=1}^{K_1}\breve{\lambda}^2_{1ji_1,\ell} + \sigma^2_{1ji_1}} \cdot \sqrt{\sum_{\ell=1}^{K_2}\breve{\lambda}^2_{2ji_2,\ell} + \sigma^2_{2ji_2}}} \\ &\le \frac{\sum_{\ell=1}^{K_1}\left|\breve{\lambda}_{1ji_1,\ell}\right|\left|\breve{\lambda}_{2ji_2,\ell}\right|\phi_{j,\ell}}{\sqrt{\sum_{\ell=1}^{K_1}\breve{\lambda}^2_{1ji_1,\ell} + \sigma^2_{1ji_1}} \cdot \sqrt{\sum_{\ell=1}^{K_2}\breve{\lambda}^2_{2ji_2,\ell} + \sigma^2_{2ji_2}}} \le \frac{\phi_{j,1}\sum_{\ell=1}^{K_1}\left|\breve{\lambda}_{1ji_1,\ell}\right|\left|\breve{\lambda}_{2ji_2,\ell}\right|}{\sqrt{\sum_{\ell=1}^{K_1}\breve{\lambda}^2_{1ji_1,\ell} + \sigma^2_{1ji_1}} \cdot \sqrt{\sum_{\ell=1}^{K_2}\breve{\lambda}^2_{2ji_2,\ell} + \sigma^2_{2ji_2}}} \\ &\le \phi_{j,1}\frac{\sqrt{\sum_{\ell=1}^{K_1}\breve{\lambda}^2_{1ji_1,\ell}}}{\sqrt{\sum_{\ell=1}^{K_1}\breve{\lambda}^2_{1ji_1,\ell} + \sigma^2_{1ji_1}}}\frac{\sqrt{\sum_{\ell=1}^{K_1}\breve{\lambda}^2_{2ji_2,\ell}}}{\sqrt{\sum_{\ell=1}^{K_2}\breve{\lambda}^2_{2ji_2,\ell} + \sigma^2_{2ji_2}}} \\ &\le \phi_{j,1}\frac{\sqrt{\sum_{\ell=1}^{K_1}\breve{\lambda}^2_{1ji_1,\ell}}}{\sqrt{\sum_{\ell=1}^{K_1}\breve{\lambda}^2_{1ji_1,\ell} + \sigma^2_{1ji_1}}}\frac{\sqrt{\sum_{\ell=1}^{K_2}\breve{\lambda}^2_{2ji_2,\ell}}}{\sqrt{\sum_{\ell=1}^{K_2}\breve{\lambda}^2_{2ji_2,\ell} + \sigma^2_{2ji_2}}} = \phi_{j,1} \cdot \sqrt{\mathcal{R}^2_{1ji_1}} \cdot \sqrt{\mathcal{R}^2_{2ji_2}}, \end{aligned}$$

where the first inequality follows from the repeated application of the triangle inequality and the fact that the canonical correlations $\phi_{j,\ell}$, for all $\ell = 1, .., K_1$, are non-negative by definition. The second inequality follows from the fact that $\phi_{j,1}$ is the largest canonical correlation, and again because all the canonical correlations are non-negative. The third inequality follows from the

Cauchy–Schwartz inequality, while the fourth inequality follows form the assumption $K_1 \leq K_2$. The last equality follows form the definition of the population R-square in equation (C.15).

Finally, assuming that $w_{gi_g} \geq 0$ for all $i_g = 1, ..., N_g$ and $g = 1, 2$, we have:

$$\begin{aligned}\left|\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{1i_1}w_{2i_2}R^0_{12ji_1i_2}\right| &\leq \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{1i_1}w_{2i_2}\frac{\left|\sum_{\ell=1}^{K_1}\breve{\lambda}_{1ji_1,\ell}\breve{\lambda}_{2ji_2,\ell}\phi_\ell\right|}{\sqrt{\sum_{\ell=1}^{K_1}\breve{\lambda}^2_{1ji_1,\ell}+\sigma^2_{1ji_1}}\cdot\sqrt{\sum_{\ell=1}^{K_2}\breve{\lambda}^2_{2ji_2,\ell}+\sigma^2_{2ji_2}}} \\ &\leq \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{1i_1}w_{2i_2}\frac{\sum_{\ell=1}^{K_1}\left|\breve{\lambda}_{1ji_1,\ell}\right|\left|\breve{\lambda}_{2ji_2,\ell}\right|\phi_{j,\ell}}{\sqrt{\sum_{\ell=1}^{K_1}\breve{\lambda}^2_{1ji_1,\ell}+\sigma^2_{1ji_1}}\cdot\sqrt{\sum_{\ell=1}^{K_2}\breve{\lambda}^2_{2ji_2,\ell}+\sigma^2_{2ji_2}}} \\ &\leq \phi_{j,1}\cdot\left(\sum_{i_1=1}^{N_1} w_{1i_1}\sqrt{\mathcal{R}^2_{1ji_1}}\right)\cdot\left(\sum_{i_2=1}^{N_2} w_{2i_2}\sqrt{\mathcal{R}^2_{2ji_2}}\right).\end{aligned}$$

■

C.2 Proof of Theorem 1

The proof of Theorem 1 requires following Auxiliary Lemmas.

LEMMA C.1. *Under Assumptions A.1 - A.8, for $g = 1, 2$, $j_g = L, H$, and $i_g = 1, \ldots, N_g$,*

$$\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)'\boldsymbol{\lambda}^0_{gj_gi_g} = \hat{\boldsymbol{V}}_{gj_g}(\theta^0_g)^{-1}\frac{\hat{\boldsymbol{\Lambda}}'_{gj_g}\boldsymbol{\Lambda}^0_{gj_g}}{N_g}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{gj_gt}(\theta^0_g)\boldsymbol{f}^0_{gt}e_{gi_gt}\right] + O_p\left(\frac{1}{C^2_{N_gT}}\right),$$

where

$$\hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g) = \frac{\boldsymbol{F}^0_{gj_g}(\theta^0_g)\boldsymbol{F}^0_{gj_g}(\theta^0_g)'}{T}\frac{\boldsymbol{\Lambda}^{0\prime}_{gj_g}\hat{\boldsymbol{\Lambda}}_{gj_g}(\theta^0_g)}{N_g}\hat{\boldsymbol{V}}_{gj_g}(\theta^0_g)^{-1} \qquad \text{(C.17)}$$

and $\hat{\boldsymbol{V}}_{gj_g}(\theta^0_g)$ is the diagonal matrix consisting of the first K_{gj_g} eigenvalues of $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{x}gj_g}(\theta_g) = \frac{1}{N_gT}\sum_{t=1}^{T}\mathbb{I}_{gj_gt}\left(\theta^0_g\right)\boldsymbol{x}_{gt}\boldsymbol{x}'_{gt}$ in decreasing order.

Proof of Lemma C.1. The result is analogous to equation (B.2) in Bai (2003) and the proof is omitted.

LEMMA C.2. *Under Assumptions A.1 - A.8, for $g = 1, 2$, and $t = 1, \dots, T$,*

$$\begin{aligned}
&\hat{\boldsymbol{f}}_{gt} - [\mathbb{I}_{g1t}(\theta_g^0)\hat{\boldsymbol{H}}_{gg11}(\theta_g^0) + \mathbb{I}_{g2t}(\theta_g^0)\hat{\boldsymbol{H}}_{gg22}(\theta_g^0)]^{-1}\hat{\boldsymbol{f}}_{gt}^0 \\
&= \mathbb{I}_{g1t}(\theta_g^0)\hat{\boldsymbol{H}}_{gg11}(\theta_g^0)'\frac{1}{N_g}\left[\sum_{i_g=1}^{N_g}\mathbb{I}_{g1t}(\theta_g^0)\boldsymbol{\lambda}_{g1i_g}^0 e_{gi_gt}\right] + \mathbb{I}_{g2t}(\theta_g^0)\hat{\boldsymbol{H}}_{gg22}(\theta_g^0)'\frac{1}{N_g}\left[\sum_{i_g=1}^{N_g}\mathbb{I}_{g2t}(\theta_g^0)\boldsymbol{\lambda}_{g2i_g}^0 e_{gi_gt}\right] \\
&+ O_p\left(\frac{1}{\sqrt{N_g}C_{N_gT}}\right) + O_p\left(\frac{1}{\sqrt{T}C_{N_gT}}\right).
\end{aligned}$$

Proof of Lemma C.2. The result is analogous to equation (A.5) in Bai (2003) and the proof is omitted.

For $g = 1, 2$, $j_g = 1, 2$, $i_g = 1, \dots, N_g$, and $t = 1, \dots, T$, let $\hat{\zeta}_{gj_gi_gt} := \hat{c}_{gj_gi_gt} - c_{gj_gi_gt}^0$. We then have

$$\begin{aligned}
\mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\hat{c}_{1j_1i_1t}\hat{c}_{2j_2i_2t} &= \mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)(c_{1j_1i_1t}^0 + \hat{\zeta}_{1j_1i_1t})(c_{2j_2i_2t}^0 + \hat{\zeta}_{2j_2i_2t}) \\
&= \mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)c_{1j_1i_1t}^0 c_{2j_2i_2t}^0 + \mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)c_{1j_1i_1t}^0\hat{\zeta}_{2j_2i_2t} \\
&\quad + \mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\hat{\zeta}_{1j_1i_1t}c_{2j_2i_2t}^0 + \mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\hat{\zeta}_{1j_1i_1t}\hat{\zeta}_{2j_2i_2t}.
\end{aligned} \tag{C.18}$$

By Theorem 3.4 in Massacci (2017), we can write $\left(\hat{\theta}_g - \theta_g^0\right) = O_p\left(T^{-1}\right)$. By continuous mapping theorem, this implies that

$$\mathbb{I}_{gj_gt}(\hat{\theta}_g) = \mathbb{I}_{gj_gt}(\theta_g^0) + O_p\left(T^{-1}\right), \quad g = 1, 2, \quad j_g = 1, 2. \tag{C.19}$$

Combining (C.18) and (C.19), it follows that

$$\begin{aligned}
\mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\hat{c}_{1j_1i_1t}\hat{c}_{2j_2i_2t} &= \mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)c_{1j_1i_1t}^0 c_{2j_2i_2t}^0 + \mathbb{I}_{1j_1t}(\theta_1^0)c_{1j_1i_1t}^0 c_{2j_2i_2t}^0 O_p\left(T^{-1}\right) \\
&\quad + \mathbb{I}_{2j_2t}(\theta_2^0)c_{1j_1i_1t}^0 c_{2j_2i_2t}^0 O_p\left(T^{-1}\right) + c_{1j_1i_1t}^0 c_{2j_2i_2t}^0 O_p\left(T^{-1}\right)O_p\left(T^{-1}\right) \\
&\quad + \mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)c_{1j_1i_1t}^0\hat{\zeta}_{2j_2i_2t} + \mathbb{I}_{1j_1t}(\theta_1^0)c_{1j_1i_1t}^0\hat{\zeta}_{2j_2i_2t}O_p\left(T^{-1}\right) \\
&\quad + \mathbb{I}_{2j_1t}(\theta_2^0)c_{1j_1i_1t}^0\hat{\zeta}_{2j_2i_2t}O_p\left(T^{-1}\right) + c_{1j_1i_1t}^0\hat{\zeta}_{2j_2i_2t}O_p\left(T^{-1}\right)O_p\left(T^{-1}\right) \\
&\quad + \mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)c_{2j_2i_2t}^0\hat{\zeta}_{1j_1i_1t} + \mathbb{I}_{1j_1t}(\theta_1^0)c_{2j_2i_2t}^0\hat{\zeta}_{1j_1i_1t}O_p\left(T^{-1}\right) \\
&\quad + \mathbb{I}_{2j_2t}(\theta_2^0)c_{2j_2i_2t}^0\hat{\zeta}_{1j_1i_1t}O_p\left(T^{-1}\right) + c_{2j_2i_2t}^0\hat{\zeta}_{1j_1i_1t}O_p\left(T^{-1}\right)O_p\left(T^{-1}\right) \\
&\quad + \mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)\hat{\zeta}_{1j_1i_1t}\hat{\zeta}_{2j_2i_2t} + \mathbb{I}_{1j_1t}(\theta_1^0)\hat{\zeta}_{1j_1i_1t}\hat{\zeta}_{2j_2i_2t}O_p\left(T^{-1}\right) \\
&\quad + \mathbb{I}_{2j_2t}(\theta_2^0)\hat{\zeta}_{1j_1i_1t}\hat{\zeta}_{2j_2i_2t}O_p\left(T^{-1}\right) + \hat{\zeta}_{1j_1i_1t}\hat{\zeta}_{2j_2i_2t}O_p\left(T^{-1}\right)O_p\left(T^{-1}\right).
\end{aligned} \tag{C.20}$$

By Assumptions A.1 and A.2 we have

$$c^0_{gj_gi_gt} \leq \mid c^0_{gj_gi_gt} \mid \leq \mid \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \boldsymbol{f}^0_{gj_gt} \mid \leq \left\| \lambda^0_{gj_gi_g} \right\| \left\| \boldsymbol{f}^0_{gj_gt} \right\| \leq \bar{\lambda} \left\| \boldsymbol{f}^0_{gj_gt} \right\| = O_p(1), \quad g = 1, 2. \tag{C.21}$$

For $g = 1, 2$, consider

$$\begin{aligned}
\hat{\zeta}_{gj_gi_gt} &= \hat{c}_{gj_gi_gt} - c^0_{gj_gi_gt} \\
&= \hat{\boldsymbol{\lambda}}'_{gj_gi_g} \hat{\boldsymbol{f}}_{gj_gt} - \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \boldsymbol{f}^0_{gj_gt} \\
&= [\hat{\boldsymbol{\lambda}}'_{gj_gi_g} \hat{\boldsymbol{f}}_{gj_gt} + \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g) \hat{\boldsymbol{f}}_{gj_gt} - \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g) \hat{\boldsymbol{f}}_{gj_gt} - \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \boldsymbol{f}^0_{gj_gt}] \\
&= \boldsymbol{\lambda}^{0\prime}_{gj_gi_gt} [\hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g) \hat{\boldsymbol{f}}_{gj_gt} - \boldsymbol{f}^0_{gj_gt}] + [\hat{\boldsymbol{\lambda}}'_{gj_gi_g} - \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)] \hat{\boldsymbol{f}}_{gj_gt} \\
&= \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g) [\hat{\boldsymbol{f}}_{gj_gt} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)^{-1} \boldsymbol{f}^0_{gj_gt}] + [\hat{\boldsymbol{\lambda}}'_{gj_gi_g} - \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)] \hat{\boldsymbol{f}}_{gj_gt} \\
&= \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g) [\hat{\boldsymbol{f}}_{gj_gt} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)^{-1} \boldsymbol{f}^0_{gj_gt}] + [\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)' \boldsymbol{\lambda}^0_{gj_gi_g}]' \hat{\boldsymbol{f}}_{gj_gt}
\end{aligned}$$

and notice that for $\mathbb{I}_{gj_gt}(\theta^0_g) = 1$

$$\begin{aligned}
[\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)' \boldsymbol{\lambda}^0_{gj_gi_g}]' \hat{\boldsymbol{f}}_{gj_gt} &= [\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)' \boldsymbol{\lambda}^0_{gj_gi_g}]' \hat{\boldsymbol{f}}_{gj_gt} \\
&\quad + [\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)' \boldsymbol{\lambda}^0_{gj_gi_g}]' \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)^{-1} \boldsymbol{f}^0_{gj_gt} \\
&\quad - [\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)' \boldsymbol{\lambda}^0_{gj_gi_g}]' \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)^{-1} \boldsymbol{f}^0_{gj_gt} \\
&= [\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)' \boldsymbol{\lambda}^0_{gj_gi_g}]' \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)^{-1} \boldsymbol{f}^0_{gj_gt} \\
&\quad + [\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)' \boldsymbol{\lambda}^0_{gj_gi_g}]' [\hat{\boldsymbol{f}}_{gj_gt} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)^{-1} \boldsymbol{f}^0_{gj_gt}] :
\end{aligned}$$

by Lemmas C.1 and C.2 and Assumptions A.7 and A.8, it follows that

$$[\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)' \boldsymbol{\lambda}^0_{gj_gi_g}]' \hat{\boldsymbol{f}}_{gj_gt} = [\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)' \boldsymbol{\lambda}^0_{gj_gi_g}]' \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)^{-1} \boldsymbol{f}^0_{gj_gt} + O_p\left(\frac{1}{\sqrt{N_gT}}\right),$$

which implies that

$$\begin{aligned}
\hat{\zeta}_{gj_gi_gt} &= \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g) [\hat{\boldsymbol{f}}_{gj_gt} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)^{-1} \boldsymbol{f}^0_{gj_gt}] \\
&\quad + [\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)' \boldsymbol{\lambda}^0_{gj_gi_g}]' \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)^{-1} \boldsymbol{f}^0_{gj_gt} + O_p\left(\frac{1}{\sqrt{N_gT}}\right) \\
&= \boldsymbol{\lambda}^{0\prime}_{gj_gi_g} \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g) [\hat{\boldsymbol{f}}_{gj_gt} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)^{-1} \boldsymbol{f}^0_{gj_gt}] \\
&\quad + \boldsymbol{f}^{0\prime}_{gt} \left[\hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)'\right]^{-1} [\hat{\boldsymbol{\lambda}}_{gj_gi_g} - \hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)' \boldsymbol{\lambda}^0_{gj_gi_g}] + O_p\left(\frac{1}{\sqrt{N_gT}}\right),
\end{aligned}$$

and, using again Lemmas C.1 and C.2 and Assumptions A.7 and A.8 we get:

$$\begin{aligned}
\hat{\zeta}_{gj_gi_gt} &= \boldsymbol{\lambda}^{0\prime}_{gj_gi_g}\hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)\hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)'\left[\frac{1}{N_g}\sum_{l_g=1}^{N_g}\mathbb{I}_{gj_gt}(\theta^0_g)\boldsymbol{\lambda}^0_{gj_gl_g}e_{gl_gt}\right] \\
&\quad+\boldsymbol{f}^{0\prime}_{gt}\left[\hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)'\right]^{-1}\boldsymbol{V}_{gj_g}(\theta^0_g)^{-1}\frac{\hat{\boldsymbol{\Lambda}}'_{gj_g}\boldsymbol{\Lambda}^0_{gj_g}}{N_g}\left[\frac{1}{T}\sum_{v=1}^{T}\mathbb{I}_{gj_gv}(\theta^0_g)\boldsymbol{f}^0_{gj_gv}e_{gi_gv}\right] \\
&\quad+O_p\left(\frac{1}{\sqrt{N_gT}}\right)+O_p\left(\frac{1}{C^2_{N_gT}}\right)+O_p\left(\frac{1}{\sqrt{N_g}C_{N_gT}}\right)+O_p\left(\frac{1}{\sqrt{T}C_{N_gT}}\right) \\
&= O_p\left(\frac{1}{\sqrt{N_g}}\right)+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{N_gT}}\right)+O_p\left(\frac{1}{C^2_{N_gT}}\right)+O_p\left(\frac{1}{\sqrt{N_g}C_{N_gT}}\right)+O_p\left(\frac{1}{\sqrt{T}C_{N_gT}}\right) \\
&= O_p\left(\frac{1}{C_{N_gT}}\right), \qquad g=1,2.
\end{aligned}$$

Combining (C.20), (C.21) and (C.22) we obtain

$$\begin{aligned}
&C_{NT}\left[\mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\hat{c}_{1j_1i_1t}\hat{c}_{2j_2i_2t}-\mathbb{I}_{1j_1t}(\theta^0_1)\mathbb{I}_{2j_2t}(\theta^0_1)c^0_{1j_1i_1t}c^0_{2j_2i_2t}\right] \\
=\;& C_{NT}\mathbb{I}_{1j_1t}(\theta^0_1)\mathbb{I}_{2j_2t}(\theta^0_2)c^0_{1j_1i_1t}\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\hat{\boldsymbol{H}}_{22j_2j_2}(\theta^0_2)\hat{\boldsymbol{H}}_{22j_2j_2}(\theta^0_2)'\left[\frac{1}{N_2}\textstyle\sum_{l_2=1}^{N_2}\mathbb{I}_{2j_2t}(\theta^0_2)\boldsymbol{\lambda}^0_{2j_2l_2}e_{2l_2t}\right] \\
&+C_{NT}\mathbb{I}_{1j_1t}(\theta^0_1)\mathbb{I}_{2j_2t}(\theta^0_2)c^0_{1j_1i_1t}\boldsymbol{f}^{0\prime}_{2j_2t}\left[\hat{\boldsymbol{H}}_{22j_2j_2}(\theta^0_2)'\right]^{-1}\boldsymbol{V}_{2j_2}(\theta^0_2)^{-1}\frac{\hat{\boldsymbol{\Lambda}}'_{2j_2}\boldsymbol{\Lambda}^0_{2j_2}}{N_2}\left[\frac{1}{T}\textstyle\sum_{v=1}^{T}\mathbb{I}_{2j_2v}(\theta^0_2)\boldsymbol{f}^0_{2j_2v}e_{2i_2v}\right] \\
&+C_{NT}\mathbb{I}_{1j_1t}(\theta^0_1)\mathbb{I}_{2j_2t}(\theta^0_2)c^0_{2j_2i_2t}\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\hat{\boldsymbol{H}}_{11j_1j_1}(\theta^0_1)\hat{\boldsymbol{H}}_{11j_1j_1}(\theta^0_1)'\left[\frac{1}{N_1}\textstyle\sum_{l_1=1}^{N_1}\mathbb{I}_{1j_1t}(\theta^0_1)\boldsymbol{\lambda}^0_{1j_1l_1}e_{1l_1t}\right] \\
&+C_{NT}\mathbb{I}_{1j_1t}(\theta^0_1)\mathbb{I}_{2j_2t}(\theta^0_2)c^0_{2j_2i_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\left[\hat{\boldsymbol{H}}_{11j_1j_1}(\theta^0_1)'\right]^{-1}\boldsymbol{V}_{1j_1}(\theta^0_1)^{-1}\frac{\hat{\boldsymbol{\Lambda}}'_{1j_1}\boldsymbol{\Lambda}^0_{1j_1}}{N_1}\left[\frac{1}{T}\textstyle\sum_{v=1}^{T}\mathbb{I}_{1j_1v}(\theta^0_1)\boldsymbol{f}^0_{1j_1v}e_{1i_1v}\right] \\
&+O_p\left(\frac{1}{C_{NT}}\right).
\end{aligned}$$

For $g=1,2$, the definition of $\hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)$ in equation (C.17) implies that

$$[\hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)']^{-1}=\left[\frac{\boldsymbol{F}^0_{gj_g}(\theta^0_g)\boldsymbol{F}^0_{gj_g}(\theta^0_g)'}{T}\right]^{-1}\left[\frac{\hat{\boldsymbol{\Lambda}}_{gj_g}(\theta^0_g)'\boldsymbol{\Lambda}^0_{gj_g}}{N_g}\right]^{-1}\hat{\boldsymbol{V}}_{gj_g}(\theta^0_g),$$

and

$$\hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)\hat{\boldsymbol{H}}_{ggj_gj_g}(\theta^0_g)'=\left(\frac{\boldsymbol{\Lambda}^{0\prime}_{gj_g}\boldsymbol{\Lambda}^0_{gj_g}}{N_g}\right)^{-1}+O_p\left(\frac{1}{C^2_{N_gT}}\right).$$

The last two equation imply:

$$
\begin{aligned}
& C_{NT}\left[\mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\hat{c}_{1j_1i_1t}\hat{c}_{2j_2i_2t} - \mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_1^0)c^0_{1j_1i_1t}c^0_{2j_2i_2t}\right] \\
= & \frac{C_{NT}}{\sqrt{N_2}}\mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)c^0_{1j_1i_1t}\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left(\frac{\boldsymbol{\Lambda}^{0\prime}_{2j_2}\boldsymbol{\Lambda}^0_{2j_2}}{N_2}\right)^{-1}\left[\frac{1}{\sqrt{N_2}}\sum_{l_2=1}^{N_2}\mathbb{I}_{2j_2t}(\theta_2^0)\boldsymbol{\lambda}^0_{2j_2l_2}e_{2l_2t}\right] \\
& +\frac{C_{NT}}{\sqrt{T}}\mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)c^0_{1j_1i_1t}\boldsymbol{f}^{0\prime}_{2j_2t}\left[\frac{\boldsymbol{F}^0_{2j_2}(\theta_2^0)\boldsymbol{F}^0_{2j_2}(\theta_2^0)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{v=1}^{T}\mathbb{I}_{2j_2v}(\theta_2^0)\boldsymbol{f}^0_{2j_2v}e_{2i_2v}\right] \\
& +\frac{C_{NT}}{\sqrt{N_1}}\mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)c^0_{2j_2i_2t}\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left(\frac{\boldsymbol{\Lambda}^{0\prime}_{1j_1}\boldsymbol{\Lambda}^0_{1j_1}}{N_1}\right)^{-1}\left[\frac{1}{\sqrt{N_1}}\sum_{l_1=1}^{N_1}\mathbb{I}_{1j_1t}(\theta_1^0)\boldsymbol{\lambda}^0_{1j_1l_1}e_{1l_1t}\right] \\
& +\frac{C_{NT}}{\sqrt{T}}\mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)c^0_{2j_2i_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\left[\frac{\boldsymbol{F}^0_{1j_1}(\theta_1^0)\boldsymbol{F}^0_{1j_1}(\theta_1^0)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{v=1}^{T}\mathbb{I}_{1j_1v}(\theta_1^0)\boldsymbol{f}^0_{1j_1v}e_{1i_1v}\right] \\
& +O_p\left(\frac{1}{C_{NT}}\right),
\end{aligned}
\tag{C.22}
$$

and the result stated in the theorem follows from Assumption A.9 as $N_1, N_2, T \to \infty$ using arguments analogous to those in the proof of Theorem 3 in Bai (2003).

