BOOTSTRAP FOR PANEL DATA MODELS*

Bertrand HOUNKANNOUNON

Université de Montréal, CIREQ

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Abstract

This paper considers bootstrap methods for panel data models with fixed regressors. It is shown that simple resampling methods (i.i.d., individual only or temporal only) are not always valid in simple cases of interest, while a double resampling that combines resampling in both individual and temporal dimensions is valid. This approach also permits to avoid multiples asymptotic theories that may occur in large panel models. In particular, it is shown that this resampling method provides valid inference in the one-way and two-way error component models and in the factor models. Simulations confirm these theoretical results.

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1 Introduction

The true probability distribution of a test statistic is rarely known. Generally, its asymptotic law is used as approximation of the true law. If the sample size is not large enough, the asymptotic behavior of the statistics could lead to a poor approximation of the true one. Using bootstrap methods, under some regularity conditions, it is possible to obtain a more accurate approximation of the distribution of the test statistic. Original bootstrap procedure has been proposed by Efron (1979) for statistical analysis of independent and identically distributed (i.i.d.) observations. It is a powerful tool for approximating the distribution of complicated statistics based on i.i.d. data. There is an extensive literature for the case of i.i.d. observations. Bickel & Freedman (1981) established some asymptotic properties for bootstrap. Freedman (1981) analyzed the use of bootstrap for least squares estimator in linear regression models.

In practice, observations are not i.i.d. Since Efron (1979) there is an extensive research to extend bootstrap to statistical analysis of non i.i.d. data. Wild bootstrap is developed in Liu (1988) following suggestions in Wu (1986) and Beran (1986) for independent but not identically distributed data. Several bootstrap procedures have been proposed for time series. The two most popular approaches are sieve bootstrap and block bootstrap. Sieve bootstrap attempts to model the dependence using a parametric model. The idea behind it is to postulate a parametric form for the data generating process, to estimate the parameters and to transform the model in order to have i.i.d. elements to resample. The weakness of this approach is that results are sensitive to model misspecification and the attractive nonparametric feature of bootstrap is lost. On the other hand, block bootstrap resamples blocks of consecutive observations. In this case, the user is not obliged to specify a particular parametric model. For an overview of bootstrap methods for dependent data, see Lahiri (2003). Application of bootstrap methods to several indices data is an embryonic research field. The expression *"several indices data"* regroups : clustered data, multilevel data, and panel data.

The term *"panel data"* refers to the pooling of observations on a cross-section of statistical units over several periods. Because of their two dimensions (individual -or cross-sectional- and temporal), panel data have the important advantage to allow to control for unobservable heterogeneity, which is a systematic difference across individuals or periods. For an overview about panel data models, see for example Baltagi (1995) or Hsiao (2003). There is an abounding literature about asymptotic theory for panel data models. Some recent developments treat of large panels, when temporal and cross-sectional dimensions are both important. Paradoxically, literature about bootstrap for panel data is rather restricted. In general, simulation results suggest that some resampling methods work well in practice but theoretical results are rather limited or exposed with strong assumptions. As previous references about bootstrap methods for panel models, it can be quoted Bellman et al. (1989), Andersson & Karlsson (2001), Carvajal (2000), Kapetanios (2004), Focarelli (2005), Everaert & Pozzi (2007) and Herwartz (2006; 2007). In error component models, Bellman et al. (1989) uses bootstrap to correct bias after feasible generalized least squares. Andersson & Karlsson (2001) presents bootstrap resampling methods for one-way error component model. For two-way error component models, Carvajal (2000) evaluates by simulations, different bootstrap resampling methods. Kapetanios (2004) presents theoretical results when cross-sectional dimension goes to infinity, under the assumption that cross-sectional vectors of regressors and errors terms are i.i.d.. This assumption does not permit time varying regressors or temporal aggregate shocks in errors terms. Focarelli (2005) and Everaert & Pozzi (2007) uses bootstrap to reduce bias in dynamic panel models with fixed effects when T is fixed and N goes to infinity, bias quoted by Nickell (1981). Herwartz (2006 & 2007) deliver a bootstrap version to Breusch-Pagan test in panel data models under cross-sectional dependence. Recently, Hounkannounon (2008) gives some theoretical results about bootstrap methods used with panel data models. Its theoretical results are about a model without regressor and concern the sample mean. This paper aims to extend these results to linear regression model. Various bootstrap resampling methods will be confronted with panel models commonly used to evaluate their validity. The paper is organized as follows. In the second section, different panel data models are presented. Section 3 presents five bootstrap resampling methods for panel data. The fourth section presents theoretical results, analyzing validity of each resampling method. In section 5, simulation results are presented and confirm theoretical results. The sixth section concludes.

2 Panel Data Models

It is practical to represent panel data as a rectangle. By convention, in this document, rows correspond to the individuals and columns represent time periods. A panel dataset with N individuals and T time periods is represented by a matrix Y of N rows and T columns. Y contains thus NT elements. y_{it} *is* $i's$ *observation at period* t.

$$\underset{(N,T)}{Y} = \begin{pmatrix} y_{11} & y_{12} & \cdots & \cdots & y_{1T} \\ y_{21} & y_{22} & \cdots & \cdots & y_{2T} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdot\cdot & \cdots \\ y_{N1} & y_{N2} & \cdot\cdot & \cdots & y_{NT} \end{pmatrix}$$

Consider the following linear model

$$y_{it} = \theta + V_i\tau + W_t\gamma + X_{it}\zeta + \nu_{it} = Z_{it}\beta + \nu_{it} \tag{2.1}$$

$$\underset{(K,1)}{\beta} = \begin{pmatrix} \theta \\ \tau \\ \gamma \\ \zeta \end{pmatrix} \tag{2.2}$$

Three kinds of variables are considered : cross-section varying variables V_i, time varying variable W_t and double dimensions varying variables X_{it}. β is an unknown vector of parameters. Inference will be about these parameters and consists in building confidence intervals for each component β_k of β. Assumptions about ν_{it} define different panel data models. The specifications of error terms commonly used can be summarized by *(2.3)* under assumptions below.

$$\nu_{it} = \mu_i + f_t + \lambda_i F_t + \varepsilon_{it} \tag{2.3}$$

Assumptions A

A1 : $(\mu_1, \mu_2, ..., \mu_N) \sim i.i.d.\left(0, \sigma_\mu^2\right),\ \sigma_\mu^2 \in (0, \infty)$

A2 : $\{f_t\}$ *is a stationary and strong* α*-mixing process*[1] *with* $E(f_t) = 0,\ \exists\ \delta \in (0,\infty)$: $E|f_t|^{2+\delta} < \infty, \sum_{j=1}^{\infty} \alpha(j)^{\delta/(2+\delta)} < \infty$, *and* $V_f^\infty = \sum_{h=-\infty}^{\infty} Cov(f_t, f_{t+h}) \in (0,\infty)$

A3 : $(\lambda_1, \lambda_2, ..., \lambda_N) \sim i.i.d.\left(0, \sigma_\lambda^2\right),\ \sigma_\lambda^2 \in (0, \infty)$

A4 : $\{\varepsilon_{it}\}_{i=1...N,\ t=1..T} \sim i.i.d.\left(0, \sigma_\varepsilon^2\right),\ \sigma_\varepsilon^2 \in (0, \infty)$

[1] See Appendix 1 for definition of an α-mixing process.

A5 : $\{F_t\}$ *is a stationary and strong* α*-mixing process with* $E(F_t) = 0$, $\exists\ \delta \in (0,\infty)$: $E|F_t|^{2+\delta} < \infty$, $\sum_{j=1}^{\infty} \alpha(j)^{\delta/(2+\delta)} < \infty$, *and* $V_F^{\infty} = \sum_{h=-\infty}^{\infty} Cov(F_t, F_{t+h}) \in (0,\infty)$

A6 : The five series are independent.