■

C.3 Proof of Theorem 2

Equation (C.19) implies

$$\mathbb{I}_{1j_1t}\left(\hat{\theta}_1\right)\mathbb{I}_{2j_2t}\left(\hat{\theta}_2\right) = \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right) + O_p\left(T^{-1}\right) \tag{C.23}$$

and

$$T_{j_1j_2}\left(\hat{\theta}_1,\hat{\theta}_2\right) = \sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\hat{\theta}_1\right)\mathbb{I}_{2j_2t}\left(\hat{\theta}_2\right) = \left[\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\right] + O_p\left(1\right). \tag{C.24}$$

From (C.22) it follows that

$$
\begin{aligned}
& \mathbb{I}_{1j_1t}(\hat{\theta}_1)\mathbb{I}_{2j_2t}(\hat{\theta}_2)\hat{c}_{i_1i_2t} \\
= \; & \mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)c^0_{i_1i_2t} + \mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)\frac{1}{\sqrt{N_2}}c^0_{1j_1i_1t}\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left(\frac{\boldsymbol{\Lambda}^{0\prime}_{2j_2}\boldsymbol{\Lambda}^0_{2j_2}}{N_2}\right)^{-1}\left[\frac{1}{\sqrt{N_2}}\sum_{i_2=1}^{N_2}\mathbb{I}_{2j_2t}(\theta_2^0)\boldsymbol{\lambda}^0_{2j_2i_2}e_{2i_2t}\right] \\
& +\mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)\frac{1}{\sqrt{T}}c^0_{1j_1i_1t}\boldsymbol{f}^{0\prime}_{2t}\left[\frac{\boldsymbol{F}^0_{2j_2}(\theta_2^0)\boldsymbol{F}^0_{2j_2}(\theta_2^0)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{2j_2t}(\theta_2^0)\boldsymbol{f}^0_{2t}e_{2i_2t}\right] \\
& +\mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)\frac{1}{\sqrt{N_1}}c^0_{2j_2i_2t}\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left(\frac{\boldsymbol{\Lambda}^{0\prime}_{1j_1}\boldsymbol{\Lambda}^0_{1j_1}}{N_1}\right)^{-1}\left[\frac{1}{\sqrt{N_1}}\sum_{i_1=1}^{N_1}\mathbb{I}_{1j_1t}(\theta_1^0)\boldsymbol{\lambda}^0_{1j_1i_1}e_{1i_1t}\right] \\
& +\mathbb{I}_{1j_1t}(\theta_1^0)\mathbb{I}_{2j_2t}(\theta_2^0)\frac{1}{\sqrt{T}}c^0_{2j_2i_2t}\boldsymbol{f}^{0\prime}_{1t}\left[\frac{\boldsymbol{F}^0_{1j_1}(\theta_1^0)\boldsymbol{F}^0_{1j_1}(\theta_1^0)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}(\theta_1^0)\boldsymbol{f}^0_{1t}e_{1i_1t}\right] \; + \; O_p\left(\frac{1}{C^2_{NT}}\right) \; .
\end{aligned}
$$

Substituting the last equation, (C.23) and (C.24) into the expression for $\hat{c}_{j_1j_2i_1i_2}$ in equation (16) we get:

$$
\begin{aligned}
\hat{c}_{j_1j_2i_1i_2} \; = \; & \left[\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right) + O_p\left(1\right)\right]^{-1} \\
& \times\sum_{t=1}^{T}\left(\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right) + O_p(T^{-1})\right) \times \Bigg\{\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)c^0_{j_1j_2i_1i_2t} \\
& +\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\frac{1}{\sqrt{N_2}}c^0_{1j_1i_1t}\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left(\frac{\boldsymbol{\Lambda}^{0\prime}_{2j_2}\boldsymbol{\Lambda}^0_{2j_2}}{N_2}\right)^{-1}\left[\frac{1}{\sqrt{N_2}}\sum_{i_2=1}^{N_2}\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{\lambda}^0_{2j_2i_2}e_{2i_2t}\right] \\
& +\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\frac{1}{\sqrt{T}}c^0_{1j_1i_1t}\boldsymbol{f}^{0\prime}_{2j_2t}\left[\frac{\boldsymbol{F}^0_{2j_2}\left(\theta_2^0\right)\boldsymbol{F}^0_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{i_2=1}^{N_2}\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}e_{2i_2t}\right] \\
& +\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\frac{1}{\sqrt{N_1}}c^0_{2j_2i_2t}\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left(\frac{\boldsymbol{\Lambda}^{0\prime}_{1j_1}\boldsymbol{\Lambda}^0_{1j_1}}{N_1}\right)^{-1}\left[\frac{1}{\sqrt{N_1}}\sum_{i_1=1}^{N_1}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{\lambda}^0_{1j_1i_1}e_{1i_1t}\right] \\
& +\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\frac{1}{\sqrt{T}}c^0_{2j_2i_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\left[\frac{\boldsymbol{F}^0_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}^0_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{i_1=1}^{N_1}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbf{f}^0_{1j_1t}e_{1i_1t}\right] \\
& +O_p\left(\frac{1}{C^2_{NT}}\right)\Bigg\}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\sqrt{T}\hat{c}_{j_1j_2i_1i_2} \quad = \quad & \left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)+O_p\left(\frac{1}{T}\right)\right]^{-1}\times\left[1+O_p\left(\frac{1}{T}\right)\right] \\
& \times\left\{\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)c^0_{j_1j_2i_1i_2t}\right]\right. \\
& +\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\frac{1}{\sqrt{N_2}}c^0_{1j_1i_1t}\right]\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left(\frac{\boldsymbol{\Lambda}^{0\prime}_{2j_2}\boldsymbol{\Lambda}^0_{2j_2}}{N_2}\right)^{-1}\left[\frac{1}{\sqrt{N_2}}\sum_{i_2=1}^{N_2}\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{\lambda}^0_{2j_2i_2}e_{2i_2t}\right] \\
& +\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\frac{1}{\sqrt{T}}c^0_{1j_1i_1t}\boldsymbol{f}^{0\prime}_{2j_2t}\right]\left[\frac{\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}e_{2i_2t}\right] \\
& +\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\frac{1}{\sqrt{N_1}}c^0_{2j_2i_2t}\right]\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left(\frac{\boldsymbol{\Lambda}^{0\prime}_{1j_1}\boldsymbol{\Lambda}^0_{1j_1}}{N_1}\right)^{-1}\left[\frac{1}{\sqrt{N_1}}\sum_{i_1=1}^{N_1}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{\lambda}^0_{1j_1i_1}e_{1i_1t}\right] \\
& +\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\frac{1}{\sqrt{T}}c^0_{2j_2i_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\right] \\
& \left.+\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)O_p\left(\frac{1}{C^2_{NT}}\right)\right\} \qquad \text{(C.25)}
\end{aligned}$$

Noting that

$$\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)+O_p\left(\frac{1}{T}\right)\right]^{-1}\times\left[1+O_p\left(\frac{1}{T}\right)\right] \quad = \quad \left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\right]^{-1}+O_p\left(\frac{1}{T}\right) \qquad \text{(C.26)}$$

we have:

$$\begin{aligned}
& \left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)+O_p\left(\frac{1}{T}\right)\right]^{-1}\times\left[1+O_p\left(\frac{1}{T}\right)\right]\times\frac{\sqrt{T}}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)c^0_{j_1j_2i_1i_2} \\
= \; & \left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\right]^{-1}\frac{\sqrt{T}}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)c^0_{j_1j_2i_1i_2} \\
& +O_p\left(\frac{1}{T}\right)\cdot\frac{\sqrt{T}}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)c^0_{j_1j_2i_1i_2} \\
= \; & \sqrt{T}c^0_{j_1j_2i_1i_2}+O_p\left(1/\sqrt{T}\right) \; .
\end{aligned}$$

The last equation, together with equation (C.25) implies:

$$\begin{aligned}
\sqrt{T}\left(\hat{c}_{j_1 j_2 i_1 i_2} - c^0_{j_1 j_2 i_1 i_2}\right) &= \left[\frac{1}{T}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right) + O_p\left(\frac{1}{T}\right)\right]^{-1} \times \left[1 + O_p\left(\frac{1}{T}\right)\right] \\
&\times \Bigg\{ \sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\cdot\left(c^0_{j_1 j_2 i_1 i_2 t} - c^0_{j_1 j_2 i_1 i_2}\right)\right] \\
&+ \frac{1}{\sqrt{N_2}}\left[\sqrt{T}\frac{1}{T}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right) c^0_{1j_1i_1t}\right] \boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left(\frac{\boldsymbol{\Lambda}^{0\prime}_{2j_2}\boldsymbol{\Lambda}^0_{2j_2}}{N_2}\right)^{-1}\left[\frac{1}{\sqrt{N_2}}\sum_{i_2=1}^{N_2} \mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{\lambda}^0_{2j_2i_2} e_{2i_2t}\right] \\
&+ \left[\frac{1}{T}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right) c^0_{1j_1i_1t}\boldsymbol{f}^{0\prime}_{2j_2t}\right]\left[\frac{\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} \mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t} e_{2i_2t}\right] \\
&+ \frac{1}{\sqrt{N_1}}\left[\sqrt{T}\frac{1}{T}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right) c^0_{2j_2i_2t}\right] \boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left(\frac{\boldsymbol{\Lambda}^{0\prime}_{1j_1}\boldsymbol{\Lambda}^0_{1j_1}}{N_1}\right)^{-1}\left[\frac{1}{\sqrt{N_1}}\sum_{i_1=1}^{N_1} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{\lambda}^0_{1j_1i_1} e_{1i_1t}\right] \\
&+ \left[\frac{1}{T}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right) c^0_{2j_2i_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t} e_{1i_1t}\right] \\
&+ \sqrt{T}\frac{1}{T}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right) O_p\left(\frac{1}{C^2_{NT}}\right)\Bigg\} + O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

Using the definition of the common component $c^0_{gj_gi_gt} = \boldsymbol{\lambda}^{0\prime}_{gj_gi_g}\boldsymbol{f}^0_{gj_gt}$ and Assumptions A.1, A.6 and A.7 we get:

$$\begin{aligned}
\sqrt{T}\left(\hat{c}_{j_1 j_2 i_1 i_2} - c^0_{j_1 j_2 i_1 i_2}\right) &= \left[\frac{1}{T}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right) + O_p\left(\frac{1}{T}\right)\right]^{-1} \\
&\times \Bigg\{ \sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\cdot\left(c^0_{j_1 j_2 i_1 i_2 t} - c^0_{j_1 j_2 i_1 i_2}\right)\right] \\
&+ \boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left[\frac{1}{T}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{1j_1t}\boldsymbol{f}^{0\prime}_{2j_2t}\right]\left[\frac{\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} \mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t} e_{2i_2t}\right] \\
&+ \boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t} e_{1i_1t}\right] \\
&+ O_p\left(\frac{1}{\sqrt{N_2}}\right) + O_p\left(\frac{1}{\sqrt{N_1}}\right) + O_p\left(\frac{\sqrt{T}}{C^2_{NT}}\right)\Bigg\} + O_p\left(\frac{1}{\sqrt{T}}\right) \\
&+ O_p\left(\frac{1}{T}\right)\left[O_p(1) + O_p\left(\frac{1}{\sqrt{N_2}}\right) + O_p\left(\frac{1}{\sqrt{N_1}}\right) + O_p\left(\frac{\sqrt{T}}{C^2_{NT}}\right)\right]
\end{aligned}$$

which implies that

$$\sqrt{T}\left(\hat{c}_{j_1j_2i_1i_2}-c^0_{j_1j_2i_1i_2}\right) = \left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)+O_p\left(\frac{1}{T}\right)\right]^{-1}$$

$$\times\left\{\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\cdot\left(c^0_{j_1j_2i_1i_2t}-c^0_{j_1j_2i_1i_2}\right)\right]\right.$$

$$+\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{1j_1t}\boldsymbol{f}^{0\prime}_{2j_2t}\right]\left[\frac{\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}e_{2i_2t}\right]$$

$$\left.+\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\right]\right\}$$

$$+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{N_1}}\right)+O_p\left(\frac{1}{\sqrt{N_2}}\right)+O_p\left(\frac{\sqrt{T}}{C^2_{NT}}\right). \tag{C.27}$$

We thus have

$$\sqrt{T}\left[\left(\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{1i_1}w_{2i_2}\hat{c}_{j_1j_2i_1i_2}\right)-\left(\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{1i_1}w_{2i_2}c^0_{j_1j_2i_1i_2}\right)\right]$$

$$= \left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)+O_p\left(\frac{1}{T}\right)\right]^{-1}$$

$$\times\left\{\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\left[\left(\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{1i_1}w_{2i_2}c^0_{j_1j_2i_1i_2t}\right)-\left(\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{1i_1}w_{2i_2}c^0_{j_1j_2i_1i_2}\right)\right]\right.$$

$$+\left(\sum_{i_1=1}^{N_1}w_{1i_1}\boldsymbol{\lambda}^0_{1j_1i_1}\right)'\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\mathbf{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]\left[\frac{\mathbf{F}_{2j_2}\left(\theta_2^0\right)\mathbf{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\sum_{i_2=1}^{N_2}w_{2i_2}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\mathbf{f}^0_{2j_2t}e_{2i_2t}\right]$$

$$\left.+\left(\sum_{i_2=1}^{N_2}w_{2i_2}\boldsymbol{\lambda}^0_{2j_2i_2}\right)'\left[\frac{\mathbf{F}_{2j_2}\left(\theta_2^0\right)\mathbf{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\mathbf{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\sum_{i_1=1}^{N_1}w_{1i_1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbf{f}^0_{1j_1t}e_{1i_1t}\right]\right\}$$

$$+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{N_1}}\right)+O_p\left(\frac{1}{\sqrt{N_2}}\right)+O_p\left(\frac{\sqrt{T}}{C^2_{NT}}\right), \tag{C.28}$$

and the result stated in the theorem follows from $C_{NT}\to\infty$ with $\sqrt{T}/N\to 0$, and Assumptions A.1 - A.3 and A.5 - A.12.

■

C.4 Proof of Theorem 3

Definition (18), and equations (C.23) - (C.24) imply:

$$
\begin{aligned}
\widehat{\sigma}^2_{\boldsymbol{x}gj_1j_2i_g}\left(\hat{\theta}_1,\hat{\theta}_2\right) &= T_{j_1j_2}\left(\hat{\theta}_1,\hat{\theta}_2\right)^{-1}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\hat{\theta}_1\right)\mathbb{I}_{2j_2t}\left(\hat{\theta}_2\right)x^2_{gi_gt} \\
&= T_{j_1j_2}\left(\hat{\theta}_1,\hat{\theta}_2\right)^{-1}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\hat{\theta}_1\right)\mathbb{I}_{2j_2t}\left[\left(\hat{\theta}_2\right)(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2+(x^2_{gi_gt}-(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2)\right] \\
&= \left[\left(\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)\right)+O_p\left(1\right)\right]^{-1} \\
&\quad\times\sum_{t=1}^{T}\left[\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)+O_p\left(T^{-1}\right)\right]\cdot\left[(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2+(x^2_{gi_gt}-(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2)\right]. \\
&= \frac{1+O_p\left(1/T\right)}{\left(\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)\right)+O_p\left(1\right)} \\
&\quad\times\left\{\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2 \quad + \quad \sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)\left[x^2_{gi_gt}-(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2\right]\right\} .
\end{aligned}
$$

From the last equation it follows:

$$
\begin{aligned}
&\sqrt{T}\widehat{\sigma}^2_{\boldsymbol{x}gj_1j_2i_g}\left(\hat{\theta}_1,\hat{\theta}_2\right)-\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2\frac{1+O_p\left(1/T\right)}{\frac{\sum_{t=1}^{T}\mathbb{I}_{1j_1t}(\theta^0_1)\mathbb{I}_{2j_2t}(\theta^0_2)}{T}+O_p\left(1/T\right)} \\
&= \frac{1+O_p\left(1/T\right)}{\frac{\sum_{t=1}^{T}\mathbb{I}_{1j_1t}(\theta^0_1)\mathbb{I}_{2j_2t}(\theta^0_2)}{T}+O_p\left(1/T\right)}\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)\left[x^2_{gi_gt}-(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2\right].
\end{aligned}
$$

From equation (C.26) we have:

$$
\begin{aligned}
&\sqrt{T}\widehat{\sigma}^2_{\boldsymbol{x}gj_1j_2i_g}\left(\hat{\theta}_1,\hat{\theta}_2\right)-\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2\left[\frac{\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)}{T}\right]^{-1} \\
&= \left[\frac{\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)}{T}\right]^{-1}\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)\left[x^2_{gi_gt}-(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2\right]+O_p\left(1/\sqrt{T}\right)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\sqrt{T}\left(\widehat{\sigma}^2_{\boldsymbol{x}gj_1j_2i_g}\left(\hat{\theta}_1,\hat{\theta}_2\right)-(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2\right) \\
&= \left(\frac{\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)}{T}\right)^{-1}\sqrt{T}\left\{\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta^0_1\right)\mathbb{I}_{2j_2t}\left(\theta^0_2\right)\left[x^2_{gi_gt}-(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2\right]\right\}+O_p\left(1/\sqrt{T}\right) ,
\end{aligned}
$$

and:

$$\widehat{\sigma}^2_{\boldsymbol{x}gj_1j_2i_g}\left(\hat{\theta}_1,\hat{\theta}_2\right)$$
$$= (\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2 + \frac{1}{\sqrt{T}}\left(\frac{\sum_{t=1}^T \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)}{T}\right)^{-1}\sqrt{T}\left\{\frac{1}{T}\sum_{t=1}^T \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\left[x^2_{gi_gt} - (\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2\right]\right\}$$
$$+O_p\left(\frac{1}{T}\right). \tag{C.29}$$

Equation (C.27) implies:

$$\hat{c}_{j_1j_2i_1i_2}$$
$$= c^0_{j_1j_2i_1i_2} \quad + \quad \frac{1}{\sqrt{T}}\left[\frac{\sum_{t=1}^T \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)}{T}\right]^{-1} \times \left\{\frac{1}{\sqrt{T}}\sum_{t=1}^T \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\left[c^0_{j_1j_2i_1i_2t} - c^0_{j_1j_2i_1i_2}\right]\right.$$
$$+\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left[\frac{1}{T}\sum_{t=1}^T \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{1j_1t}\boldsymbol{f}^{0\prime}_{2j_2t}\right]\left[\frac{\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^T \mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}e_{2i_2t}\right]$$
$$\left.+\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{t=1}^T \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^T \mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\right]\right\}$$
$$+o_p\left(\frac{1}{\sqrt{T}}\right). \tag{C.30}$$

A first order expansion of $\widehat{R}_{j_1j_2i_1i_2}$ defined in equation (17), together with equations (C.29) and

(C.30), implies:

$$\widehat{R}_{j_1 j_2 i_1 i_2} = \frac{\hat{c}_{j_1 j_2 i_1 i_2}}{\hat{\sigma}_{\boldsymbol{x}1 j_1 j_2 i_1}\hat{\sigma}_{\boldsymbol{x}2 j_1 j_2 i_2}}$$

$$= \frac{c^0_{j_1 j_2 i_1 i_2}}{\sigma^0_{\boldsymbol{x}1 j_1 j_2 i_1}\sigma^0_{\boldsymbol{x}2 j_1 j_2 i_2}}$$

$$+\frac{1}{\sigma^0_{\boldsymbol{x}1 j_1 j_2 i_1}\sigma^0_{\boldsymbol{x}2 j_1 j_2 i_2}}\frac{1}{\sqrt{T}}\left[\frac{\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)}{T}\right]^{-1} \times \left\{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\left[c^0_{j_1 j_2 i_1 i_2 t} - c^0_{j_1 j_2 i_1 i_2}\right]\right.$$

$$+\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{1j_1t}\boldsymbol{f}^{0\prime}_{2j_2t}\right]\left[\frac{\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}e_{2i_2t}\right]$$

$$\left.+\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\right]\right\}$$

$$-\frac{c^0_{j_1 j_2 i_1 i_2}}{2(\sigma^0_{\boldsymbol{x}1 j_1 j_2 i_1})^3\sigma^0_{\boldsymbol{x}2 j_1 j_2 i_2}}\frac{1}{\sqrt{T}}\left(\frac{\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)}{T}\right)^{-1}\sqrt{T}\left\{\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\left[x^2_{1i_1t} - (\sigma^0_{\boldsymbol{x}1 j_1 j_2 i_1})^2\right]\right\}$$

$$-\frac{c^0_{j_1 j_2 i_1 i_2}}{2\sigma^0_{\boldsymbol{x}1 j_1 j_2 i_1}(\sigma^0_{\boldsymbol{x}2 j_1 j_2 i_2})^3}\frac{1}{\sqrt{T}}\left(\frac{\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)}{T}\right)^{-1}\sqrt{T}\left\{\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\left[x^2_{2i_2t} - (\sigma^0_{\boldsymbol{x}2 j_1 j_2 i_2})^2\right]\right\}$$

$$+o_p\left(\frac{1}{\sqrt{T}}\right).$$

The last equation and the definition of $R^0_{j_1 j_2 i_1 i_2}$ imply:

$$\sqrt{T}\left(\widehat{R}_{j_1 j_2 i_1 i_2} - R^0_{j_1 j_2 i_1 i_2}\right)$$

$$= \frac{1}{\sigma^0_{\boldsymbol{x}1 j_1 j_2 i_1}\sigma^0_{\boldsymbol{x}2 j_1 j_2 i_2}}\left[\frac{\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)}{T}\right]^{-1}\left\{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\left[c^0_{j_1 j_2 i_1 i_2 t} - c^0_{j_1 j_2 i_1 i_2}\right]\right.$$

$$+\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{1j_1t}\boldsymbol{f}^{0\prime}_{2j_2t}\right]\left[\frac{\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}e_{2i_2t}\right]$$

$$+\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\right]$$

$$-\frac{c^0_{j_1 j_2 i_1 i_2}}{2(\sigma^0_{\boldsymbol{x}1 j_1 j_2 i_1})^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\left[x^2_{1i_1t} - (\sigma^0_{\boldsymbol{x}1 j_1 j_2 i_1})^2\right]$$

$$\left.-\frac{c^0_{j_1 j_2 i_1 i_2}}{2(\sigma^0_{\boldsymbol{x}2 j_1 j_2 i_2})^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\left[x^2_{2i_2t} - (\sigma^0_{\boldsymbol{x}2 j_1 j_2 i_2})^2\right]\right\} + o_p\left(1\right).$$