These assumptions can seem strong. They are made in order to have strong convergence and to simplify demonstrations. Considering special cases with different combinations of processes in *(2.3.)*, gives the following panel data models : one-way error component model, two-way error component model and factor model.

One-way ECM

$$\nu_{it} = \mu_i + \varepsilon_{it} \tag{2.4}$$

$$\nu_{it} = f_t + \varepsilon_{it} \tag{2.5}$$

Two specifications are considered. The term, one-way error component model (ECM), comes from the structure of error terms : only one kind of heterogeneity, that is systematic differences across cross-sectional units or time periods, is taken into account. The specification *2.4 (resp. 2.5)* allows to control unobservable individual (resp. temporal) heterogeneity. The specification *(2.4)* is called *individual one-way ECM*, *(2.5)* is *temporal one-way ECM*. It is important to emphasize that here, unobservable heterogeneity is a random variable, not a parameter to be estimated. The alternative is to use *fixed effects model* in which heterogeneity must be estimated.

Two-way ECM

$$\nu_{it} = \mu_i + f_t + \varepsilon_{it} \tag{2.6}$$

Two-way error component model allows to control for individual and temporal heterogeneity, hence the term *two-way*. Like in one-way ECM, individual and temporal heterogeneities are random variables. Classical papers on error component models include Balestra & Nerlove (1966), Fuller & Battese (1974) and Mundlak (1978).

Factor Model

$$\nu_{it} = \mu_i + \lambda_i F_t + \varepsilon_{it} \tag{2.7}$$

In *(2.7)*, the difference with one-way ECM, is the term $\lambda_i F_t$. The product allows the common factor F_t to have differential effects on cross-section units. This specification is used by Bai & Ng (2004), Moon & Perron (2004) and Phillips & Sul (2003). It is a way to introduce dependence among cross-sectional units. An other way is to use spatial model in which, the structure of the dependence can be related to geographic, economic or social distance (see Anselin (1988))[2].

[2] Bootstrap methods studied in this paper do not take into account spatial dependence. Reader insterested by resampling methods for spatial data, can see for example Lahiri (2003), chap. 12.

3 Bootstrap Methods

This section presents the bootstrap methodology and five ways to resample panel data.

Bootstrap Methodology

From initial data (Y, X) , create pseudo data (Y^*, X^*) by resampling with replacement elements of (Y, X). This operation must be repeated B times in order to have $B+1$ pseudo-samples : $\{Y_b^*, X_b^*\}_{b=1..B+1}$. Statistics are computed with these pseudo-samples in order to make inference. In this paper, inference is about β and consists in building confidence intervals and testing hypothesis for each parameter of the element of the vector β. There are two main bootstrap approaches with regression models :the residual-based bootstrap an the paired bootstrap. The paired bootstrap resamples dependent variable and regressors whereas residuals resamples first step residuals to compute pseudo values of the dependent variable. This paper analyzes only the residual-based bootstrap which steps are the followings :

Step 1 : Run pooling regression to obtain OLS estimator $\widehat{\beta}$ and the residuals $\widehat{\nu}_{it}$

$$\widehat{\beta} = \left(\widetilde{Z}^{/}\widetilde{Z}\right)^{-1}\widetilde{Z}^{/}\widetilde{Y}$$

$$\widehat{\nu}_{it} = y_{it} - Z_{it}\widehat{\beta}$$

Step 2 : Rescale the residuals in order to have better properties in small samples.

$$u_{it} = \frac{NT}{NT-K}\,\widehat{\nu}_{it}$$

By OLS properties, the residuals have mean equal to zero : centering is not necessary. The $N \times T$ matrix of rescaled residuals is noted U.