Therefore we get:

$$
\begin{aligned}
&\sqrt{T}\left(\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{1i_1}w_{2i_2}\widehat{R}_{j_1j_2i_1i_2} - \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{1i_1}w_{2i_2}R^0_{j_1j_2i_1i_2}\right)\\
&= \left[\frac{\sum\limits_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)}{T}\right]^{-1}\left\{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{1i_1}w_{2i_2}\frac{c^0_{j_1j_2i_1i_2t}-c^0_{j_1j_2i_1i_2}}{\sigma^0_{\boldsymbol{x}1j_1j_2i_1}\sigma^0_{\boldsymbol{x}2j_1j_2i_2}}\right.\\
&+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{1i_1}w_{2i_2}\frac{1}{\sigma^0_{\boldsymbol{x}1j_1j_2i_1}\sigma^0_{\boldsymbol{x}2j_1j_2i_2}}\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left[\frac{1}{T}\sum_{s=1}^{T}\mathbb{I}_{1j_1s}\left(\theta_1^0\right)\mathbb{I}_{2j_2s}\left(\theta_2^0\right)\boldsymbol{f}^0_{1j_1s}\boldsymbol{f}^{0\prime}_{2j_2s}\right]\\
&\times\left[\frac{\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\boldsymbol{f}^0_{2j_2t}e_{2i_2t}\\
&+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{1i_1}w_{2i_2}\frac{1}{\sigma^0_{\boldsymbol{x}1j_1j_2i_1}\sigma^0_{\boldsymbol{x}2j_1j_2i_2}}\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{s=1}^{T}\mathbb{I}_{1j_1s}\left(\theta_1^0\right)\mathbb{I}_{2j_2s}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2s}\boldsymbol{f}^{0\prime}_{1j_1s}\right]\\
&\times\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\\
&-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{1i_1}w_{2i_2}\frac{c^0_{j_1j_2i_1i_2}}{2(\sigma^0_{\boldsymbol{x}1j_1j_2i_1})^3\ \sigma^0_{\boldsymbol{x}2j_1j_2i_2}}\left[x^2_{1i_1t}-(\sigma^0_{\boldsymbol{x}1j_1j_2i_1})^2\right]\\
&\left.-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{1i_1}w_{2i_2}\frac{c^0_{j_1j_2i_1i_2}}{2(\sigma^0_{\boldsymbol{x}2j_1j_2i_2})^3\ \sigma^0_{\boldsymbol{x}1j_1j_2i_1}}\left[x^2_{2i_2t}-(\sigma^0_{\boldsymbol{x}2j_1j_2i_2})^2\right]\right\}+o_p\left(1\right).
\end{aligned}
$$

Using the definition $w_{\boldsymbol{\sigma},gj_1j_2i_g}=\frac{w_{gi_g}}{\sigma^0_{\boldsymbol{x}gj_1j_2i_g}}$, with $g=1,\ 2$, provided in Theorem 3, we can rewrite the last equation as:

$$
\begin{aligned}
&\sqrt{T}\left(\widehat{R}_{j_1j_2\boldsymbol{w}_{1i_1}\boldsymbol{w}_{2i_2}}-R^0_{j_1j_2\boldsymbol{w}_{1i_1}\boldsymbol{w}_{2i_2}}\right)\\
&= \left[\frac{\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)}{T}\right]^{-1}\left\{\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{\boldsymbol{\sigma},1i_1}w_{\boldsymbol{\sigma},2i_2}\left(c^0_{j_1j_2i_1i_2t}-c^0_{j_1j_2i_1i_2}\right)\right.\\
&+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{\boldsymbol{\sigma},1j_1j_2i_1}w_{\boldsymbol{\sigma},2j_1j_2i_2}\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left[\frac{1}{T}\sum_{s=1}^{T}\mathbb{I}_{1j_1s}\left(\theta_1^0\right)\mathbb{I}_{2j_2s}\left(\theta_2^0\right)\boldsymbol{f}^0_{1j_1s}\boldsymbol{f}^{0\prime}_{2j_2s}\right]\left[\frac{\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\boldsymbol{f}^0_{2j_2t}e_{2i_2t}\\
&+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{\boldsymbol{\sigma},1j_1j_2i_1}w_{\boldsymbol{\sigma},2j_1j_2i_2}\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{s=1}^{T}\mathbb{I}_{1j_1s}\left(\theta_1^0\right)\mathbb{I}_{2j_2s}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2s}\boldsymbol{f}^{0\prime}_{1j_1s}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\\
&\left.-\frac{1}{2\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{\boldsymbol{\sigma},1j_1j_2i_1}w_{\boldsymbol{\sigma},2j_1j_2i_2}c^0_{j_1j_2i_1i_2}\left[\frac{x^2_{1i_1t}}{(\sigma^0_{\boldsymbol{x}1j_1j_2i_1})^2}+\frac{x^2_{2i_2t}}{(\sigma^0_{\boldsymbol{x}2j_1j_2i_2})^2}-2\right]\right\}+o_p\left(1\right),
\end{aligned}
$$

or, equivalently:

$$
\begin{aligned}
&\sqrt{T}\left(\widehat{R}_{j_1j_2\boldsymbol{w}_{1i_1}\boldsymbol{w}_{2i_2}}-R^0_{j_1j_2\boldsymbol{w}_{1i_1}\boldsymbol{w}_{2i_2}}\right)\\
&=\quad\left[\frac{\sum_{t=1}^T\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)}{T}\right]^{-1}\\
&\times\Bigg\{\frac{1}{\sqrt{T}}\sum_{t=1}^T\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{\boldsymbol{\sigma},1j_1j_2i_1}w_{\boldsymbol{\sigma},2j_1j_2i_2}\left[c^0_{j_1j_2i_1i_2t}-\frac{1}{2}c^0_{j_1j_2i_1i_2}\left(\frac{x^2_{1i_1t}}{(\sigma^0_{\boldsymbol{x}1j_1j_2i_1})^2}+\frac{x^2_{2i_2t}}{(\sigma^0_{\boldsymbol{x}2j_1j_2i_2})^2}\right)\right]\\
&+\frac{1}{\sqrt{T}}\sum_{t=1}^T\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{\boldsymbol{\sigma},1j_1j_2i_1}w_{\boldsymbol{\sigma},2j_1j_2i_2}\boldsymbol{\lambda}^{0\prime}_{1j_1i_1}\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\mathbf{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]\left[\frac{\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)\boldsymbol{F}_{2j_2}\left(\theta_2^0\right)'}{T}\right]^{-1}\boldsymbol{f}^0_{2j_2t}e_{2i_2t}\\
&+\frac{1}{\sqrt{T}}\sum_{t=1}^T\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2}w_{\boldsymbol{\sigma},1j_1j_2i_1}w_{\boldsymbol{\sigma},2j_1j_2i_2}\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{\mathbf{F}_{2j_2}\left(\theta_2^0\right)\mathbf{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\Bigg\}\\
&+o_p\left(1\right).
\end{aligned}
\tag{C.31}
$$

In order to compute the asymptotic variance of the RHS of equation (C.31) we need to compute, for $g=1,2$, covariance terms like:

$$
\begin{aligned}
&Cov\Bigg(\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{t=1}^T\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^T\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\ ,\\
&\frac{c^0_{j_1j_2i_1i_2}}{2(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2}\frac{1}{\sqrt{T}}\sum_{t=1}^T\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)x^2_{gi_gt}\Bigg)\\
&=\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{t=1}^T\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\\
&\times\left\{\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T\sum_{v=1}^T\text{Cov}\left(\mathbb{I}_{1j_1v}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1v}e_{1i_1v}\ ,\ \frac{c^0_{j_1j_2i_1i_2}}{2(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)x^2_{gi_gt}\right)\right\}\\
&=\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{t=1}^T\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\\
&\times\Bigg\{\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T\sum_{v=1}^T\text{Cov}\left(\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\ ,\ \frac{c^0_{j_1j_2i_1i_2}}{2(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2}\mathbb{I}_{1j_1v}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)(\boldsymbol{\lambda}'_{gj_gi_g}\boldsymbol{f}_{gj_gv})^2\right)\\
&+\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T\sum_{v=1}^T\text{Cov}\left(\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\ ,\ \frac{c^0_{j_1j_2i_1i_2}}{2(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2}\mathbb{I}_{1j_1v}\left(\theta_1^0\right)\mathbb{I}_{2j_2v}\left(\theta_2^0\right)e^2_{g,i_gv}\right)\\
&+2\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T\sum_{v=1}^T\text{Cov}\left(\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\ ,\ \frac{c^0_{j_1j_2i_1i_2}}{2(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2}\mathbb{I}_{1j_1v}\left(\theta_1^0\right)\mathbb{I}_{2j_2v}\left(\theta_2^0\right)\boldsymbol{\lambda}'_{gj_gi_g}\boldsymbol{f}_{gj_gv}e_{g,i_gv}\right)\Bigg\}\\
&=\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{\mathbf{F}_{2j_2}\left(\theta_2^0\right)\mathbf{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\\
&\times\left\{\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T\sum_{v=1}^T\text{Cov}\left(\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\ ,\ \frac{c^0_{j_1j_2i_1i_2}}{(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2}\mathbb{I}_{1j_1v}\left(\theta_1^0\right)\mathbb{I}_{2j_2v}\left(\theta_2^0\right)\boldsymbol{\lambda}'_{gj_gi_g}\boldsymbol{f}_{gj_gv}e_{g,i_gv}\right)\right\},
\end{aligned}
$$

where the first equality follows from the definitions $x_{gi_gt} = \boldsymbol{\lambda}'_{gj_gi_g}\boldsymbol{f}_{gj_gt} + e_{g,i_gt}$, and the third equality follows from the independence of the factors from the innovations - that is Assumption A.12 - and the assumption that the factors have zero expected value, see Assumption A.1. Therefore we have

$$\begin{aligned}
&Cov\Bigg(\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\ ,\\
&\frac{c^0_{j_1j_2i_1i_2}}{2(\sigma^0_{\boldsymbol{x}gj_1j_2i_g})^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)x^2_{gi_gt}\Bigg)\\
&=\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\sum_{v=1}^{T}\mathrm{Cov}\Bigg(\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{\mathbf{F}_{2j_2}\left(\theta_2^0\right)\mathbf{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\ ,\\
&\frac{c^0_{j_1j_2i_1i_2}}{(\sigma^0_{\boldsymbol{x}1j_1j_2i_1})^2}\mathbb{I}_{1j_1v}\left(\theta_1^0\right)\mathbb{I}_{2j_2v}\left(\theta_2^0\right)\boldsymbol{\lambda}'_{1j_1i_1}\boldsymbol{f}_{1j_1v}e_{1,i_1v}\Bigg)
\end{aligned}$$

and, using again the independence across groups of the innovations - that is Assumption A.9, we have:

$$\begin{aligned}
&Cov\Bigg(\boldsymbol{\lambda}^{0\prime}_{2j_2i_2}\left[\frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)\boldsymbol{f}^0_{2j_2t}\boldsymbol{f}^{0\prime}_{1j_1t}\right]\left[\frac{\mathbf{F}_{1j_1}\left(\theta_1^0\right)\boldsymbol{F}_{1j_1}\left(\theta_1^0\right)'}{T}\right]^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\boldsymbol{f}^0_{1j_1t}e_{1i_1t}\ ,\\
&\frac{c^0_{j_1j_2i_1i_2}}{2(\sigma^0_{\boldsymbol{x}2j_1j_2i_2})^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbb{I}_{1j_1t}\left(\theta_1^0\right)\mathbb{I}_{2j_2t}\left(\theta_2^0\right)x^2_{2i_2t}\Bigg) \quad = \quad 0.
\end{aligned}$$

Using analogous arguments for al the covariances between the elements in the RHS of equation (C.31), the result of the theorem follows.

■

C.5 Proof of Theorem 4

Consider $\sqrt{T}\left[\left(\hat{c}_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2} - c^0_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2}\right) - \left(\hat{c}_{j_1^*j_2^*\boldsymbol{w}_1\boldsymbol{w}_2} - c^0_{j_1^*j_2^*\boldsymbol{w}_1\boldsymbol{w}_2}\right)\right]$. Under $\mathcal{H}_0^c$,

$$\sqrt{T}\left[\left(\hat{c}_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2} - c^0_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2}\right) - \left(\hat{c}_{j_1^*j_2^*\boldsymbol{w}_1\boldsymbol{w}_2} - c^0_{j_1^*j_2^*\boldsymbol{w}_1\boldsymbol{w}_2}\right)\right] = \sqrt{T}\left(\hat{c}_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2} - \hat{c}_{j_1^*j_2^*\boldsymbol{w}_1\boldsymbol{w}_2}\right),$$

and the asymptotic distribution under the null hypothesis follows from equation (C.28), Theorem 2 and Assumptions A.9 - A.12, and the computation of the covariance terms between $\sqrt{T}\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{1i_1}w_{2i_2}$ $\times(\hat{c}_{j_1j_2i_1i_2} - c^0_{j_1j_2i_1i_2})$ and $\sqrt{T}\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2} w_{1i_1}w_{2i_2}(\hat{c}_{j_1^*j_2^*i_1i_2} - c^0_{j_1^*j_2^*i_1i_2})$. Under $\mathcal{H}_1^c$, notice that

$$\lim_{T,N_1,N_2\to\infty}\Pr\left(\sqrt{T}\left|c^0_{j_1j_2\boldsymbol{w}_1\boldsymbol{w}_2} - c^0_{j_1^*j_2^*\boldsymbol{w}_1\boldsymbol{w}_2}\right| \to \infty\right) = 1,$$

which completes the proof of the theorem.

■

C.6 Proof of Theorem 5

Consider $\sqrt{T}\left[\left(\hat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}\right) - \left(\hat{R}_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2} - R^0_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}\right)\right]$. Under $\mathcal{H}_0^c$,

$$\sqrt{T}\left[\left(\hat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}\right) - \left(\hat{R}_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2} - R^0_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}\right)\right] = \sqrt{T}\left(\hat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - \hat{R}_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}\right),$$

and the asymptotic distribution under the null hypothesis follows from equation (C.31), Theorem 3 and Assumptions A.9 - A.12, and the computation of the covariance terms between $\sqrt{T}\left(\hat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}\right)$ and $\sqrt{T}\left(\hat{R}_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2} - R^0_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}\right)$. Under $\mathcal{H}_1^c$, notice that

$$\lim_{T, N_1, N_2 \to \infty} \Pr\left(\sqrt{T}\left|\hat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - \hat{R}_{j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}\right| \to \infty\right) = 1,$$

which completes the proof of the theorem.

■

D Monte Carlo: Tables

Table D.1 – Finite sample distribution of the recentered and standardized statistic $\hat{c}_{i_1 i_2 t}$ in Theorem 1 with $\pi_0 = 0.75$

$a_F = 0$		**Design 1**				**Design 2**				**Design 3**				**Design 4**			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.017	0.001	1.329	1.477	-0.035	-0.018	1.167	1.456	-0.063	-0.020	1.213	1.537	-0.035	-0.027	1.232	1.582
30	200	-0.055	-0.022	1.155	1.418	0.007	-0.002	1.377	1.456	-0.069	-0.015	1.167	1.478	-0.061	-0.043	1.169	1.471
30	500	-0.024	-0.019	1.205	1.442	0.010	0.025	1.090	1.406	-0.020	-0.026	1.102	1.393	-0.054	-0.039	1.135	1.508
50	100	-0.032	-0.022	1.181	1.488	-0.025	-0.008	1.175	1.466	-0.051	-0.022	1.218	1.480	-0.044	-0.027	1.173	1.485
50	200	-0.016	-0.042	1.103	1.431	-0.012	0.012	1.104	1.414	-0.059	-0.039	1.119	1.425	-0.037	-0.040	1.718	1.442
50	500	-0.010	0.006	1.068	1.407	0.004	-0.010	1.081	1.436	-0.049	-0.015	1.075	1.412	-0.020	-0.003	1.064	1.414
100	100	-0.025	-0.019	1.126	1.445	-0.031	-0.009	1.197	1.444	-0.040	-0.012	1.266	1.497	-0.049	-0.032	1.160	1.491
100	200	-0.027	-0.027	1.025	1.380	0.000	0.016	1.065	1.413	-0.007	-0.037	2.907	1.390	-0.032	-0.008	1.090	1.371
100	500	-0.011	0.018	1.037	1.377	-0.029	-0.033	1.025	1.387	-0.022	-0.020	1.085	1.365	-0.004	0.003	1.043	1.371
300	100	-0.045	-0.024	1.128	1.424	0.020	0.018	1.147	1.443	-0.003	0.022	1.277	1.428	-0.016	-0.019	1.200	1.422
300	200	-0.014	0.006	1.064	1.382	-0.008	-0.017	1.040	1.369	-0.016	0.001	1.103	1.385	-0.048	-0.021	1.104	1.449
300	500	-0.015	-0.005	1.038	1.374	-0.023	-0.016	1.011	1.366	-0.016	-0.005	1.036	1.390	0.001	0.002	1.017	1.326
500	100	0.005	0.001	1.159	1.473	-0.018	-0.017	1.139	1.456	-0.066	-0.021	1.174	1.449	-0.033	-0.009	1.153	1.479
500	500	0.003	0.001	1.022	1.364	-0.015	-0.013	1.024	1.333	-0.002	0.018	1.016	1.363	-0.032	-0.040	1.022	1.345
500	1000	-0.006	0.010	1.024	1.352	-0.007	-0.033	1.010	1.338	-0.025	-0.020	1.237	1.342	-0.019	0.003	1.070	1.343

$a_F = 0.5$		**Design 5**				**Design 6**				**Design 7**				**Design 8**			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.075	-0.053	1.230	1.474	-0.012	0.020	1.165	1.457	-0.051	-0.014	1.186	1.565	-0.062	-0.026	1.248	1.561
30	200	-0.052	-0.039	1.137	1.460	-0.038	-0.036	1.138	1.433	-0.032	-0.018	1.203	1.506	-0.071	-0.069	1.127	1.491
30	500	-0.027	-0.022	1.124	1.436	-0.020	-0.018	1.134	1.487	-0.076	-0.056	1.135	1.489	-0.056	-0.045	1.147	1.465
50	100	0.010	0.027	1.172	1.457	-0.068	-0.047	1.383	1.466	-0.026	-0.039	2.266	1.484	-0.041	-0.013	1.385	1.494
50	200	-0.010	0.005	1.104	1.420	-0.041	-0.029	1.095	1.432	-0.026	-0.003	1.091	1.445	-0.049	-0.039	1.196	1.487
50	500	-0.011	-0.002	1.062	1.412	-0.006	0.015	1.084	1.437	-0.017	-0.009	1.083	1.425	-0.006	0.025	1.069	1.395
100	100	-0.043	-0.036	1.141	1.454	-0.057	-0.054	1.315	1.470	-0.066	-0.032	1.248	1.485	-0.031	-0.021	1.176	1.489
100	200	-0.020	-0.005	1.089	1.404	-0.018	-0.030	1.091	1.399	0.008	0.018	1.071	1.407	-0.035	-0.025	1.094	1.424
100	500	0.001	0.008	1.041	1.358	-0.023	-0.009	1.045	1.416	-0.022	-0.005	1.047	1.402	-0.027	-0.013	1.067	1.410
300	100	-0.028	0.004	1.128	1.480	-0.044	-0.029	1.146	1.434	-0.023	0.028	1.279	1.538	-0.036	-0.006	1.202	1.479
300	200	-0.039	-0.024	1.077	1.420	0.022	0.028	1.069	1.401	-0.020	-0.013	1.110	1.422	-0.032	-0.023	1.083	1.385
300	500	0.008	0.002	1.022	1.383	-0.003	-0.012	1.020	1.371	0.010	0.026	1.029	1.347	-0.036	-0.017	1.036	1.369
500	100	-0.008	-0.011	1.158	1.488	-0.029	-0.006	1.183	1.484	-0.030	-0.007	1.204	1.476	-0.066	-0.045	1.209	1.446
500	500	-0.010	0.008	1.033	1.361	0.015	0.032	1.034	1.364	-0.045	-0.046	1.050	1.402	-0.016	-0.028	1.055	1.392
500	1000	0.012	0.001	1.019	1.366	-0.019	-0.007	1.009	1.375	-0.026	-0.016	1.024	1.380	-0.010	-0.003	1.001	1.349

This table reports the mean (*m.*), the median (*med.*), standard deviation (*std.*) and interquartile range (*iqr.*) of the empirical distribution of the recentered and standardized statistic $\hat{c}_{i_1 i_2 t}$ in Theorem 1 defined as: $c_{NT}\left(\hat{c}_{i_1 i_2 t} - c^0_{i_1 i_2 t}\right)/\sqrt{\hat{\boldsymbol{Q}}_{i_1 i_2 t}}$. The standardized statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters (Designs 1 - 8). The asymptotic distribution of the statistic is $N(0,1)$ under the Assumptions of Theorem 1 and has interquartile ≈ 1.349. The empirical distributions are obtained by recomputing the statistics with 4000 Monte Carlo simulations.