Step 3 : Use a resampling method to create pseudo-sample of residuals U^*.

$$U = \{u_{it}\} \overset{resampling}{\rightarrow} U^* = \{u_{it}^*\}$$

Use the pseudo- residuals to pseudo-values of the dependent variable.

$$y_{it}^* = Z_{it}\widehat{\beta} + u_{it}^*$$

Run pooling regression with (Y^*, X)

$$\widehat{\beta}^* = \left(\widetilde{Z}^{/}\widetilde{Z}\right)^{-1}\widetilde{Z}^{/}\widetilde{Y}^*$$

Step 4 : Repeat *step 3* B times in order to have $B+1$ realizations of $\{Y^*, X\}$ and $\widehat{\beta}^*$. These realizations are quoted $\{Y_b^*, X\}_{b=1..B+1}$and $\left\{\widehat{\beta}_b^*\right\}_{b=1..B+1}$.

The probability measure induced by the resampling method conditionally on U is noted P^*. $E^*()$ and $Var^*()$ are respectively expectation and variance associated to P^*. In this paper, The resampling methods used to compute pseudo-samples are exposed below.

Iid Bootstrap

In this document i.i.d bootstrap refers to original bootstrap as defined by Efron (1979). It was designed for one dimensional data, but it's easy to adapt it to panel data. For $N \times T$ matrix U, i.i.d. resampling is the operation of constructing a $N \times T$ matrix U^* where each element u^*_{it} is selected with replacement from Y. Conditionally on U, all the elements of U^* are independent and identically distributed. There is a probability $1/NT$ that each u^*_{it} is one of the NT elements u_{it} of U.

Individual Bootstrap

For a $N \times T$ matrix U, *individual resampling* is the operation of constructing a $N \times T$ matrix U^* with rows obtained by resampling with replacement rows of U. Conditionally on U, the rows of U^* are independent and identically distributed. Contrary to i.i.d. bootstrap case, u^*_{it} cannot take any value. u^*_{it} can just take one of the N values $\{u_{it}\}_{i=1,\ldots N}$.

Temporal Bootstrap

For $N \times T$ matrix U, temporal resampling is the operation of constructing a $N \times T$ matrix Y^* with columns obtained by resampling with replacement columns of Y. Conditionally on U, the columns of U^* are independent and identically distributed. u^*_{it} can just take one of the T values $\{u_{it}\}_{t=1,\ldots T}$.

Block Bootstrap

Block bootstrap for panel data is a direct accommodation of non-overlapping block bootstrap for time series, due to Carlstein (1986). The idea is to resample in temporal dimension, but not single period like in temporal bootstrap case, but blocks of consecutive periods in order to capture temporal dependence. Assume that $T = Kl$, with l the length of a block, then there are K non-overlapping blocks. For $N \times T$ matrix Y, block bootstrap resampling is the operation of constructing a $N \times T$ matrix U^* with columns obtained by resampling with replacement the K blocks of columns of U. Note that temporal bootstrap is a special case of block bootstrap, when $l = 1$. Moving block bootstrap (Kunsch (1989), Liu & Singh (1992)), circular block bootstrap (Politis & Romano (1992)) and stationary block bootstrap (Politis & Romano (1994)) can also be accommodated to panel data.

Double Resampling Bootstrap

For a $N \times T$ matrix Y, double resampling is the operation of construction a $N \times T$ matrix U^{**} with columns and rows obtained by resampling columns and rows of U. Two schemes are explored. The first scheme is a *combination of individual and temporal bootstrap.* The second scheme is a *combination of individual and block bootstrap.* Carvajal (2000) and Kapetanios (2004) improve this resampling method by Monte Carlo simulations, but give no theoretical support. Double stars $**$ are used to distinguish estimator, probability measure, expectation and variance induced by double resampling.

4 Theoretical Results

This section presents theoretical results about resampling methods exposed in section 3, using models specified in section 2.

Multiple Asymptotics

In the study of asymptotic distributions for panel data, there are many possibilities. One index can be fixed and the other goes to infinity. In the second case, how N and T go to infinity, is not always without consequence. Hsiao (2003 p. 296) distinguishes three approaches : *sequential limit, diagonal path limit* and *joint limit.* A sequential limit is obtained when an index is fixed and the other passes to infinity, to have intermediate result. The final result is obtained by passing the fixed index to infinity. In case of diagonal path limit, N and T pass to infinity along a specific path, for example $T = T(N)$ and $N \to \infty$. With joint limit, N and T pass to infinity simultaneously without a specific restrictions. In some times it can be necessary to control relative expansion rate of N and T. It is obvious that joint limit implies diagonal path limit. For equivalence conditions between sequential and joint limits, see Phillips & Moon (1999). In practice, it is not always clear how to choose among these multiple asymptotic distributions which may be different.

Assumptions B *(Regressors)*
B1 : The regressors are fixed.
B2 : (Rank condition) : The $K \times K$ *matrix* $\widetilde{Z}^{/}\widetilde{Z}$ *is not singular.*
B3 :

$$\frac{\widetilde{Z}^{/}\widetilde{Z}}{NT} \underset{NT\to\infty}{\to} \underset{(K,K)}{Q} > 0 \tag{4.1}$$

B4 :

$$\frac{\overline{Z}^{/}\overline{Z}}{N} \underset{N\to\infty}{\to} \overline{Q} \tag{4.2}$$

B5 :

$$\frac{\underline{Z}^{/}\underline{Z}}{T} \underset{T\to\infty}{\to} \underline{Q} \tag{4.3}$$

B6 :

$$\Omega > 0 \tag{4.4}$$

In addition to these assumptions, some Lindeberg conditions will be appropriately assumed, depending on the case of analysis. Assumption *A4* implies that time varying regressors are excluded. Assumptions *A5* excludes cross-section varying regressors.

Table 1 summarizes asymptotic distributions for the different panel models. For i.i.d. panel model, $NT \to \infty$ summarizes three cases of asymptotic : N is fixed and T goes to infinity, T is fixed and N goes to infinity, and finally N and T pass to infinity simultaneously. Two asymptotic theories are available for one-way ECM. In the case of two-way ECM, N and T must go to infinity. The relative convergence rate between the two indexes, δ defines a continuum of asymptotic distributions. Finally, factor has a unique asymptotic distribution, when the two dimensions go to infinity. Details about these convergences are exposed in appendix 1.

Model	*Asymptotic distribution*	*Variance* (Ω)
Individual One − way	$\sqrt{N}\left(\widehat{\beta}-\beta\right) \underset{N\to\infty}{\Longrightarrow} N(0,\Omega)$	$\sigma_\mu^2\left(Q^{-1}\overline{Q}Q^{-1}\right)+\frac{\sigma_\varepsilon^2}{T}Q^{-1}$
ECM	$\sqrt{N}\left(\widehat{\beta}-\beta\right) \underset{N,T\to\infty}{\Longrightarrow} N(0,\Omega)$	$\sigma_\mu^2\left(Q^{-1}\overline{Q}Q^{-1}\right)$
Temporal One − way	$\sqrt{T}\left(\widehat{\beta}-\beta\right) \underset{T\to\infty}{\Longrightarrow} N(0,\Omega)$	$\sigma_f^2\left(Q^{-1}\underline{Q}Q^{-1}\right)$ $\frac{\sigma_\varepsilon^2}{N}Q^{-1}$
ECM	$\sqrt{T}\left(\widehat{\beta}-\beta\right) \underset{N,T\to\infty}{\Longrightarrow} N(0,\Omega)$	$\sigma_f^2\left(Q^{-1}\underline{Q}Q^{-1}\right)$
Two − way	$\sqrt{N}\left(\widehat{\beta}-\beta\right) \underset{\substack{N,T\to\infty \\ \frac{N}{T}\to\delta\in[0,\infty)}}{\Longrightarrow} N(0,\Omega)$	$\sigma_\mu^2\left(Q^{-1}\overline{Q}Q^{-1}\right)+\delta.\Sigma_f^\infty$
ECM	$\sqrt{T}\left(\widehat{\beta}-\beta\right) \underset{\substack{N,T\to\infty \\ \frac{N}{T}\to\infty}}{\Longrightarrow} N(0,\Omega)$	Σ_f^∞
Factor model	$\sqrt{N}\left(\widehat{\beta}-\beta\right) \underset{N,T\to\infty}{\Longrightarrow} N(0,\Omega)$	$\sigma_\mu^2\left(Q^{-1}\overline{Q}Q^{-1}\right)$