Table D.2 – Finite sample distribution of the recentered and standardized statistic $\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 2 with $\pi_0 = 0.75$

$a_F = 0$		Design 1				Design 2				Design 3				Design 4			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.121	-0.054	1.025	1.348	-0.147	-0.093	1.011	1.359	-0.214	-0.095	1.090	1.460	-0.194	-0.102	1.080	1.372
30	200	-0.080	-0.039	1.023	1.376	-0.100	-0.044	1.037	1.396	-0.138	-0.075	1.055	1.450	-0.140	-0.073	1.040	1.343
30	500	-0.073	-0.041	1.041	1.440	-0.051	-0.021	0.994	1.361	-0.109	-0.096	1.008	1.368	-0.104	-0.066	1.006	1.352
50	100	-0.154	-0.074	1.036	1.414	-0.134	-0.068	1.044	1.345	-0.195	-0.104	1.066	1.404	-0.197	-0.100	1.095	1.445
50	200	-0.107	-0.038	1.026	1.352	-0.087	-0.026	1.040	1.398	-0.131	-0.044	1.046	1.397	-0.131	-0.070	1.040	1.394
50	500	-0.078	-0.074	1.009	1.333	-0.034	-0.025	0.993	1.356	-0.095	-0.056	0.996	1.344	-0.092	-0.049	1.036	1.362
100	100	-0.152	-0.076	1.033	1.416	-0.153	-0.077	1.052	1.454	-0.205	-0.134	1.043	1.371	-0.209	-0.120	1.090	1.438
100	200	-0.085	-0.042	1.004	1.354	-0.085	-0.042	1.008	1.365	-0.159	-0.106	1.042	1.386	-0.142	-0.068	1.058	1.343
100	500	-0.069	-0.058	1.015	1.356	-0.046	-0.029	1.008	1.379	-0.069	-0.020	1.010	1.369	-0.092	-0.046	1.014	1.379
300	100	-0.126	-0.054	1.033	1.364	-0.106	-0.033	1.051	1.401	-0.198	-0.090	1.072	1.411	-0.193	-0.106	1.076	1.417
300	200	-0.107	-0.062	1.015	1.379	-0.142	-0.083	1.025	1.393	-0.148	-0.096	1.012	1.314	-0.155	-0.097	1.021	1.305
300	500	-0.034	-0.016	0.995	1.335	-0.070	-0.036	1.020	1.396	-0.094	-0.049	1.019	1.364	-0.081	-0.037	1.022	1.371
500	100	-0.137	-0.083	1.043	1.409	-0.109	-0.050	1.014	1.343	-0.196	-0.097	1.066	1.398	-0.179	-0.093	1.078	1.441
500	500	-0.071	-0.036	1.017	1.375	-0.100	-0.064	1.011	1.352	-0.086	-0.024	1.012	1.381	-0.065	-0.019	1.023	1.393
500	1000	-0.041	-0.007	0.991	1.339	-0.079	-0.076	1.009	1.336	-0.100	-0.081	1.035	1.386	-0.050	-0.013	1.024	1.381

$a_F = 0.5$		Design 5				Design 6				Design 7				Design 8			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.209	-0.108	1.164	1.601	-0.229	-0.130	1.168	1.574	-0.309	-0.171	1.209	1.548	-0.313	-0.175	1.214	1.578
30	200	-0.164	-0.118	1.114	1.547	-0.147	-0.080	1.088	1.476	-0.229	-0.119	1.185	1.525	-0.248	-0.160	1.148	1.498
30	500	-0.073	-0.031	1.079	1.479	-0.077	-0.033	1.087	1.414	-0.150	-0.094	1.090	1.471	-0.126	-0.065	1.112	1.506
50	100	-0.192	-0.096	1.136	1.561	-0.214	-0.141	1.115	1.506	-0.319	-0.204	1.217	1.588	-0.303	-0.179	1.213	1.567
50	200	-0.126	-0.071	1.096	1.455	-0.143	-0.060	1.112	1.472	-0.193	-0.089	1.141	1.461	-0.231	-0.140	1.145	1.524
50	500	-0.101	-0.070	1.082	1.475	-0.093	-0.049	1.074	1.424	-0.128	-0.060	1.118	1.491	-0.174	-0.092	1.096	1.460
100	100	-0.202	-0.120	1.150	1.584	-0.230	-0.155	1.139	1.559	-0.323	-0.197	1.211	1.558	-0.300	-0.165	1.210	1.595
100	200	-0.152	-0.081	1.117	1.456	-0.170	-0.108	1.101	1.493	-0.204	-0.116	1.120	1.532	-0.243	-0.145	1.131	1.485
100	500	-0.082	-0.034	1.082	1.468	-0.113	-0.060	1.054	1.438	-0.099	-0.048	1.082	1.486	-0.151	-0.092	1.084	1.424
300	100	-0.216	-0.126	1.120	1.521	-0.207	-0.123	1.145	1.562	-0.266	-0.169	1.173	1.499	-0.313	-0.178	1.220	1.604
300	200	-0.151	-0.082	1.082	1.475	-0.153	-0.080	1.113	1.472	-0.238	-0.133	1.157	1.476	-0.200	-0.080	1.127	1.488
300	500	-0.102	-0.052	1.081	1.451	-0.099	-0.038	1.089	1.467	-0.162	-0.102	1.103	1.497	-0.162	-0.101	1.102	1.459
500	100	-0.221	-0.132	1.138	1.532	-0.238	-0.164	1.153	1.512	-0.317	-0.177	1.214	1.595	-0.283	-0.141	1.202	1.555
500	500	-0.091	-0.083	1.070	1.436	-0.091	-0.050	1.076	1.448	-0.143	-0.091	1.105	1.483	-0.141	-0.066	1.071	1.464
500	1000	-0.062	-0.020	1.084	1.476	-0.090	-0.077	1.070	1.446	-0.087	-0.062	1.055	1.380	-0.078	-0.018	1.078	1.453

This table reports the mean (*m.*), the median (*med.*), standard deviation (*std.*) and interquartile range (*iqr.*) of the empirical distribution of the recentered and standardized statistic $\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 2 defined as: $\sqrt{T}\left(\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}\right) / \sqrt{\hat{\boldsymbol{Q}}_{j_1 j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2)}$, with $\boldsymbol{w}_1 = [1, 0, 0, ..., 0]$ and $\boldsymbol{w}_2 = [1, 0, 0, ..., 0]$. The standardized statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters (Designs 1 - 8). The asymptotic distribution of the statistic is $N(0, 1)$ under the Assumptions of Theorem 2 and has interquartile ≈ 1.349. The empirical distributions are obtained by recomputing the statistics with 4000 Monte Carlo simulations.

Table D.3 – Empirical size and power of the test of change in comovement based on $\widehat{\mathcal{T}}^R_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 5 with $\pi_0 = 0.75$ and $\pi_0 = 0.50$

$\pi_0 = 0.75$		**Design 1**				**Design 2**				**Design 3**				**Design 4**			
		size			**power**	**size**			**power**	**size**			**power**	**size**			**power**
$N_1 = N_2$	T	1%	5%	10%	5%	1%	5%	10%	5%	1%	5%	10%	5%	1%	5%	10%	5%
30	100	0.030	0.090	0.154	0.699	0.045	0.113	0.168	0.700	0.049	0.119	0.179	0.668	0.049	0.120	0.188	0.668
30	200	0.018	0.062	0.121	0.929	0.019	0.068	0.128	0.934	0.028	0.084	0.151	0.823	0.029	0.090	0.151	0.873
30	500	0.012	0.059	0.112	0.999	0.013	0.051	0.107	1.000	0.019	0.066	0.128	0.991	0.021	0.067	0.127	0.995
50	100	0.031	0.097	0.157	0.680	0.038	0.110	0.175	0.712	0.044	0.113	0.188	0.674	0.052	0.126	0.185	0.709
50	200	0.019	0.071	0.125	0.893	0.023	0.071	0.130	0.927	0.028	0.086	0.142	0.827	0.028	0.090	0.157	0.893
50	500	0.017	0.070	0.124	0.997	0.017	0.063	0.115	0.999	0.021	0.075	0.127	0.999	0.020	0.076	0.137	0.998
100	100	0.031	0.093	0.154	0.688	0.035	0.094	0.150	0.720	0.044	0.121	0.185	0.675	0.050	0.120	0.192	0.698
100	200	0.026	0.072	0.128	0.889	0.018	0.067	0.118	0.916	0.028	0.095	0.162	0.864	0.032	0.087	0.144	0.891
100	500	0.014	0.061	0.113	1.000	0.014	0.058	0.113	0.999	0.019	0.067	0.129	0.998	0.021	0.072	0.127	0.999
300	100	0.032	0.101	0.161	0.691	0.042	0.104	0.163	0.727	0.046	0.129	0.194	0.637	0.052	0.124	0.191	0.704
300	200	0.015	0.069	0.128	0.879	0.020	0.072	0.119	0.919	0.031	0.096	0.153	0.869	0.028	0.090	0.152	0.888
300	500	0.013	0.061	0.116	0.998	0.015	0.062	0.116	1.000	0.018	0.072	0.124	0.996	0.016	0.064	0.120	1.000
500	100	0.033	0.089	0.144	0.677	0.038	0.100	0.157	0.717	0.052	0.121	0.191	0.659	0.050	0.127	0.192	0.687
500	500	0.013	0.058	0.111	0.999	0.013	0.058	0.107	0.999	0.017	0.064	0.117	0.997	0.019	0.071	0.124	0.999
500	1000	0.013	0.051	0.105	1.000	0.012	0.052	0.105	1.000	0.012	0.059	0.117	1.000	0.013	0.062	0.122	1.000

$\pi_0 = 0.50$		**Design 1**				**Design 2**				**Design 3**				**Design 4**			
		size			**power**	**size**			**power**	**size**			**power**	**size**			**power**
$N_1 = N_2$	T	1%	5%	10%	5%	1%	5%	10%	5%	1%	5%	10%	5%	1%	5%	10%	5%
30	100	0.018	0.068	0.124	0.804	0.026	0.070	0.128	0.879	0.037	0.103	0.170	0.746	0.034	0.104	0.170	0.817
30	200	0.013	0.059	0.108	0.976	0.017	0.066	0.124	0.986	0.026	0.086	0.144	0.920	0.024	0.084	0.142	0.965
30	500	0.014	0.061	0.113	1.000	0.011	0.052	0.108	1.000	0.021	0.077	0.126	1.000	0.021	0.074	0.133	1.000
50	100	0.017	0.071	0.131	0.805	0.021	0.077	0.133	0.843	0.036	0.101	0.168	0.775	0.036	0.103	0.165	0.821
50	200	0.015	0.056	0.107	0.914	0.014	0.057	0.116	0.992	0.021	0.079	0.141	0.916	0.025	0.086	0.144	0.965
50	500	0.012	0.054	0.100	1.000	0.011	0.055	0.104	1.000	0.015	0.073	0.132	0.999	0.016	0.070	0.128	1.000
100	100	0.021	0.071	0.122	0.812	0.024	0.071	0.135	0.899	0.040	0.104	0.169	0.710	0.032	0.105	0.166	0.852
100	200	0.011	0.061	0.112	0.975	0.015	0.057	0.115	0.989	0.024	0.084	0.148	0.946	0.026	0.082	0.142	0.984
100	500	0.014	0.055	0.105	1.000	0.010	0.051	0.101	1.000	0.018	0.067	0.127	1.000	0.020	0.072	0.122	1.000
300	100	0.021	0.065	0.121	0.798	0.019	0.073	0.126	0.861	0.033	0.094	0.161	0.741	0.038	0.106	0.172	0.822
300	200	0.014	0.059	0.111	0.970	0.016	0.060	0.114	0.987	0.027	0.085	0.148	0.930	0.027	0.084	0.145	0.964
300	500	0.011	0.049	0.105	1.000	0.012	0.055	0.107	1.000	0.017	0.075	0.130	0.999	0.016	0.067	0.124	1.000
500	100	0.019	0.074	0.130	0.770	0.020	0.070	0.120	0.865	0.035	0.100	0.157	0.727	0.041	0.105	0.162	0.826
500	500	0.013	0.057	0.105	1.000	0.013	0.058	0.105	1.000	0.021	0.069	0.126	1.000	0.018	0.068	0.129	1.000
500	1000	0.011	0.049	0.099	1.000	0.008	0.053	0.113	1.000	0.017	0.068	0.122	1.000	0.015	0.064	0.118	1.000

This table reports the empirical size and power of the two-tailed test of a change in comovement across regimes based on the test statistic $\widehat{\mathcal{T}}^R_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 5, with $\boldsymbol{w}_1 = [1/N_1, ..., 1/N_1]$ and $\boldsymbol{w}_2 = [1/N_2, ..., 1/N_2]$. The statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters under the null hypothesis of no change in comovement (Designs 1 H_0 - 4 H_0) and under the alternative of change in comovement (Designs 1 H_1 - 4 H_1). The empirical size is assessed at the α levels of 1%, 5% and 10% using Designs 1 H_0 - 4 H_0, while the empirical power is assessed at the α level of 5% using Designs 1 H_1 - 4 H_1. The empirical distributions are obtained by recomputing the statistics with 4000 MC simulations.

Supplementary Material
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E Monte Carlo: Additional Results

This section provides additional MC results.

Table E.1 – Parameters of Monte Carlo simulation designs for Theorems 1 and 2

Design / Param.	$K_L^C = K_H^C$	$K_L^s = K_H^s$	$\delta_{i_g}^c = \delta_{i_g}^s$	c_{gj}^*	β	a_F	a_z	π_0
Design 9	0	3	0.25	**1.0**	0	0	0	0.75
Design 10	0	3	1.00	**1.0**	0	0	0	0.75
Design 11	0	3	0.25	0.5	**0.2**	0	0	0.75
Design 12	0	3	1.00	0.5	**0.2**	0	0	0.75
Design 13	0	3	0.25	0.5	0	0	0	**0.50**
Design 14	0	3	1.00	0.5	0	0	0	**0.50**
Design 15	0	3	0.25	0.5	0	0	**0.50**	0.75
Design 16	0	3	1.00	0.5	0	0	**0.50**	0.75
Design 17	0	**1**	0.25	0.5	0	0	0	0.75
Design 18	0	**1**	1.00	0.5	0	0	0	0.75
Design 19	**1**	0	0.25	0.5	0	0	0	0.75
Design 20	**1**	0	1.00	0.5	0	0	0	0.75

Table E.1 provides values of the parameters in the DGP described in Section 5 for each of the MC simulation designs considered in the Online Appendix and used to assess the properties of the statistics in Theorems 1 and 2. In all simulation designs we also set $\sigma_{gj}^c = \sigma_{gj}^s = 1$ and $\alpha = 1$. In Designs 9 - 16, $\Phi_L^s = diag(0.4,\ 0.2,\ 0.1)$ and $\Phi_H^s = diag(0.8,\ 0.4,\ 0.2)$. In Designs 17 - 18, Φ_L^s and Φ_H^s reduce to two scalar parameters that we assume to be $\Phi_L^s = 0.4$, $\Phi_H^s = 0.8$. In Designs 19 - 20, Φ_L^s and Φ_H^s are empty matrices.

Table E.2 – Finite sample distribution of the recentered and standardized statistic $\widehat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 3 with $\pi_0 = 0.75$

$a_F = 0$		Design 1				Design 2				Design 3				Design 4			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	0.008	0.010	1.057	1.413	-0.011	-0.013	1.062	1.407	-0.019	-0.014	1.080	1.394	-0.039	-0.023	1.068	1.449
30	200	0.003	-0.017	1.032	1.358	-0.024	-0.041	1.043	1.407	-0.012	0.006	1.027	1.384	-0.026	-0.004	1.035	1.355
30	500	0.005	-0.001	1.015	1.346	0.008	-0.001	1.025	1.370	-0.031	-0.042	0.997	1.356	-0.015	-0.007	1.029	1.393
50	100	0.013	0.014	1.064	1.381	-0.034	-0.016	1.062	1.396	-0.052	-0.037	1.090	1.418	-0.018	0.037	1.067	1.350
50	200	-0.011	-0.002	1.053	1.388	0.002	-0.018	1.025	1.375	-0.000	0.006	1.035	1.381	-0.003	-0.003	1.036	1.442
50	500	0.027	0.027	1.015	1.343	-0.003	0.023	1.005	1.368	-0.015	-0.026	1.017	1.348	-0.017	-0.026	1.023	1.333
100	100	-0.027	-0.020	1.081	1.430	-0.009	0.021	1.071	1.423	-0.043	-0.034	1.076	1.447	-0.069	-0.049	1.067	1.362
100	200	-0.005	0.016	1.026	1.346	0.007	0.023	1.056	1.407	-0.019	-0.008	1.035	1.378	-0.022	-0.006	1.023	1.351
100	500	-0.013	-0.021	1.038	1.433	0.008	-0.011	1.020	1.379	-0.010	-0.017	1.025	1.383	-0.033	-0.020	1.026	1.356
300	100	-0.001	0.009	1.063	1.427	0.006	0.025	1.043	1.388	-0.011	-0.002	1.064	1.405	-0.008	0.000	1.074	1.359
300	200	0.019	0.022	1.033	1.369	-0.031	-0.036	1.036	1.441	-0.035	-0.044	1.030	1.351	-0.031	-0.015	1.039	1.368
300	500	0.027	0.013	1.025	1.387	-0.004	-0.020	1.024	1.372	-0.004	0.014	1.001	1.345	-0.007	0.004	1.016	1.374
500	100	-0.018	-0.000	1.071	1.397	0.019	0.012	1.053	1.428	-0.049	-0.031	1.082	1.406	-0.029	0.007	1.098	1.456
500	500	-0.016	-0.020	1.024	1.355	-0.029	-0.015	1.021	1.378	-0.010	-0.010	1.009	1.400	-0.021	-0.025	1.015	1.332
500	1000	0.015	-0.006	1.020	1.336	-0.008	-0.014	1.017	1.375	-0.014	0.009	1.020	1.362	0.002	0.014	1.016	1.378

$a_F = 0.5$		Design 5				Design 6				Design 7				Design 8			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.048	-0.044	1.199	1.566	-0.004	0.005	1.199	1.550	-0.078	-0.061	1.201	1.564	-0.079	-0.039	1.206	1.572
30	200	-0.002	0.004	1.133	1.518	-0.049	-0.043	1.120	1.467	-0.046	-0.014	1.116	1.515	-0.032	-0.017	1.129	1.512
30	500	-0.026	-0.012	1.089	1.425	0.006	0.022	1.102	1.477	-0.045	-0.035	1.107	1.508	-0.025	-0.011	1.088	1.461
50	100	-0.014	-0.038	1.203	1.580	-0.024	-0.042	1.198	1.587	-0.107	-0.094	1.194	1.526	-0.106	-0.078	1.203	1.577
50	200	-0.048	-0.027	1.148	1.529	-0.017	-0.008	1.140	1.507	-0.060	-0.033	1.145	1.522	-0.048	-0.048	1.139	1.526
50	500	0.011	0.022	1.074	1.451	-0.035	-0.010	1.104	1.529	-0.055	-0.038	1.102	1.502	-0.023	-0.023	1.091	1.455
100	100	0.004	-0.014	1.210	1.628	-0.007	-0.024	1.210	1.566	-0.058	-0.044	1.194	1.588	-0.084	-0.051	1.209	1.535
100	200	-0.011	0.001	1.151	1.525	-0.016	-0.016	1.144	1.504	-0.072	-0.081	1.142	1.516	-0.036	0.009	1.126	1.515
100	500	-0.036	-0.027	1.079	1.455	0.033	0.025	1.068	1.477	-0.013	-0.012	1.096	1.473	-0.005	0.028	1.067	1.438
300	100	-0.051	-0.040	1.233	1.634	-0.010	-0.013	1.229	1.612	-0.093	-0.065	1.201	1.523	-0.071	-0.035	1.234	1.607
300	200	-0.043	-0.039	1.148	1.558	-0.003	-0.002	1.138	1.500	-0.033	0.006	1.159	1.538	-0.047	-0.023	1.134	1.500
300	500	-0.002	-0.022	1.083	1.440	-0.021	-0.023	1.081	1.444	-0.047	-0.024	1.093	1.465	-0.035	-0.028	1.077	1.468
500	100	0.022	0.036	1.211	1.594	-0.043	-0.018	1.216	1.589	-0.080	-0.041	1.213	1.532	-0.076	-0.035	1.216	1.569
500	500	-0.032	-0.010	1.067	1.415	-0.021	-0.007	1.091	1.483	-0.019	-0.006	1.092	1.470	-0.057	-0.067	1.091	1.474
500	1000	0.003	0.029	1.061	1.414	-0.016	-0.018	1.069	1.449	-0.044	-0.041	1.067	1.430	-0.038	-0.033	1.041	1.367

This table reports the mean (*m.*), the median (*med.*), standard deviation (*std.*) and interquartile range (*iqr.*) of the empirical distribution of the recentered and standardized statistic $\widehat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 3 defined as: $\sqrt{T}\left(\widehat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}\right) / \sqrt{\hat{\boldsymbol{Q}}_{R,j_1 j_2}(\hat{\boldsymbol{w}}_{\sigma,1j_1j_2}, \hat{\boldsymbol{w}}_{\sigma,2j_1j_2})}$, with $\boldsymbol{w}_1 = [1/N_1, ..., 1/N_1]$ and $\boldsymbol{w}_2 = [1/N_2, ..., 1/N_2]$. The standardized statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters (Designs 1 - 8). The asymptotic distribution of the statistic is $N(0,1)$ under the Assumptions of Theorem 3 and has interquartile ≈ 1.349. The empirical distributions are obtained by recomputing the statistics with 4000 Monte Carlo simulations.

Table E.3 – Finite sample distribution of the test statistic $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4 with $\pi_0 = 0.75$

		Design 1 H_0				**Design 2 H_0**				**Design 3 H_0**				**Design 4 H_0**			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.037	-0.036	1.024	1.448	-0.006	-0.004	1.014	1.426	0.007	-0.002	1.075	1.523	-0.021	-0.017	1.083	1.546
30	200	-0.005	-0.013	0.987	1.345	0.013	0.014	1.005	1.378	0.003	0.029	1.050	1.476	-0.011	-0.003	1.063	1.492
30	500	-0.030	-0.024	0.979	1.276	-0.015	-0.013	0.996	1.335	-0.014	0.005	1.046	1.450	0.004	0.017	1.031	1.404
50	100	-0.011	-0.020	1.006	1.421	-0.036	-0.053	1.027	1.443	-0.019	-0.016	1.089	1.510	-0.017	0.004	1.079	1.508
50	200	-0.007	0.010	1.010	1.363	-0.004	-0.016	0.999	1.387	-0.031	-0.039	1.054	1.455	-0.000	-0.005	1.046	1.459
50	500	0.008	0.005	1.013	1.391	-0.001	-0.016	1.018	1.417	-0.026	-0.025	1.041	1.395	0.011	0.018	1.030	1.391
100	100	0.041	0.039	1.003	1.383	-0.008	0.014	1.014	1.434	0.024	0.016	1.061	1.510	0.006	0.003	1.070	1.519
100	200	0.004	-0.003	1.003	1.365	0.022	0.035	1.013	1.368	0.010	-0.007	1.029	1.391	0.014	0.003	1.051	1.446
100	500	0.024	0.034	0.997	1.319	0.032	0.065	0.997	1.323	-0.034	-0.029	1.057	1.475	0.016	0.042	1.032	1.426
300	100	0.015	0.005	0.999	1.435	-0.003	0.003	1.007	1.409	0.000	0.000	1.053	1.404	-0.003	-0.004	1.083	1.537
300	200	-0.010	-0.005	1.000	1.418	0.004	0.020	1.005	1.397	0.006	-0.007	1.056	1.446	-0.006	-0.004	1.058	1.455
300	500	-0.000	0.026	0.983	1.335	0.015	0.024	0.988	1.330	0.007	0.025	1.039	1.372	-0.006	-0.021	1.031	1.397
500	100	-0.020	-0.028	1.021	1.439	-0.009	-0.008	1.007	1.418	-0.014	-0.021	1.081	1.515	0.022	0.009	1.068	1.524
500	500	0.014	0.013	1.013	1.386	0.016	-0.006	0.997	1.355	-0.016	-0.026	1.045	1.433	0.001	0.000	1.014	1.359
500	1000	0.011	0.022	0.989	1.326	0.006	0.011	0.998	1.357	-0.016	-0.049	1.015	1.403	0.018	0.014	1.011	1.400

This table reports the mean (*m.*), the median (*med.*), standard deviation (*std.*) and interquartile range (*iqr.*) of the empirical distribution of the test statistic $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4, with $\boldsymbol{w}_1 = [1, 0, 0, ..., 0]$ and $\boldsymbol{w}_2 = [1, 0, 0, ..., 0]$. The statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters under the null hypothesis of no change in comovement across regimes (Designs 1 H_0 - 4 H_0). The asymptotic distribution of the statistic is $N(0, 1)$ under the Assumptions of Theorem 4 and has interquartile ≈ 1.349. The empirical distributions are obtained by recomputing the statistics with 4000 MC simulations.

Table E.4 – Empirical size and power of the test of change in comovement based on $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4 with $\pi_0 = 0.75$

		Design 1 H_0 , H_1				**Design 2 H_0 , H_1**				**Design 3 H_0 , H_1**				**Design 4 H_0 , H_1**			
		size			**power**	**size**			**power**	**size**			**power**	**size**			**power**
$N_1 = N_2$	T	1%	5%	10%	5%	1%	5%	10%	5%	1%	5%	10%	5%	1%	5%	10%	5%
30	100	0.009	0.050	0.105	0.428	0.008	0.051	0.104	0.504	0.014	0.066	0.122	0.424	0.012	0.067	0.128	0.516
30	200	0.006	0.050	0.098	0.799	0.008	0.047	0.103	0.868	0.011	0.059	0.114	0.761	0.016	0.059	0.113	0.835
30	500	0.009	0.043	0.096	0.995	0.009	0.050	0.100	0.999	0.015	0.059	0.114	0.993	0.012	0.058	0.111	0.998
50	100	0.007	0.050	0.098	0.456	0.008	0.051	0.108	0.522	0.016	0.067	0.130	0.421	0.011	0.066	0.127	0.518
50	200	0.009	0.053	0.104	0.805	0.008	0.050	0.097	0.869	0.012	0.060	0.117	0.764	0.010	0.061	0.115	0.825
50	500	0.009	0.051	0.107	0.998	0.010	0.051	0.107	1.000	0.015	0.061	0.115	0.993	0.010	0.057	0.112	0.997
100	100	0.007	0.045	0.102	0.438	0.005	0.050	0.105	0.528	0.012	0.060	0.123	0.429	0.010	0.065	0.122	0.515
100	200	0.009	0.051	0.103	0.797	0.007	0.047	0.108	0.870	0.012	0.054	0.112	0.764	0.010	0.061	0.122	0.840
100	500	0.008	0.051	0.100	0.997	0.009	0.049	0.099	0.999	0.014	0.064	0.120	0.995	0.011	0.057	0.110	0.998
300	100	0.005	0.042	0.095	0.444	0.009	0.046	0.102	0.535	0.011	0.061	0.121	0.426	0.011	0.067	0.134	0.499
300	200	0.005	0.043	0.098	0.818	0.008	0.046	0.095	0.873	0.012	0.065	0.122	0.760	0.015	0.059	0.118	0.846
300	500	0.008	0.042	0.094	0.998	0.007	0.050	0.097	1.000	0.014	0.061	0.114	0.993	0.010	0.053	0.112	0.999
500	100	0.008	0.047	0.103	0.439	0.007	0.045	0.101	0.525	0.013	0.066	0.127	0.431	0.010	0.060	0.120	0.512
500	500	0.010	0.053	0.102	0.996	0.009	0.050	0.102	0.998	0.011	0.053	0.121	0.994	0.011	0.051	0.105	0.999
500	1000	0.009	0.049	0.098	1.000	0.007	0.051	0.100	1.000	0.010	0.050	0.105	1.000	0.007	0.052	0.107	1.000

This table reports the empirical size and power of the two-tailed test of a change in comovement across regimes based on the test statistic $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4, with $\boldsymbol{w}_1 = [1, 0, 0, ..., 0]$ and $\boldsymbol{w}_2 = [1, 0, 0, ..., 0]$. The statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters under the null hypothesis of no change in comovement (Designs 1 H_0 - 4 H_0) and under the alternative of change in comovement (Designs 1 H_1 - 4 H_1). The empirical size is assessed at the α levels of 1%, 5% and 10% using Designs 1 H_0 - 4 H_0, while the empirical power is assessed at the α level of 5% using Designs 1 H_1 - 4 H_1. The empirical distributions are obtained by recomputing the statistics with 4000 MC simulations.

Table E.5 – Finite sample distribution of the test statistic $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4 with $\pi_0 = 0.75$

		Design 1 H_0				**Design 2 H_0**				**Design 3 H_0**				**Design 4 H_0**			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.018	-0.029	1.063	1.446	-0.004	0.010	1.085	1.522	-0.004	-0.008	1.127	1.572	-0.007	-0.013	1.139	1.573
30	200	0.005	-0.017	1.016	1.402	0.008	0.012	1.041	1.405	0.029	0.036	1.097	1.495	0.007	0.007	1.084	1.483
30	500	-0.025	-0.023	1.020	1.387	0.007	0.003	1.030	1.388	-0.011	-0.008	1.046	1.442	-0.003	-0.013	1.042	1.360
50	100	0.006	-0.012	1.061	1.441	0.015	0.030	1.059	1.465	0.010	0.047	1.145	1.549	0.016	-0.014	1.126	1.556
50	200	0.002	-0.010	1.028	1.363	0.010	0.018	1.027	1.416	-0.018	-0.019	1.085	1.459	0.008	-0.015	1.086	1.476
50	500	0.005	-0.002	1.010	1.371	-0.034	-0.044	1.006	1.382	0.004	-0.003	1.065	1.443	0.001	-0.002	1.040	1.391
100	100	0.003	0.004	1.058	1.447	-0.005	-0.004	1.100	1.484	0.001	0.014	1.158	1.600	-0.032	-0.029	1.129	1.572
100	200	-0.006	0.011	1.037	1.443	-0.010	0.004	1.030	1.426	-0.004	-0.004	1.106	1.515	-0.025	-0.038	1.097	1.512
100	500	-0.017	-0.014	1.014	1.361	0.019	0.038	1.013	1.354	-0.009	-0.016	1.052	1.405	-0.004	-0.040	1.030	1.398
300	100	-0.005	-0.021	1.051	1.445	0.017	0.003	1.079	1.473	-0.012	-0.022	1.166	1.616	0.001	-0.000	1.149	1.578
300	200	-0.005	-0.010	1.028	1.412	0.006	0.034	1.051	1.452	0.031	0.031	1.094	1.482	-0.005	0.012	1.072	1.473
300	500	-0.005	-0.003	1.016	1.380	0.006	0.010	1.014	1.328	-0.001	0.020	1.060	1.409	-0.014	-0.020	1.073	1.426
500	100	-0.002	-0.003	1.052	1.432	-0.005	-0.010	1.085	1.466	-0.011	-0.001	1.133	1.534	-0.024	-0.018	1.149	1.610
500	500	-0.002	0.019	0.999	1.366	0.028	0.021	1.036	1.386	-0.007	-0.008	1.060	1.403	-0.002	-0.007	1.061	1.436
500	1000	-0.016	-0.012	0.995	1.362	-0.006	-0.015	1.028	1.427	0.005	0.020	1.039	1.419	-0.024	-0.029	1.035	1.407

This table reports the mean (*m.*), the median (*med.*), standard deviation (*std.*) and interquartile range (*iqr.*) of the empirical distribution of the test statistic $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4, with $\boldsymbol{w}_1 = [1/N_1, ..., 1/N_1]$ and $\boldsymbol{w}_2 = [1/N_2, ..., 1/N_2]$. The statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters under the null hypothesis of no change in comovement across regimes (Designs 1 H_0 - 4 H_0). The asymptotic distribution of the statistic is $N(0,1)$ under the Assumptions of Theorem 4 and has interquartile ≈ 1.349. The empirical distributions are obtained by recomputing the statistics with 4000 MC simulations.

Table E.6 – Empirical size and power of the test of change in comovement based on $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4 with $\pi_0 = 0.75$

		Design 1 H_0 , H_1				**Design 2 H_0 , H_1**				**Design 3 H_0 , H_1**				**Design 4 H_0 , H_1**			
		size			**power**	**size**			**power**	**size**			**power**	**size**			**power**
$N_1 = N_2$	T	1%	5%	10%	5%	1%	5%	10%	5%	1%	5%	10%	5%	1%	5%	10%	5%
30	100	0.014	0.065	0.124	0.512	0.013	0.070	0.128	0.661	0.021	0.077	0.141	0.544	0.022	0.083	0.144	0.585
30	200	0.009	0.054	0.103	0.781	0.011	0.060	0.112	0.894	0.017	0.072	0.135	0.831	0.018	0.066	0.127	0.842
30	500	0.012	0.057	0.105	0.994	0.013	0.058	0.113	1.000	0.013	0.060	0.119	0.988	0.014	0.064	0.118	0.999
50	100	0.015	0.059	0.120	0.589	0.014	0.061	0.119	0.590	0.027	0.086	0.145	0.474	0.022	0.078	0.147	0.530
50	200	0.013	0.060	0.110	0.885	0.009	0.053	0.107	0.916	0.019	0.072	0.128	0.740	0.017	0.072	0.130	0.911
50	500	0.010	0.052	0.100	0.999	0.011	0.050	0.096	1.000	0.014	0.066	0.124	0.983	0.012	0.058	0.114	1.000
100	100	0.014	0.060	0.120	0.616	0.017	0.074	0.135	0.599	0.022	0.087	0.155	0.514	0.022	0.079	0.139	0.642
100	200	0.014	0.054	0.107	0.850	0.011	0.054	0.108	0.933	0.018	0.074	0.140	0.832	0.018	0.072	0.131	0.858
100	500	0.010	0.052	0.103	0.999	0.012	0.054	0.103	1.000	0.014	0.066	0.114	0.995	0.014	0.054	0.105	1.000
300	100	0.012	0.061	0.118	0.589	0.015	0.068	0.129	0.638	0.026	0.095	0.155	0.535	0.021	0.085	0.155	0.628
300	200	0.014	0.056	0.106	0.901	0.015	0.061	0.111	0.926	0.018	0.072	0.134	0.833	0.016	0.069	0.124	0.898
300	500	0.012	0.051	0.102	0.999	0.014	0.057	0.105	0.999	0.017	0.066	0.124	0.996	0.019	0.066	0.129	0.998
500	100	0.012	0.058	0.115	0.562	0.014	0.071	0.130	0.628	0.022	0.085	0.145	0.549	0.020	0.087	0.153	0.590
500	500	0.010	0.049	0.105	0.999	0.012	0.061	0.120	1.000	0.018	0.067	0.121	0.997	0.015	0.062	0.116	0.998
500	1000	0.007	0.044	0.097	1.000	0.014	0.052	0.104	1.000	0.015	0.060	0.109	1.000	0.009	0.059	0.116	1.000

This table reports the empirical size and power of the two-tailed test of a change in comovement across regimes based on the test statistic $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4, with $\boldsymbol{w}_1 = [1/N_1, ..., 1/N_1]$ and $\boldsymbol{w}_2 = [1/N_2, ..., 1/N_2]$. The statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters under the null hypothesis of no change in comovement (Designs 1 H_0 - 4 H_0) and under the alternative of change in comovement (Designs 1 H_1 - 4 H_1). The empirical size is assessed at the α levels of 1%, 5% and 10% using Designs 1 H_0 - 4 H_0, while the empirical power is assessed at the α level of 5% using Designs 1 H_1 - 4 H_1. The empirical distributions are obtained by recomputing the statistics with 4000 MC simulations.

Table E.7 – Finite sample distribution of the test statistic $\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 5 with $\pi_0 = 0.75$

		Design 1 H_0				**Design 2 H_0**				**Design 3 H_0**				**Design 4 H_0**			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.045	-0.043	1.159	1.500	-0.009	0.003	1.238	1.520	-0.022	-0.022	1.271	1.606	0.006	0.007	1.284	1.602
30	200	0.004	0.009	1.068	1.447	0.004	0.010	1.089	1.458	-0.020	-0.017	1.156	1.543	-0.029	-0.027	1.157	1.539
30	500	0.006	0.016	1.043	1.406	0.011	0.011	1.016	1.328	-0.011	-0.014	1.081	1.489	-0.006	-0.013	1.075	1.430
50	100	0.028	0.010	1.166	1.518	0.020	-0.001	1.226	1.549	0.018	0.032	1.261	1.629	-0.040	-0.042	1.307	1.655
50	200	0.023	0.015	1.079	1.453	-0.014	-0.021	1.096	1.390	-0.007	-0.005	1.146	1.508	0.001	-0.002	1.166	1.526
50	500	0.014	0.021	1.069	1.428	0.005	-0.016	1.048	1.383	0.004	0.016	1.074	1.436	-0.010	0.012	1.115	1.471
100	100	0.003	0.009	1.180	1.508	-0.004	-0.020	1.187	1.487	-0.019	-0.035	1.268	1.679	-0.000	-0.011	1.299	1.623
100	200	0.033	0.031	1.084	1.360	0.033	0.030	1.070	1.417	-0.008	-0.005	1.168	1.503	0.021	0.035	1.150	1.478
100	500	-0.023	-0.018	1.036	1.374	0.003	0.000	1.038	1.370	0.013	0.021	1.087	1.469	0.002	0.017	1.087	1.442
300	100	0.014	0.007	1.186	1.566	0.030	0.038	1.233	1.553	0.017	0.021	1.292	1.693	-0.029	-0.021	1.297	1.619
300	200	0.046	0.057	1.068	1.420	0.002	0.007	1.091	1.474	-0.012	-0.012	1.177	1.532	0.035	0.046	1.158	1.475
300	500	0.020	0.025	1.032	1.354	-0.005	-0.007	1.041	1.368	0.033	0.025	1.077	1.412	0.002	0.008	1.072	1.430
500	100	0.012	0.009	1.156	1.486	0.015	0.008	1.195	1.481	-0.013	0.001	1.282	1.616	0.040	0.028	1.315	1.650
500	500	-0.003	-0.009	1.032	1.356	0.026	0.007	1.031	1.367	-0.044	-0.032	1.061	1.428	-0.004	-0.033	1.083	1.444
500	1000	-0.011	-0.006	1.014	1.369	-0.035	-0.052	1.020	1.372	-0.025	-0.001	1.052	1.435	-0.002	0.011	1.055	1.442

This table reports the mean (*m.*), the median (*med.*), standard deviation (*std.*) and interquartile range (*iqr.*) of the empirical distribution of the test statistic $\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 5, with $\boldsymbol{w}_1 = [1/N_1, ..., 1/N_1]$ and $\boldsymbol{w}_2 = [1/N_2, ..., 1/N_2]$. The statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters under the null hypothesis of no change in comovement across regimes (Designs 1 H_0 - 4 H_0). The asymptotic distribution of the statistic is $N(0,1)$ under the Assumptions of Theorem 4 and has interquartile ≈ 1.349. The empirical distributions are obtained by recomputing the statistics with 4000 MC simulations.

Table E.8 – Finite sample distribution of the test statistic $\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 5 with $\pi_0 = 0.5$

		Design 1 H_0				**Design 2 H_0**				**Design 3 H_0**				**Design 4 H_0**			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	0.001	-0.027	1.065	1.379	0.029	0.046	1.092	1.396	-0.002	-0.008	1.211	1.567	-0.013	-0.015	1.206	1.595
30	200	-0.017	-0.022	1.031	1.386	-0.003	-0.015	1.060	1.404	0.022	0.003	1.129	1.515	0.011	0.006	1.125	1.480
30	500	-0.010	-0.031	1.039	1.401	0.007	0.005	1.015	1.333	0.003	-0.007	1.091	1.460	0.013	0.014	1.099	1.467
50	100	0.002	0.008	1.086	1.437	0.004	0.031	1.109	1.447	0.001	-0.001	1.197	1.571	-0.001	-0.001	1.204	1.546
50	200	-0.024	-0.014	1.032	1.359	-0.012	-0.010	1.040	1.399	0.005	-0.011	1.102	1.444	0.020	0.028	1.131	1.490
50	500	0.008	-0.001	0.998	1.323	-0.001	0.011	1.009	1.333	-0.018	-0.022	1.088	1.469	0.006	-0.003	1.077	1.427
100	100	-0.014	-0.025	1.072	1.381	-0.004	0.004	1.102	1.437	-0.010	0.009	1.211	1.585	0.017	0.037	1.207	1.601
100	200	0.003	0.007	1.030	1.387	-0.013	-0.015	1.039	1.392	-0.001	0.006	1.130	1.496	0.007	0.014	1.130	1.465
100	500	-0.002	0.011	1.023	1.365	0.004	0.009	1.002	1.348	0.024	0.014	1.071	1.423	0.008	0.007	1.090	1.461
300	100	-0.002	-0.010	1.069	1.370	0.026	0.022	1.081	1.433	0.015	0.016	1.178	1.542	-0.015	-0.004	1.226	1.592
300	200	-0.003	-0.014	1.029	1.337	-0.014	-0.016	1.046	1.393	-0.046	-0.034	1.143	1.550	0.004	0.005	1.133	1.479
300	500	-0.005	-0.013	1.010	1.401	-0.007	-0.006	1.010	1.363	0.007	0.023	1.092	1.445	-0.003	0.015	1.082	1.476
500	100	0.014	0.008	1.093	1.430	0.025	0.033	1.064	1.389	0.004	0.001	1.188	1.558	0.008	0.011	1.233	1.590
500	500	-0.007	-0.002	1.017	1.363	0.016	0.021	1.024	1.381	-0.003	0.012	1.077	1.415	0.010	0.012	1.087	1.468
500	1000	0.001	0.004	1.008	1.359	0.014	0.033	1.008	1.339	-0.002	-0.004	1.077	1.450	0.005	-0.012	1.057	1.416

This table reports the mean (*m.*), the median (*med.*), standard deviation (*std.*) and interquartile range (*iqr.*) of the empirical distribution of the test statistic $\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 5, with $\boldsymbol{w}_1 = [1/N_1, ..., 1/N_1]$ and $\boldsymbol{w}_2 = [1/N_2, ..., 1/N_2]$. The statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters under the null hypothesis of no change in comovement across regimes (Designs 1 H_0 - 4 H_0). The asymptotic distribution of the statistic is $N(0,1)$ under the Assumptions of Theorem 4 and has interquartile ≈ 1.349. The empirical distributions are obtained by recomputing the statistics with 4000 MC simulations.

Table E.9 – Finite sample distribution of the recentered and standardized statistic $\hat{c}_{i_1 i_2 t}$ in Theorem 1

		Design 9				Design 10				Design 11				Design 12			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.062	-0.055	1.352	1.590	-0.067	-0.032	1.312	1.554	0.008	-0.013	1.303	1.638	-0.003	0.001	1.277	1.464
30	200	-0.009	0.002	1.234	1.465	-0.045	-0.015	1.474	1.534	-0.007	-0.007	1.176	1.461	-0.024	-0.051	1.229	1.557
30	500	-0.066	-0.040	1.233	1.468	-0.053	-0.071	1.682	1.501	-0.009	0.003	1.250	1.658	0.020	0.027	1.232	1.505
50	100	-0.007	0.003	1.233	1.539	-0.006	-0.001	1.303	1.524	0.004	0.016	1.201	1.520	-0.047	-0.014	1.466	1.597
50	200	-0.078	-0.045	1.378	1.453	-0.041	-0.011	1.144	1.391	-0.020	-0.013	1.202	1.549	-0.037	-0.033	1.136	1.493
50	500	0.034	-0.002	1.164	1.330	-0.004	-0.019	1.290	1.434	-0.044	-0.047	1.136	1.432	0.005	0.021	1.138	1.492
100	100	-0.039	-0.010	1.348	1.523	-0.028	-0.015	1.336	1.439	-0.004	-0.019	1.144	1.464	-0.037	-0.027	1.296	1.529
100	200	-0.038	-0.036	1.129	1.371	-0.040	-0.023	1.190	1.430	-0.024	-0.005	1.224	1.475	-0.017	-0.008	1.122	1.432
100	500	0.047	0.040	1.063	1.359	-0.031	-0.018	1.075	1.363	0.002	0.012	1.077	1.415	-0.029	-0.015	1.082	1.415
300	100	-0.018	0.006	1.276	1.427	-0.026	-0.006	1.306	1.378	-0.023	-0.016	1.158	1.438	-0.023	-0.026	1.130	1.461
300	200	-0.026	0.002	1.066	1.392	-0.014	0.014	1.096	1.436	-0.006	-0.011	1.079	1.412	-0.030	-0.038	1.060	1.417
300	500	0.004	0.008	1.042	1.326	0.006	-0.014	1.069	1.421	-0.014	-0.004	1.070	1.441	-0.000	0.000	1.059	1.430
500	100	-0.025	-0.021	1.174	1.420	-0.042	-0.026	1.204	1.439	-0.035	-0.025	1.172	1.421	-0.019	-0.014	1.075	1.426
500	500	0.008	0.012	1.045	1.386	-0.043	-0.032	1.044	1.393	0.043	0.055	1.086	1.397	-0.016	-0.001	1.086	1.460
500	1000	0.011	0.038	1.023	1.372	-0.005	-0.014	1.055	1.383	0.001	-0.003	1.070	1.459	-0.022	-0.002	1.050	1.410

		Design 13				Design 14				Design 15				Design 16			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.003	-0.018	1.387	1.546	-0.000	-0.005	1.199	1.515	-0.003	0.005	1.230	1.507	-0.049	-0.042	1.173	1.479
30	200	0.005	-0.002	1.152	1.513	-0.009	-0.010	1.139	1.447	-0.042	-0.011	1.239	1.442	-0.007	0.011	1.132	1.497
30	500	-0.015	-0.007	1.150	1.436	-0.016	0.001	1.118	1.441	-0.005	-0.019	1.210	1.456	-0.015	-0.014	1.175	1.487
50	100	-0.084	-0.046	1.340	1.457	-0.039	-0.029	1.136	1.445	0.014	0.013	1.156	1.448	-0.033	-0.008	1.125	1.441
50	200	-0.016	-0.007	1.169	1.385	-0.009	0.000	1.095	1.408	-0.008	0.009	1.098	1.348	0.001	0.002	1.131	1.464
50	500	-0.027	-0.010	1.096	1.391	-0.044	-0.047	1.097	1.426	-0.010	-0.017	1.122	1.402	0.002	0.016	1.119	1.448
100	100	0.015	0.006	1.136	1.419	-0.032	-0.032	1.113	1.392	-0.026	-0.007	1.140	1.407	-0.023	0.002	1.493	1.456
100	200	-0.003	0.017	1.069	1.404	0.009	0.006	1.091	1.418	-0.017	-0.032	1.068	1.426	-0.016	-0.035	1.067	1.417
100	500	-0.010	0.001	1.050	1.394	0.013	0.028	1.046	1.371	0.010	-0.008	1.072	1.391	-0.008	-0.014	1.065	1.414
300	100	-0.013	0.002	1.130	1.473	-0.008	-0.009	1.115	1.391	0.023	0.033	1.111	1.462	-0.023	-0.028	1.141	1.438
300	200	0.005	0.009	1.045	1.407	0.026	0.049	1.057	1.409	-0.043	-0.033	1.045	1.356	-0.035	-0.009	1.062	1.385
300	500	-0.011	0.007	1.033	1.364	-0.009	0.005	1.032	1.365	-0.012	-0.012	1.026	1.374	0.014	0.021	1.042	1.369
500	100	-0.003	-0.000	1.133	1.452	-0.017	-0.012	1.109	1.472	0.010	0.012	1.127	1.400	-0.032	-0.022	1.100	1.417
500	500	0.002	0.010	1.008	1.369	-0.012	-0.023	1.042	1.404	-0.011	0.001	1.048	1.395	-0.023	-0.036	1.015	1.370
500	1000	0.006	0.016	1.019	1.363	0.038	0.039	1.017	1.337	-0.021	-0.039	1.014	1.365	0.024	0.040	1.215	1.373

		Design 17				Design 18				Design 19				Design 20			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.028	-0.005	1.163	1.470	-0.052	-0.044	1.319	1.419	-0.183	-0.108	1.137	1.401	-0.226	-0.112	1.275	1.365
30	200	-0.032	0.007	1.341	1.446	-0.066	-0.044	1.154	1.421	-0.157	-0.062	1.188	1.428	-0.170	-0.115	1.354	1.364
30	500	-0.027	-0.018	1.110	1.400	-0.008	0.013	1.097	1.464	0.226	0.113	1.222	1.385	-0.161	-0.098	1.144	1.318
50	100	0.014	0.007	1.128	1.366	-0.003	0.009	1.083	1.361	-0.196	-0.086	1.274	1.430	0.162	0.075	1.202	1.383
50	200	-0.076	-0.064	1.065	1.405	0.083	0.050	1.099	1.409	-0.157	-0.060	1.334	1.350	-0.158	-0.079	1.080	1.335
50	500	-0.007	0.009	1.050	1.366	-0.039	-0.016	1.066	1.406	0.137	0.061	1.099	1.370	-0.116	-0.055	1.073	1.352
100	100	0.016	0.003	1.406	1.419	-0.030	-0.009	1.000	1.321	-0.133	-0.072	1.077	1.379	0.057	0.027	1.145	1.386
100	200	-0.028	-0.015	1.106	1.385	-0.021	-0.009	1.051	1.375	-0.124	-0.067	1.102	1.416	0.037	0.042	1.075	1.371
100	500	0.018	0.015	1.041	1.372	-0.026	-0.028	1.026	1.356	-0.120	-0.050	1.073	1.330	-0.105	-0.056	1.010	1.321
300	100	0.003	0.027	1.044	1.353	-0.007	0.017	1.057	1.369	-0.131	-0.102	1.131	1.422	-0.109	-0.063	1.057	1.358
300	200	-0.051	-0.001	1.069	1.368	-0.031	-0.018	1.051	1.386	-0.081	-0.039	1.124	1.356	-0.072	-0.017	1.072	1.358
300	500	0.010	0.013	1.018	1.377	-0.007	-0.028	0.997	1.314	-0.075	-0.052	1.029	1.360	-0.082	-0.038	1.035	1.335
500	100	0.026	-0.001	1.052	1.351	-0.025	0.004	1.100	1.434	-0.094	-0.032	1.061	1.356	-0.073	-0.017	1.068	1.423
500	500	0.006	-0.011	1.011	1.369	-0.007	0.002	1.014	1.356	-0.069	-0.050	1.024	1.370	-0.086	-0.047	1.050	1.408
500	1000	-0.031	-0.035	1.011	1.338	0.023	0.038	1.003	1.350	0.070	0.036	1.026	1.340	0.047	0.022	1.018	1.358

This table reports the mean (*m.*), the median (*med.*), standard deviation (*std.*) and interquartile range (*iqr.*) of the empirical distribution of the recentered and standardized statistic $\hat{c}_{i_1 i_2 t}$ in Theorem 1 defined as: $c_{NT}\left(\hat{c}_{i_1 i_2 t} - c^0_{i_1 i_2 t}\right) / \sqrt{\hat{Q}_{i_1 i_2 t}}$. The standardized statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters (Designs 9 - 20). The asymptotic distribution of the statistic is $N(0, 1)$ under the Assumptions of Theorem 1 and has interquartile ≈ 1.349. The empirical distributions are obtained by recomputing the statistics with 4000 Monte Carlo simulations.

Table E.10 – Finite sample distribution of recentered and standardized statistic $\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 2

		Design 9				**Design 10**				**Design 11**				**Design 12**			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.211	-0.142	1.035	1.384	-0.226	-0.130	1.042	1.373	-0.132	-0.050	1.035	1.408	-0.164	-0.104	1.029	1.408
30	200	-0.186	-0.117	1.055	1.407	-0.164	-0.121	1.041	1.383	-0.075	-0.020	1.009	1.339	-0.086	-0.042	1.009	1.325
30	500	-0.134	-0.097	1.028	1.371	-0.112	-0.042	1.054	1.418	-0.098	-0.065	1.024	1.368	-0.121	-0.088	0.997	1.344
50	100	-0.209	-0.131	1.034	1.397	-0.217	-0.112	1.069	1.380	-0.159	-0.082	1.032	1.382	-0.156	-0.096	1.040	1.358
50	200	-0.161	-0.091	1.050	1.426	-0.157	-0.066	1.033	1.383	-0.080	-0.010	1.021	1.380	-0.115	-0.092	1.006	1.348
50	500	-0.095	-0.025	1.034	1.395	-0.121	-0.057	1.028	1.412	-0.054	-0.029	1.007	1.346	-0.098	-0.051	1.006	1.335
100	100	-0.203	-0.102	1.031	1.383	-0.192	-0.116	1.036	1.385	-0.155	-0.078	1.025	1.366	-0.160	-0.085	1.041	1.400
100	200	-0.155	-0.069	1.010	1.368	-0.171	-0.099	1.037	1.348	-0.100	-0.053	1.035	1.401	-0.101	-0.073	1.013	1.359
100	500	-0.106	-0.061	1.008	1.347	-0.098	-0.054	0.998	1.333	-0.063	-0.018	1.015	1.383	-0.065	-0.055	1.004	1.360
300	100	-0.207	-0.108	1.032	1.382	-0.246	-0.169	1.026	1.358	-0.113	-0.039	1.041	1.410	-0.116	-0.060	1.038	1.381
300	200	-0.145	-0.067	1.022	1.337	-0.150	-0.065	1.030	1.380	-0.115	-0.081	1.016	1.372	-0.070	-0.012	1.001	1.319
300	500	-0.114	-0.069	1.010	1.355	-0.088	-0.049	1.007	1.352	-0.040	0.013	1.009	1.409	-0.026	0.001	0.995	1.337
500	100	-0.183	-0.075	1.021	1.360	-0.204	-0.119	1.020	1.344	-0.155	-0.100	1.042	1.456	-0.156	-0.083	1.041	1.387
500	500	-0.092	-0.037	1.025	1.371	-0.089	-0.010	1.015	1.326	-0.034	0.021	0.999	1.348	-0.062	-0.039	0.993	1.301
500	1000	-0.081	-0.036	0.981	1.354	-0.081	-0.031	1.002	1.338	-0.010	0.011	1.024	1.363	-0.046	-0.023	1.022	1.384

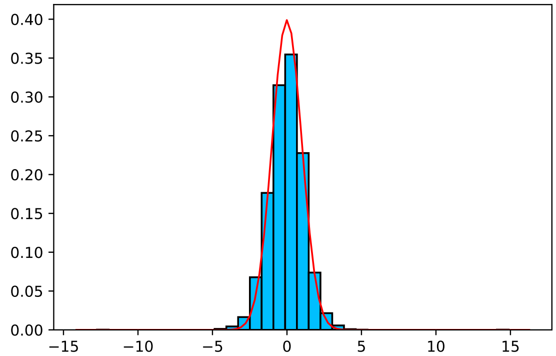
		Design 13				**Design 14**				**Design 15**				**Design 16**			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.116	-0.062	1.039	1.437	-0.196	-0.133	1.053	1.371	-0.159	-0.086	1.026	1.386	-0.117	-0.043	1.032	1.404
30	200	-0.112	-0.055	1.017	1.391	-0.103	-0.021	1.048	1.426	-0.094	-0.044	1.034	1.410	-0.089	-0.039	1.013	1.340
30	500	-0.074	-0.025	1.009	1.323	-0.066	-0.052	1.025	1.351	-0.057	-0.038	1.009	1.372	-0.046	-0.027	1.020	1.398
50	100	-0.151	-0.052	1.072	1.406	-0.163	-0.077	1.062	1.424	-0.134	-0.089	1.022	1.352	-0.146	-0.054	1.039	1.383
50	200	-0.133	-0.090	1.023	1.390	-0.115	-0.055	1.011	1.394	-0.127	-0.071	1.013	1.356	-0.086	-0.057	1.004	1.378
50	500	-0.073	-0.045	1.016	1.393	-0.072	-0.041	0.993	1.337	-0.076	-0.040	1.023	1.378	-0.076	-0.038	1.025	1.382
100	100	-0.175	-0.101	1.032	1.372	-0.185	-0.109	1.056	1.421	-0.130	-0.059	1.051	1.397	-0.154	-0.069	1.053	1.419
100	200	-0.115	-0.044	1.035	1.382	-0.129	-0.061	1.019	1.415	-0.099	-0.046	1.043	1.364	-0.107	-0.067	1.030	1.383
100	500	-0.066	-0.022	1.010	1.319	-0.071	-0.033	1.015	1.389	-0.015	0.014	0.999	1.353	-0.061	-0.041	1.003	1.357
300	100	-0.162	-0.086	1.058	1.428	-0.162	-0.083	1.059	1.408	-0.139	-0.048	1.057	1.414	-0.140	-0.078	1.021	1.372
300	200	-0.114	-0.040	1.047	1.417	-0.111	-0.040	1.017	1.391	-0.108	-0.038	1.034	1.381	-0.084	-0.015	1.046	1.470
300	500	-0.080	-0.036	1.023	1.392	-0.097	-0.069	1.006	1.348	-0.065	-0.044	1.021	1.376	-0.074	-0.040	1.006	1.363
500	100	-0.124	-0.042	1.043	1.415	-0.143	-0.064	1.046	1.425	-0.128	-0.070	1.022	1.396	-0.171	-0.105	1.052	1.414
500	500	-0.079	-0.041	1.020	1.364	-0.050	-0.005	0.998	1.350	-0.070	-0.038	1.021	1.374	-0.084	-0.066	1.005	1.339
500	1000	-0.085	-0.061	1.006	1.328	-0.059	-0.024	0.992	1.349	-0.064	-0.052	1.004	1.359	-0.069	-0.058	1.003	1.364

		Design 17				**Design 18**				**Design 19**				**Design 20**			
$N_1 = N_2$	T	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.	m.	med.	std.	iqr.
30	100	-0.235	-0.116	1.103	1.467	-0.209	-0.131	1.070	1.409	-0.252	-0.124	1.120	1.419	-0.275	-0.161	1.115	1.414
30	200	-0.150	-0.096	1.043	1.381	-0.150	-0.099	1.058	1.414	-0.159	-0.080	1.054	1.406	-0.169	-0.082	1.038	1.389
30	500	-0.104	-0.041	1.030	1.390	-0.078	-0.036	1.021	1.374	-0.086	-0.048	1.011	1.366	-0.113	-0.037	1.055	1.436
50	100	-0.256	-0.131	1.117	1.434	-0.205	-0.115	1.076	1.374	-0.267	-0.156	1.107	1.430	-0.275	-0.165	1.146	1.458
50	200	-0.155	-0.100	1.042	1.407	-0.178	-0.101	1.078	1.404	-0.145	-0.076	1.084	1.421	-0.124	-0.040	1.053	1.367
50	500	-0.112	-0.056	1.044	1.372	-0.066	0.001	1.035	1.401	-0.118	-0.052	1.026	1.360	-0.103	-0.041	1.030	1.390
100	100	-0.237	-0.156	1.103	1.396	-0.226	-0.134	1.094	1.407	-0.268	-0.162	1.119	1.445	-0.243	-0.143	1.107	1.379
100	200	-0.124	-0.047	1.037	1.349	-0.118	-0.036	1.030	1.350	-0.185	-0.105	1.075	1.427	-0.180	-0.074	1.070	1.367
100	500	-0.084	-0.033	1.047	1.410	-0.072	-0.028	1.034	1.388	-0.113	-0.064	1.038	1.374	-0.122	-0.072	1.032	1.393
300	100	-0.229	-0.113	1.110	1.456	-0.218	-0.126	1.101	1.433	-0.251	-0.135	1.138	1.469	-0.259	-0.136	1.121	1.420
300	200	-0.134	-0.054	1.035	1.390	-0.129	-0.081	1.040	1.379	-0.184	-0.074	1.074	1.389	-0.188	-0.111	1.072	1.391
300	500	-0.086	-0.032	1.028	1.361	-0.107	-0.067	1.026	1.344	-0.105	-0.068	1.005	1.328	-0.096	-0.031	1.003	1.334
500	100	-0.202	-0.095	1.103	1.445	-0.228	-0.118	1.113	1.416	-0.248	-0.132	1.117	1.418	-0.233	-0.110	1.130	1.441
500	500	-0.112	-0.074	1.005	1.372	-0.074	-0.025	0.997	1.341	-0.110	-0.046	1.052	1.377	-0.089	-0.039	1.035	1.388
500	1000	-0.060	-0.036	1.019	1.357	-0.074	-0.050	1.011	1.364	-0.061	-0.025	1.014	1.340	-0.073	-0.005	1.035	1.362

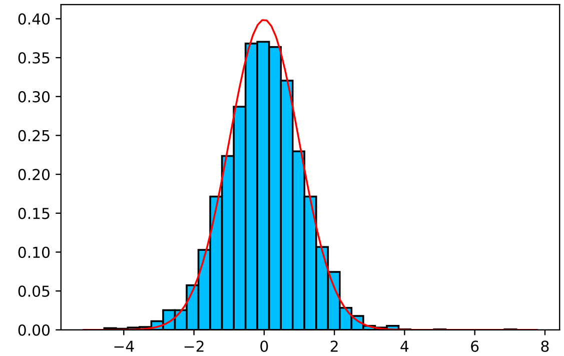
This table reports the mean (*m.*), the median (*med.*), standard deviation (*std.*) and interquartile range (*iqr.*) of the empirical distribution of the recentered standardized statistic $\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 2 defined as: $\sqrt{T}\left(\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}\right) / \sqrt{\hat{\boldsymbol{Q}}_{j_1 j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2)}$, with $\boldsymbol{w}_1 = [1, 0, 0, ..., 0]$ and $\boldsymbol{w}_2 = [1, 0, 0, ..., 0]$. The standardized statistic is computed for different sample sizes (N_1, N_2, T) and different values of the DGP parameters (Designs 9 - 20). The asymptotic distribution of the statistic is $N(0, 1)$ under the Assumptions of Theorem 2 and has interquartile ≈ 1.349. The empirical distributions are obtained by recomputing the statistics with 4000 MC simulations.

Figure E.1 – Finite sample distribution of the recentered and standardized statistic $\hat{c}_{i_1 i_2 t}$ in Theorem 1

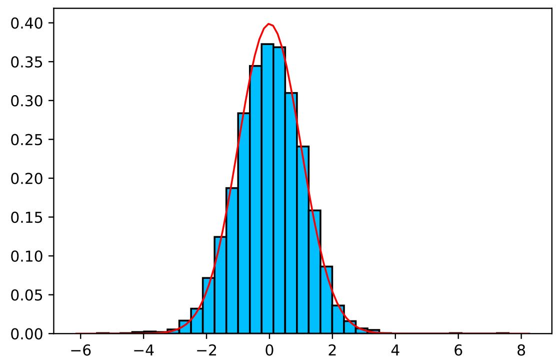
(a) Design 2, $N_1 = N_2 = 30$, $T = 100$



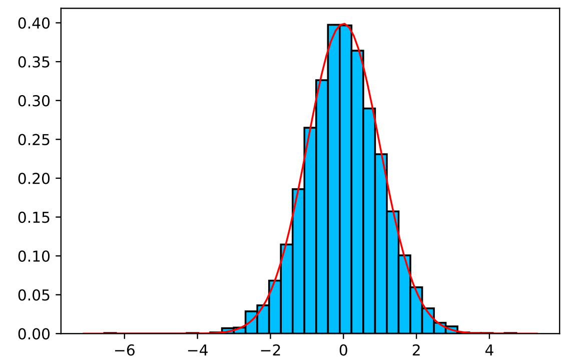
(b) Design 2, $N_1 = N_2 = 50$, $T = 200$



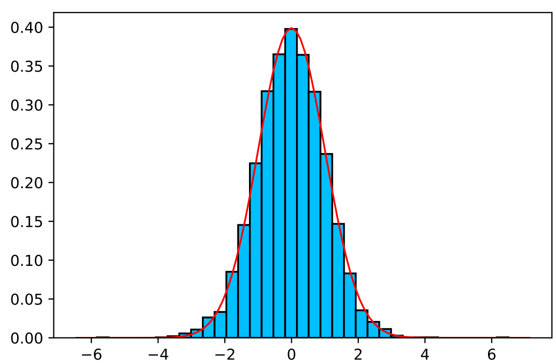
(c) Design 2, $N_1 = N_2 = 100$, $T = 200$



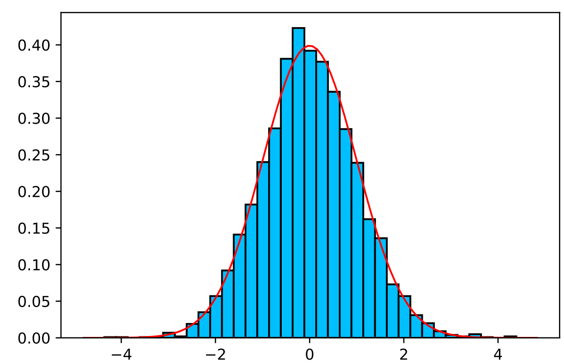
(d) Design 2, $N_1 = N_2 = 300$, $T = 200$



(e) Design 2, $N_1 = N_2 = 500$, $T = 500$



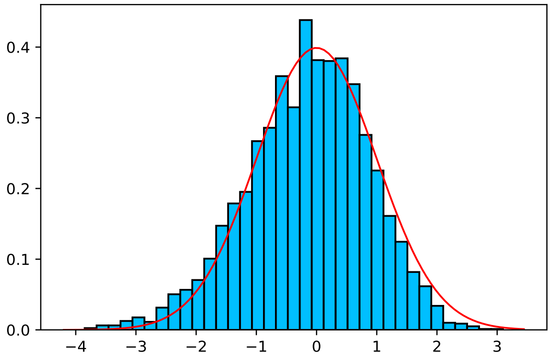
(f) Design 2, $N_1 = N_2 = 500$, $T = 1000$



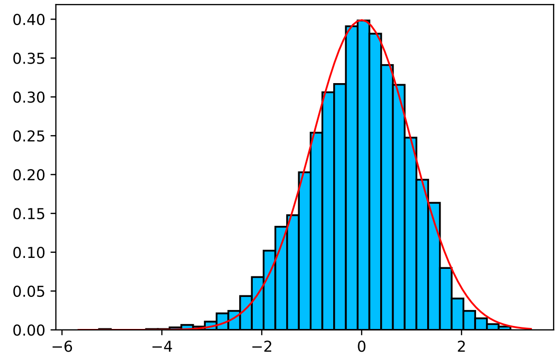
This figure shows the simulated empirical distribution of the recentered and standardized statistic $\hat{c}_{i_1 i_2 t}$ in Theorem 1 defined as: $C_{NT}\left(\hat{c}_{i_1 i_2 t} - c^0_{i_1 i_2 t}\right) / \sqrt{\hat{\boldsymbol{Q}}_{i_1 i_2 t}}$. The standardized statistic is computed for different sample sizes (N_1, N_2, T) and for the values of the DGP parameters in Design 2 of Table 1. Under the Assumptions of Theorem 1, the asymptotic distribution of the statistic is $N(0,1)$ (solid red line). The empirical distributions are obtained by recomputing the statistics with 4000 Monte Carlo simulations.

Figure E.2 – Finite sample distribution of recentered and standardized statistic $\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 2

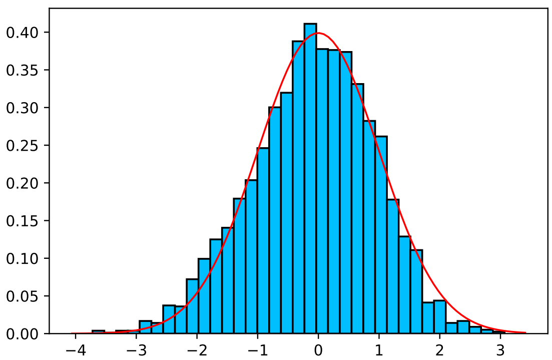
(a) Design 2, $N_1 = N_2 = 30$, $T = 100$



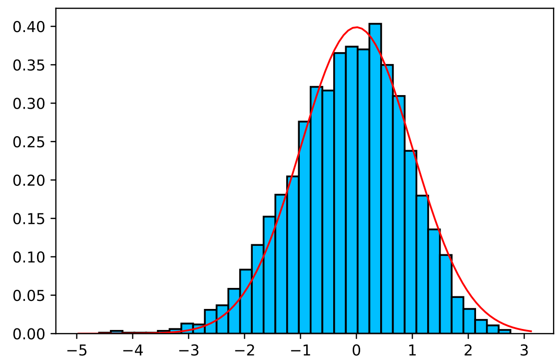
(b) Design 2, $N_1 = N_2 = 50$, $T = 200$



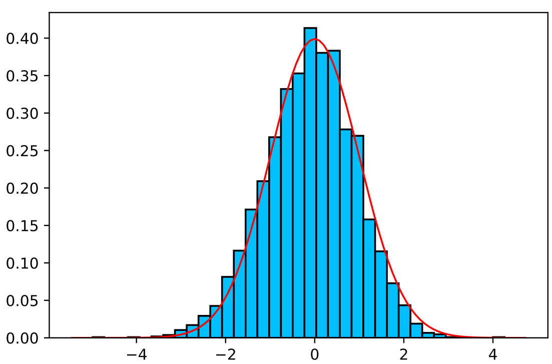
(c) Design 2, $N_1 = N_2 = 100$, $T = 200$



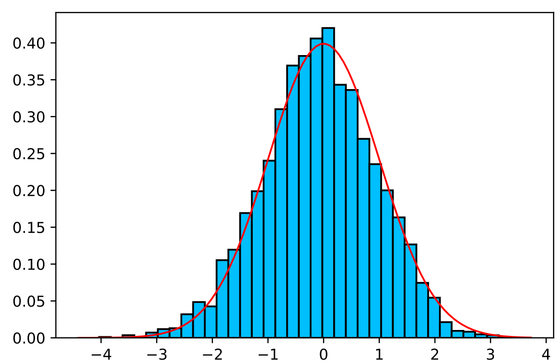
(d) Design 2, $N_1 = N_2 = 300$, $T = 200$



(e) Design 2, $N_1 = N_2 = 500$, $T = 500$



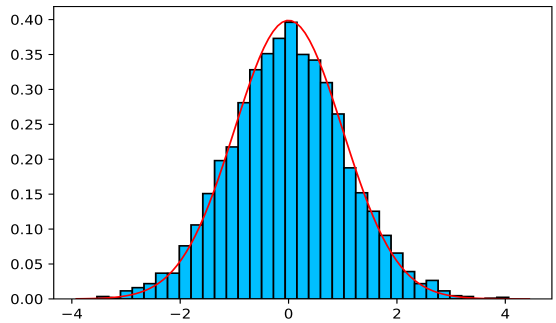
(f) Design 2, $N_1 = N_2 = 500$, $T = 1000$



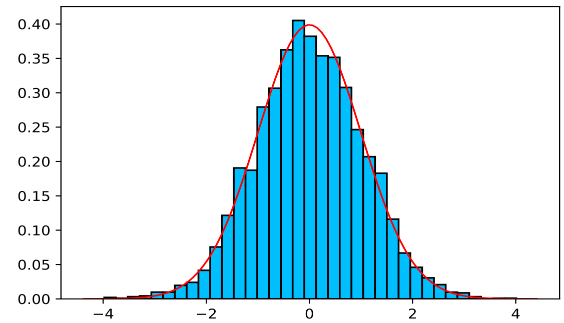
This figure shows the simulated empirical distribution of the recentered and standardized statistic $\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 2 defined as: $\sqrt{T}\left(\hat{c}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - c^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}\right)/\sqrt{\hat{\boldsymbol{Q}}_{j_1 j_2}(\boldsymbol{w}_1, \boldsymbol{w}_2)}$, with $\boldsymbol{w}_1 = [1, 0, 0, ..., 0]$ and $\boldsymbol{w}_2 = [1, 0, 0, ..., 0]$. The statistic is computed for different sample sizes (N_1, N_2, T) and for the values of the DGP parameters in Design 2 of Table 1. Under the Assumptions of Theorem 2, the asymptotic distribution of the statistic is $N(0, 1)$ (solid red line). The empirical distributions are obtained with 4000 MC simulations.

Figure E.3 – Finite sample distribution of recentered and standardized statistic $\widehat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 3

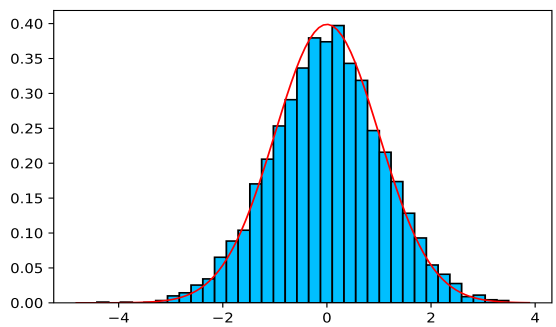
(a) Design 2, $N_1 = N_2 = 30$, $T = 100$



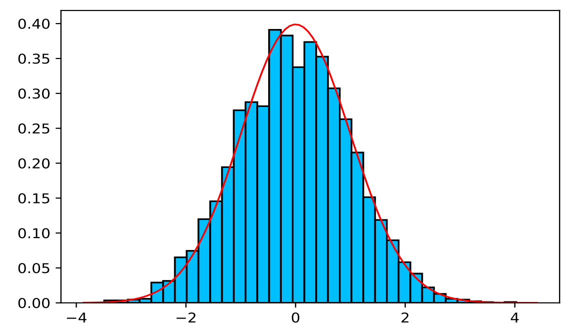
(b) Design 2, $N_1 = N_2 = 50$, $T = 200$



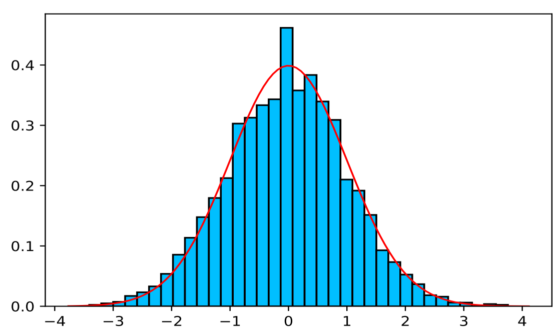
(c) Design 2, $N_1 = N_2 = 100$, $T = 200$



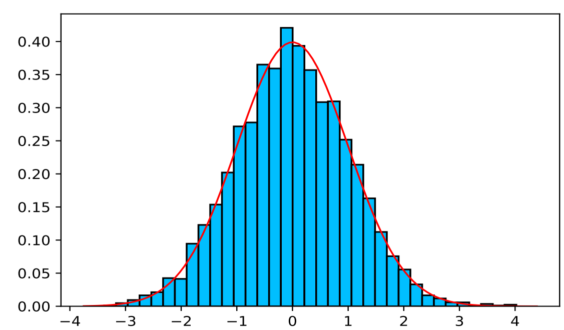
(d) Design 2, $N_1 = N_2 = 300$, $T = 200$



(e) Design 2, $N_1 = N_2 = 500$, $T = 500$

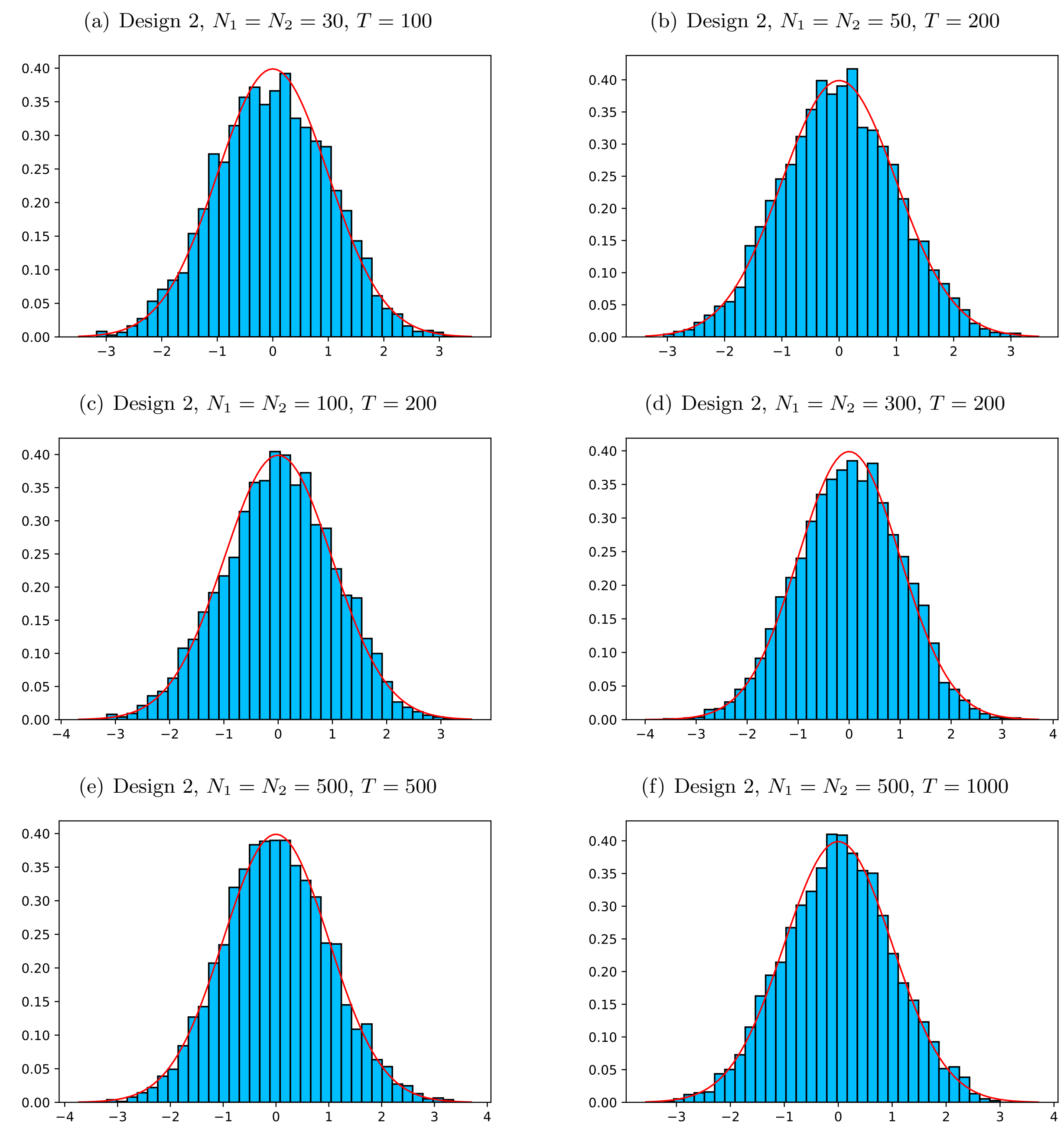


(f) Design 2, $N_1 = N_2 = 500$, $T = 1000$



This figure shows the simulated empirical distribution of the recentered and standardized statistic $\widehat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 3 defined as: $\sqrt{T}\left(\widehat{R}_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2} - R^0_{j_1 j_2 \boldsymbol{w}_1 \boldsymbol{w}_2}\right) / \sqrt{\hat{\boldsymbol{Q}}_{R, j_1 j_2}(\hat{\boldsymbol{w}}_{\sigma,1 j_1 j_2}, \hat{\boldsymbol{w}}_{\sigma,2 j_1 j_2})}$, with $\boldsymbol{w}_1 = [1/N_1, ..., 1/N_1]$ and $\boldsymbol{w}_2 = [1/N_2, ..., 1/N_2]$. The standardized statistic is computed for different sample sizes $(N_1,\ N_2,\ T)$ and for the values of the DGP parameters in Design 2 of Table 1. Under the Assumptions of Theorem 3, the asymptotic distribution of the statistic is $N(0, 1)$ (solid red line). The empirical distributions are obtained by recomputing the statistics with 4000 Monte Carlo simulations.

Figure E.4 – Finite sample distribution of the test statistic $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4



(a) Design 2, $N_1 = N_2 = 30$, $T = 100$

(b) Design 2, $N_1 = N_2 = 50$, $T = 200$

(c) Design 2, $N_1 = N_2 = 100$, $T = 200$

(d) Design 2, $N_1 = N_2 = 300$, $T = 200$

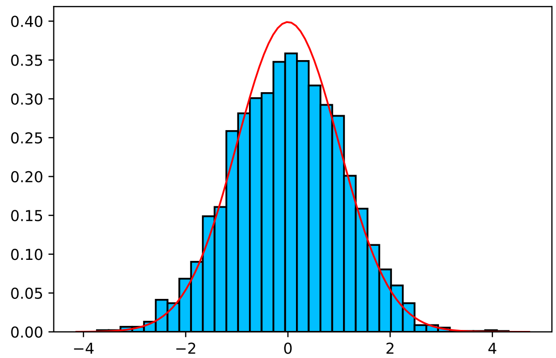
(e) Design 2, $N_1 = N_2 = 500$, $T = 500$

(f) Design 2, $N_1 = N_2 = 500$, $T = 1000$

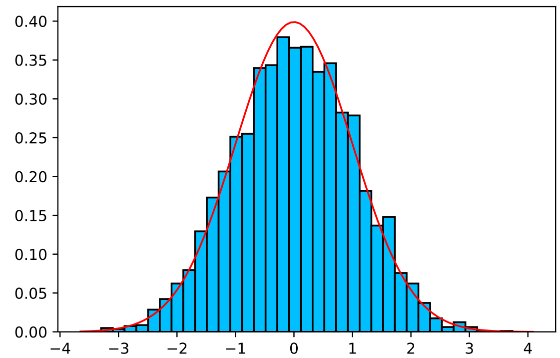
This figure shows the simulated empirical distribution of the test statistic $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4, with $\boldsymbol{w}_1 = [1, 0, 0, ..., 0]$ and $\boldsymbol{w}_2 = [1, 0, 0, ..., 0]$. The test statistic is computed for different sample sizes (N_1, N_2, T) and for the values of the DGP parameters in Design 2 of Table 2. Under the Assumptions of Theorem 4, the asymptotic distribution of the test statistic is $N(0, 1)$ (solid red line). The empirical distributions are obtained by recomputing the statistics with 4000 Monte Carlo simulations.

Figure E.5 – Finite sample distribution of the test statistic $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4

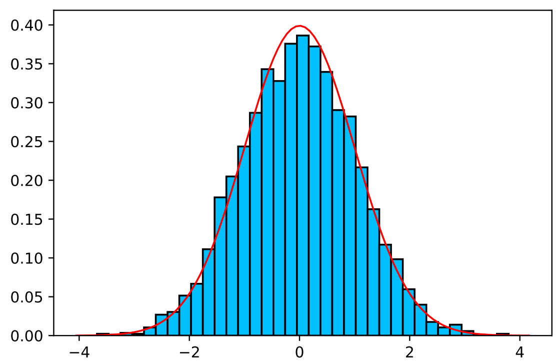
(a) Design 2, $N_1 = N_2 = 30$, $T = 100$



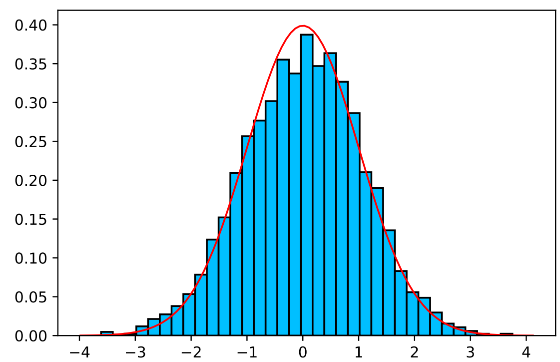
(b) Design 2, $N_1 = N_2 = 50$, $T = 200$



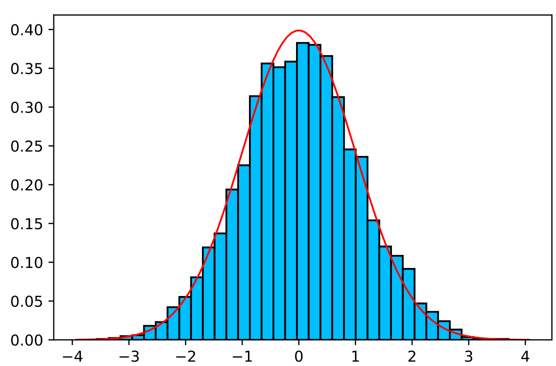
(c) Design 2, $N_1 = N_2 = 100$, $T = 200$



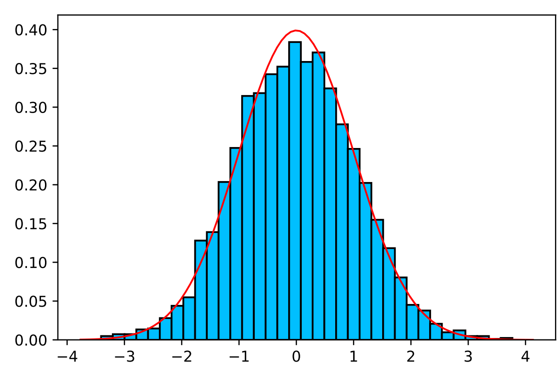
(d) Design 2, $N_1 = N_2 = 300$, $T = 200$



(e) Design 2, $N_1 = N_2 = 500$, $T = 500$



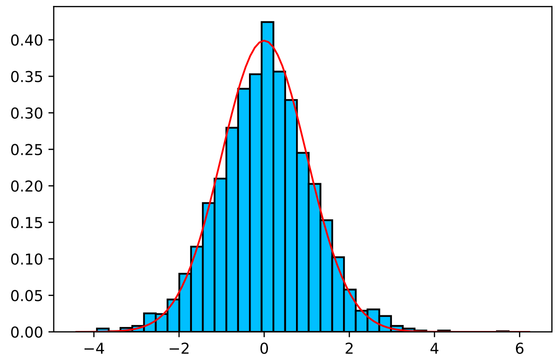
(f) Design 2, $N_1 = N_2 = 500$, $T = 1000$



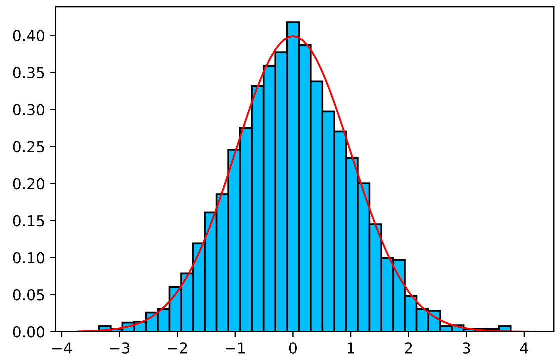
This figure shows the simulated empirical distribution of the test statistic $\widehat{\mathcal{T}}^{c}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 4, with $\boldsymbol{w}_1 = [1/N_1, ..., 1/N_1]$ and $\boldsymbol{w}_2 = [1/N_2, ..., 1/N_2]$. The test statistic is computed for different sample sizes (N_1, N_2, T) and for the values of the DGP parameters in Design 2 of Table 2. Under the Assumptions of Theorem 4, the asymptotic distribution of the test statistic is $N(0,1)$ (solid red line). The empirical distributions are obtained by recomputing the statistics with 4000 Monte Carlo simulations.

Figure E.6 – Finite sample distribution of the test statistic $\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 5 with $\pi_0 = 0.5$

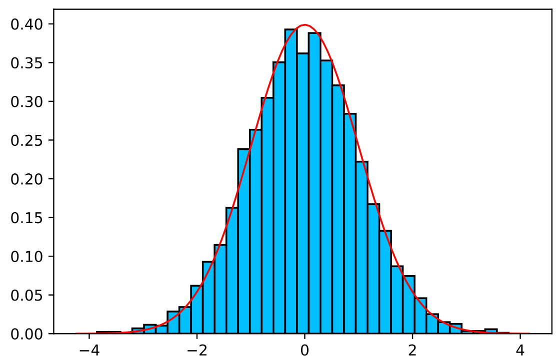
(a) Design 2, $N_1 = N_2 = 30$, $T = 100$



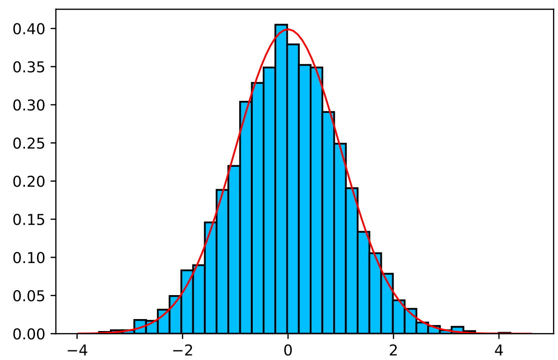
(b) Design 2, $N_1 = N_2 = 50$, $T = 200$



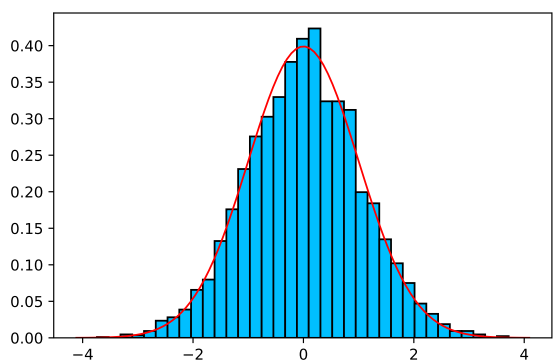
(c) Design 2, $N_1 = N_2 = 100$, $T = 200$



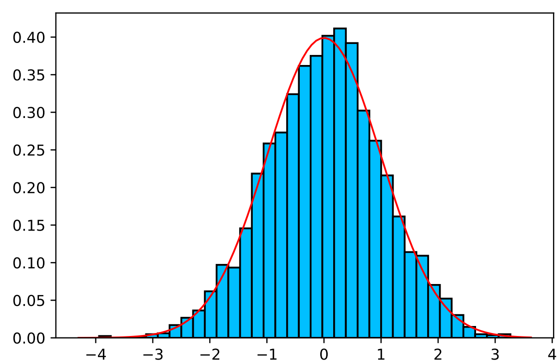
(d) Design 2, $N_1 = N_2 = 300$, $T = 200$



(e) Design 2, $N_1 = N_2 = 500$, $T = 500$



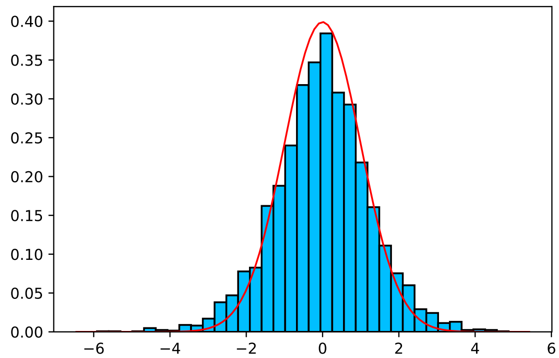
(f) Design 2, $N_1 = N_2 = 500$, $T = 1000$



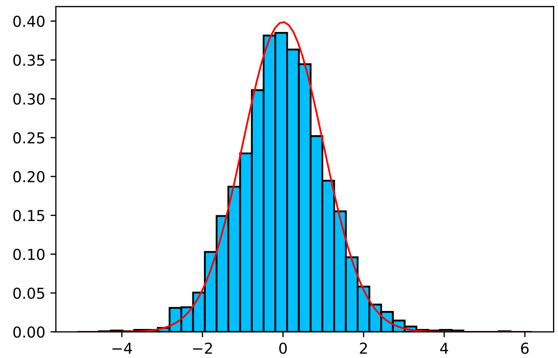
This figure shows the simulated empirical distribution of the test statistic $\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 5, with $\boldsymbol{w}_1 = [1/N_1, ..., 1/N_1]$ and $\boldsymbol{w}_2 = [1/N_2, ..., 1/N_2]$. The test statistic is computed for different sample sizes (N_1, N_2, T) and for the values of the DGP parameters in Design 2 of Table 2. Under the Assumptions of Theorem 5, the asymptotic distribution of the test statistic is $N(0, 1)$ (solid red line). The empirical distributions are obtained by recomputing the statistics with 4000 Monte Carlo simulations.

Figure E.7 – Finite sample distribution of the test statistic $\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 5

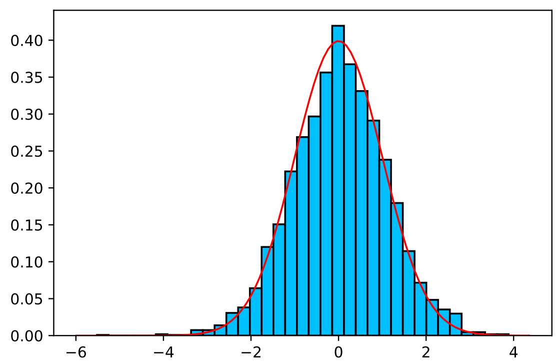
(a) Design 2, $N_1 = N_2 = 30$, $T = 100$



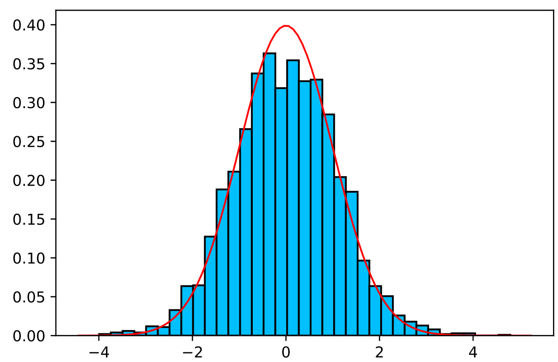
(b) Design 2, $N_1 = N_2 = 50$, $T = 200$



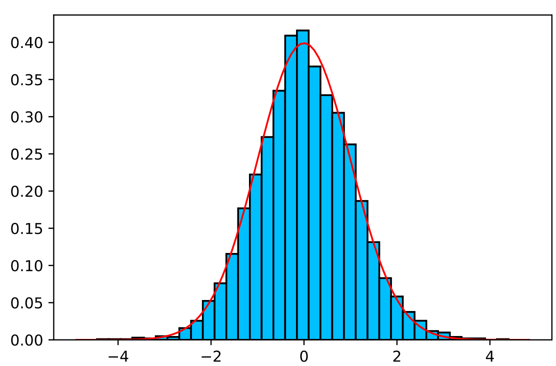
(c) Design 2, $N_1 = N_2 = 100$, $T = 200$



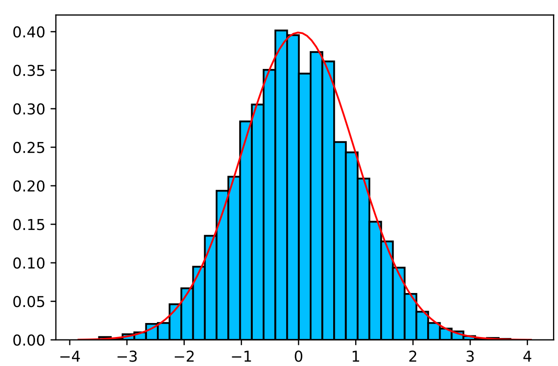
(d) Design 2, $N_1 = N_2 = 300$, $T = 200$



(e) Design 2, $N_1 = N_2 = 500$, $T = 500$



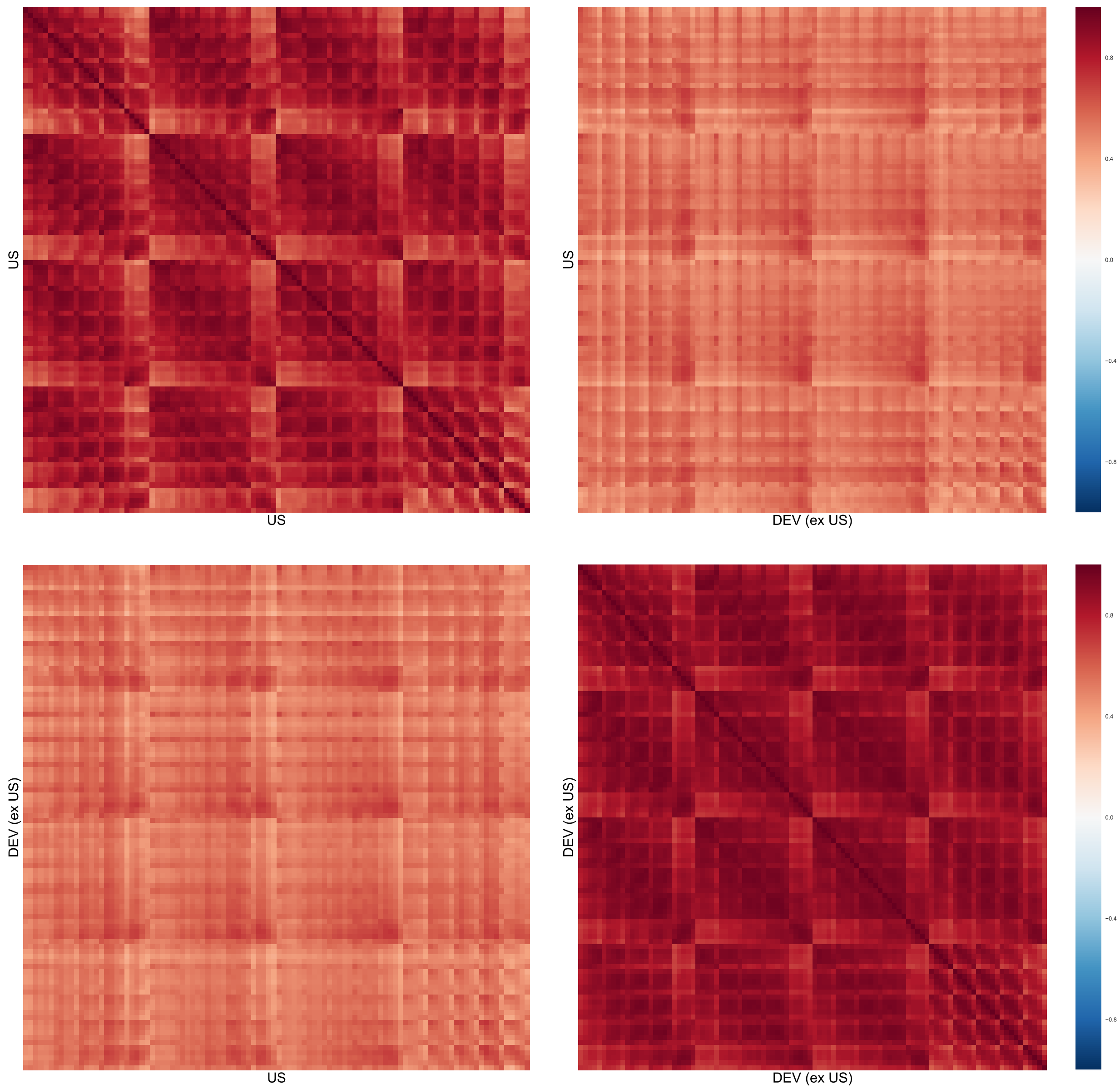
(f) Design 2, $N_1 = N_2 = 500$, $T = 1000$



This figure shows the simulated empirical distribution of the test statistic $\widehat{\mathcal{T}}^{R}_{j_1 j_2 j_1^* j_2^* \boldsymbol{w}_1 \boldsymbol{w}_2}$ in Theorem 5, with $\boldsymbol{w}_1 = [1/N_1, ..., 1/N_1]$ and $\boldsymbol{w}_2 = [1/N_2, ..., 1/N_2]$. The test statistic is computed for different sample sizes (N_1, N_2, T) and for the values of the DGP parameters in Design 2 of Table 2. Under the Assumptions of Theorem 5, the asymptotic distribution of the test statistic is $N(0, 1)$ (solid red line). The empirical distributions are obtained by recomputing the statistics with 4000 Monte Carlo simulations.

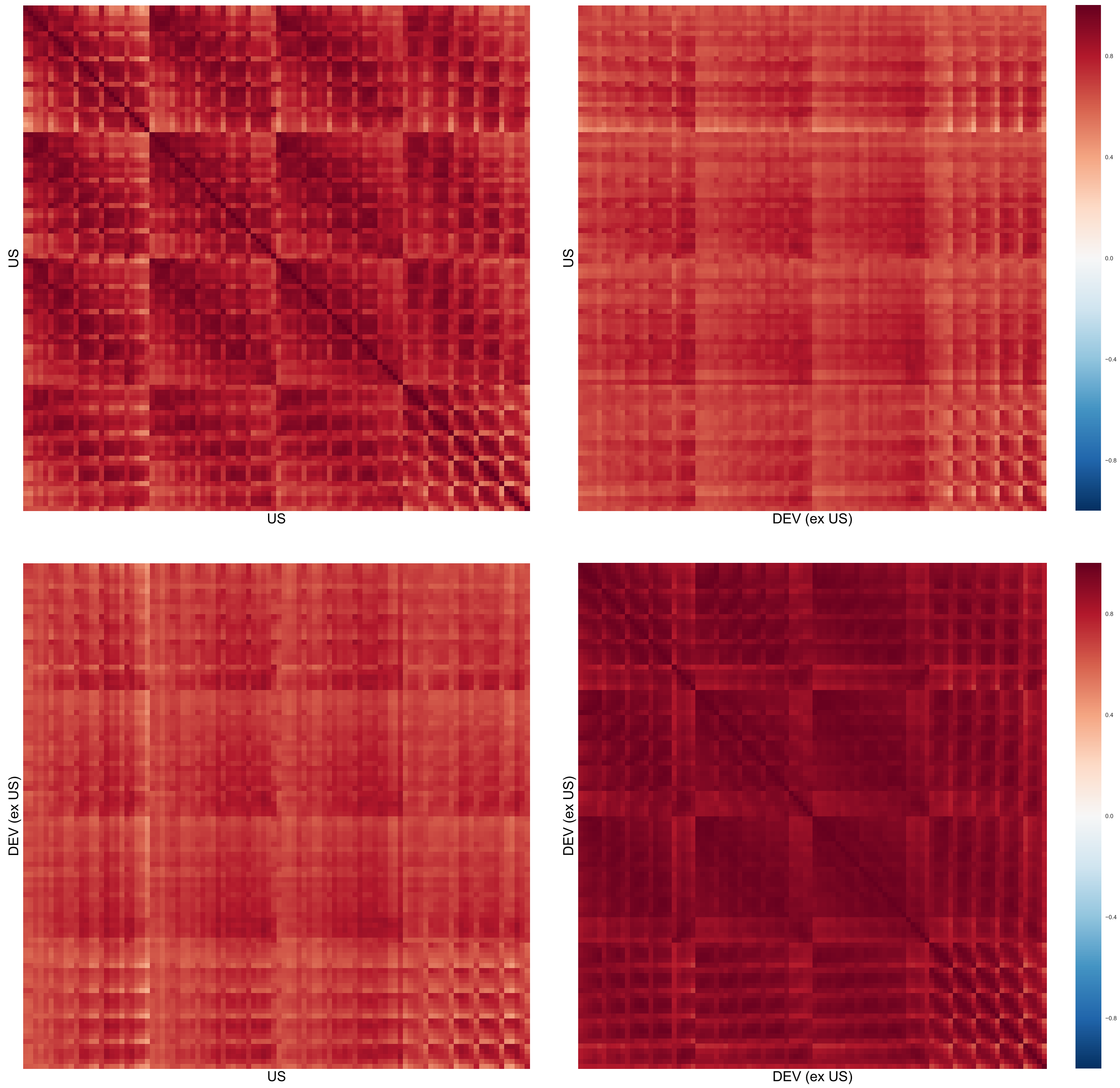
E Empirical Application: Additional Results

Figure E.1 – Actual correlations within and across panels when $z_t \leq \hat{\theta}_1 = \hat{\theta}_2$



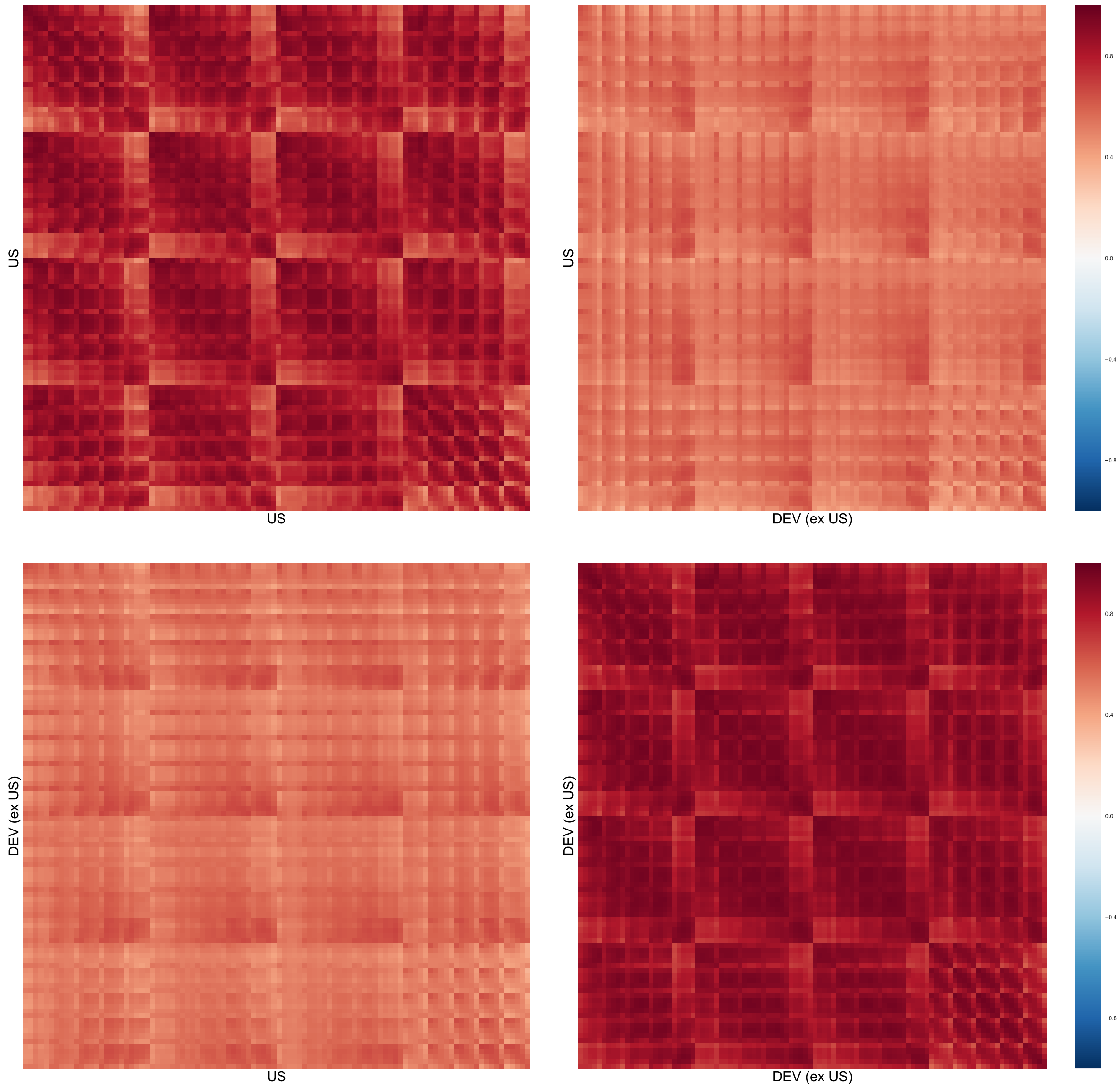
The figure displays the sample correlations within and across panels of monthly excess returns on the U.S. equity portfolios (Group 1) and on the developed (ex US) equity portfolios (Group 2) when $z_t \leq \hat{\theta}_1 = \hat{\theta}_2$, that is for all dates such that $\mathbb{U}^{\mathrm{M}} \leq 0.674$. Average sample correlations are $\bar{R}_{11} = 0.802$, $\bar{R}_{12} = \bar{R}_{21} = 0.543$, and $\bar{R}_{22} = 0.871$.

Figure E.2 – Actual correlations within and across panels when $z_t > \hat{\theta}_1 = \hat{\theta}_2$



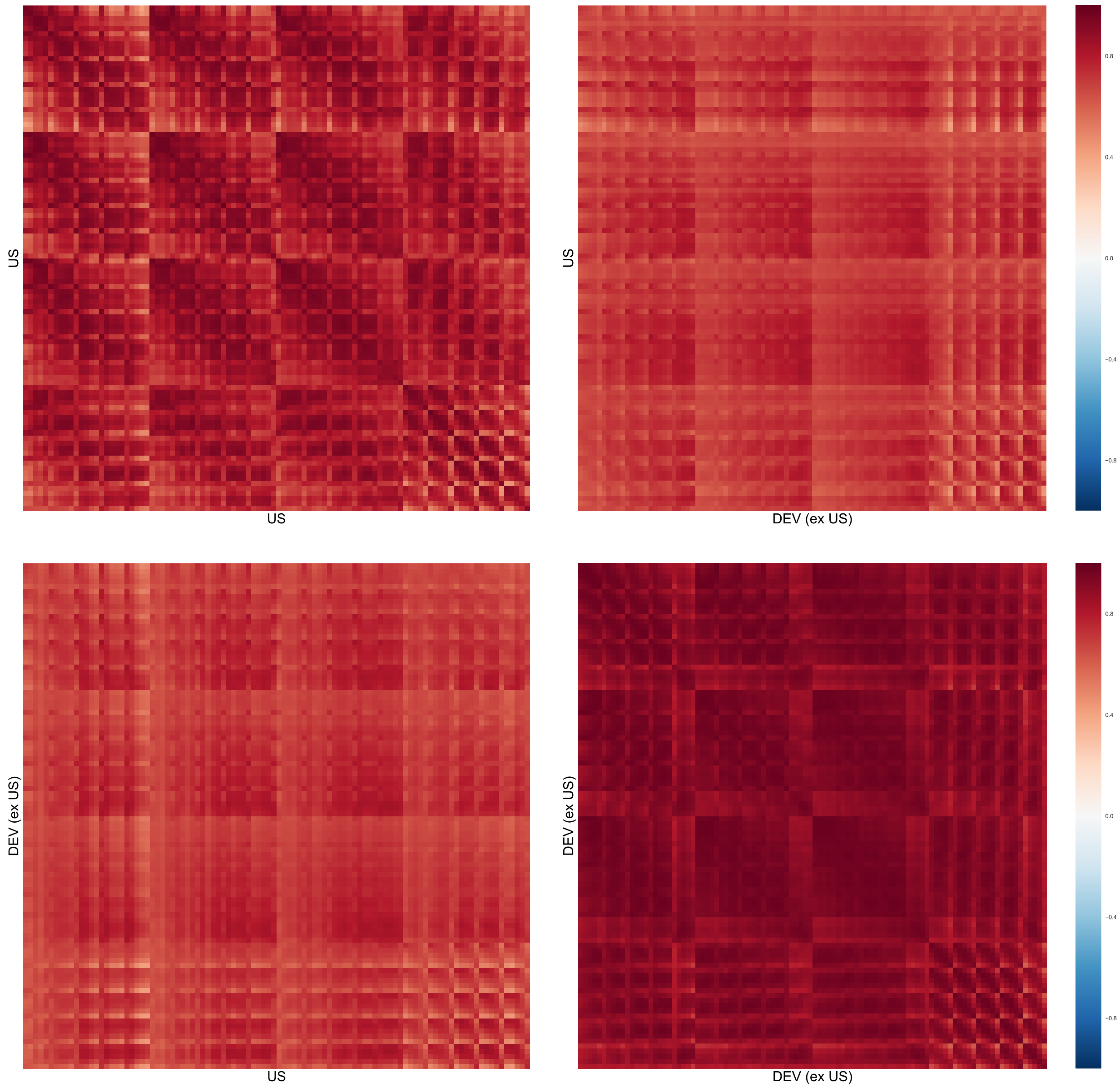
The figure displays the sample correlations within and across panels of monthly excess returns on the U.S. equity portfolios (Group 1) and the international (ex U.S.) equity portfolios (Group 2) when $z_t > \hat{\theta}_1 = \hat{\theta}_2$, that is for all dates such that $\mathbb{U}^{\mathrm{M}} > 0.674$. Average sample correlations are $\bar{R}_{11} = 0.808$, $\bar{R}_{12} = \bar{R}_{21} = 0.707$, and $\bar{R}_{22} = 0.917$.

Figure E.3 – Estimated systematic correlations within and across panels when $z_t \leq \hat{\theta}_1 = \hat{\theta}_2$



The figure displays the estimated systematic correlations within and across panels of monthly excess returns on the U.S. equity portfolios (Group 1) and the international (ex U.S.) equity portfolios (Group 2) when $z_t \leq \hat{\theta}_1 = \hat{\theta}_2$, that is for all dates such that $\mathbb{U}^{\mathrm{M}} \leq 0.674$. Average systematic correlations are $\widehat{R}^{w}_{11} = 0.802$, $\widehat{R}_{LL\boldsymbol{w}_1\boldsymbol{w}_2} = 0.543$, and $\widehat{R}^{w}_{22} = 0.871$.

Figure E.4 – Estimated systematic correlations within and across panels when $z_t > \hat{\theta}_1 = \hat{\theta}_2$



The figure displays the estimated systematic correlations within and across panels of monthly excess returns on the U.S. equity portfolios (Group 1) and the international (ex U.S.) equity portfolios (Group 2) when $z_t > \hat{\theta}_1 = \hat{\theta}_2$, that is for all dates such that $\mathbb{U}^{\mathrm{M}} > 0.674$. Average systematic correlations are $\widehat{R}_{11}^{w} = 0.808$, $\widehat{R}_{HH\boldsymbol{w}_1\boldsymbol{w}_2} = 0.708$, and $\widehat{R}_{22}^{w} = 0.917$.

Figure E.5 – Return loadings on $F_{US}^{(1)}$, the first pervasive factor of the U.S. equity portfolios.

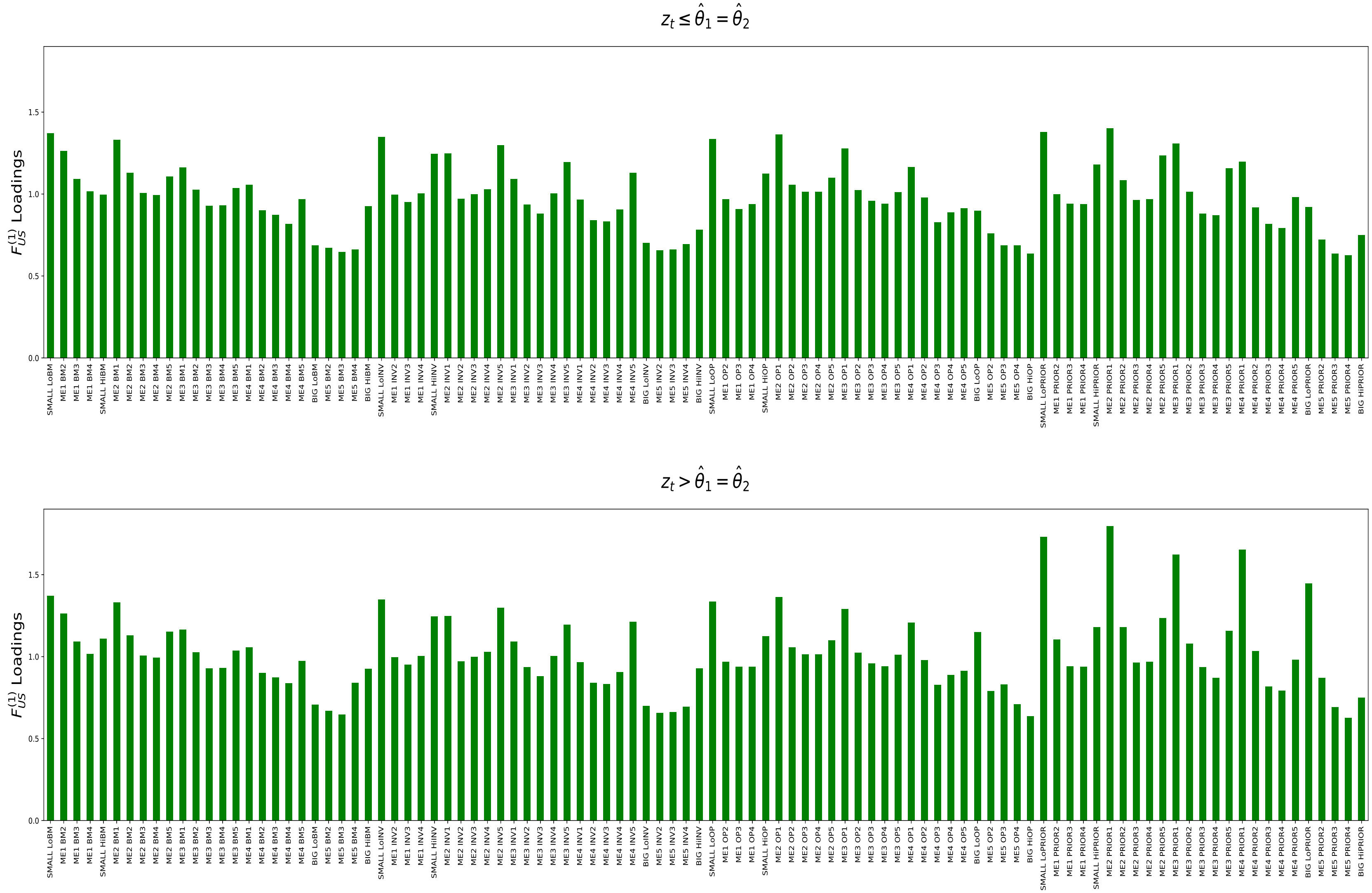


Figure E.6 – Return loadings on $F_{US}^{(2)}$, the second pervasive factor of the U.S. equity portfolios.

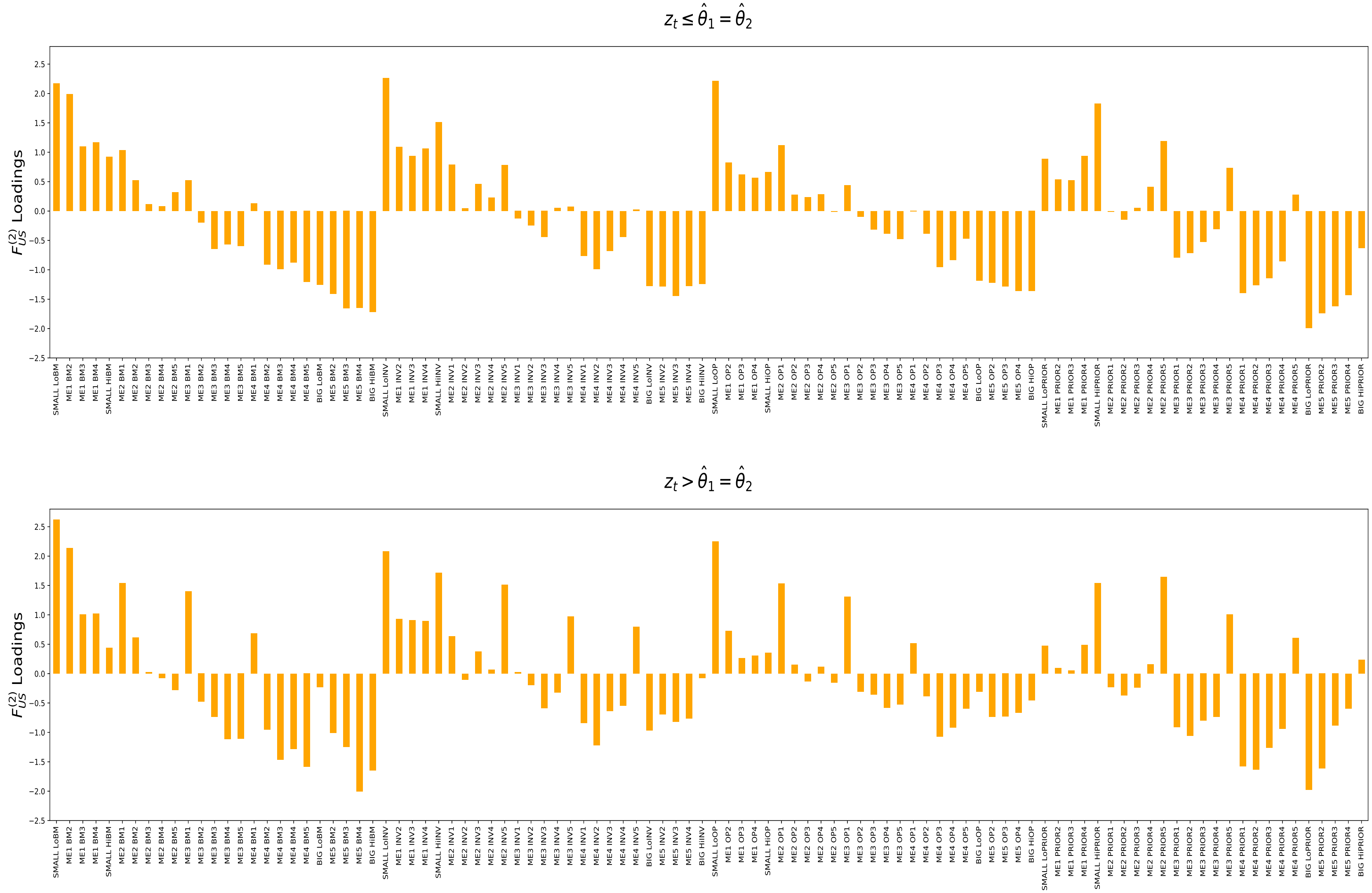


Figure E.7 – Return loadings on $F_{DEV}^{(1)}$, the first pervasive factor of the developed (ex US) equity portfolios.

Figure E.8 – Return loadings on $F^{(2)}_{DEV}$, the second pervasive factor of the developed (ex US) equity portfolios.