Table 1 : Asymptotic distributions

Bootstrap Confidence Intervals

In the literature, there are several bootstrap confidence interval. The methods commonly used are *percentile interval* and *percentile-t interval.*

Bootstrap Percentile Interval
With each pseudo-sample Y_b^*, compute $\widehat{\beta}_b^*$ and the K statistics

$$r_k^{b*} = \widehat{\beta}_k^{b*} - \widehat{\beta}_k \qquad (4.5)$$

The empirical distribution of these $(B+1)$ realizations is :

$$\widehat{R}_k^*(x) = \frac{1}{B+1}\sum_{b=1}^{B+1} I\left(r_k^{b*} \leq x\right) \qquad (4.6)$$

The *percentile-t* confidence interval of level $(1-\alpha)$ for the parameter $\widehat{\beta}_k$ is:

$$CI_{1-\alpha,k}^* = \left[\widehat{\beta}_k + r_{k,\frac{\alpha}{2}}^* ; \widehat{\beta}_k + r_{k,1-\frac{\alpha}{2}}^*\right] \qquad (4.7)$$

where $r_{k,\alpha/2}^*$ and $r_{k,1-\alpha/2}^*$ are respectively is the lower empirical $\alpha/2$-percentage point and $(1-\alpha/2)$-percentage point of $\widehat{R}_k^*$. B must be chosen so that $\alpha(B+1)/2$ is an integer. Using equality 4.5, 4.7 becomes :

$$CI_{1-\alpha,k}^* = \left[\widehat{\beta}_{k,\frac{\alpha}{2}}^* ; \widehat{\beta}_{k,1-\frac{\alpha}{2}}^*\right] \qquad (4.9)$$

where $\widehat{\beta}_{k,\alpha/2}^*$ and $\widehat{\beta}_{k,1-\alpha/2}^*$ are respectively is the lower empirical $\alpha/2$-percentage point and $(1-\alpha/2)$-percentage point of the empirical distribution of $\left\{\widehat{\beta}_{k,b}^*\right\}_{b=1..B+1}$.

Bootstrap Percentile-t Interval
With each pseudo-sample Y_b^*, compute $\widehat{\beta}_b^*$ and the K statistics

$$t_k^{b*} = \frac{\widehat{\beta}_k^{b*} - \widehat{\beta}_k}{\sqrt{\widehat{V}ar^*\left(\widehat{\beta}_k^{b*}\right)}} \qquad (4.10)$$

In the denominator of *(4.10)*, there is an estimator of the bootstrap-variance that must be computed for every bootstrap resample. The empirical distribution of the $(B+1)$ realizations of t_k^{b*} is :

$$\widehat{F}_k^*(x) = \frac{1}{B+1}\sum_{b=1}^{B+1} I\left(t_k^{b*} \leq x\right) \qquad (4.11)$$

The *percentile-t* confidence interval of level $(1-\alpha)$ for the parameter $\widehat{\beta}_k$ is:

$$CI_{1-\alpha,k}^* = \left[\widehat{\beta}_k - \sqrt{Var^*\left(\widehat{\beta}_k^{b*}\right)}.t_{k,1-\frac{\alpha}{2}}^* ; \widehat{\beta}_k - \sqrt{Var^*\left(\widehat{\beta}_k^{b*}\right)}.t_{k,\frac{\alpha}{2}}^*\right] \qquad (4.12)$$

The strength of percentile-t is that it permits theoretical demonstrations about asymptotic refinements.

Bootstrap Consistency

There are several ways to prove consistency of a resampling method. For an overview, see Shao & Tu (1995, chap. 3). The method commonly used is to show that the distance between the cumulative distribution function on the classical estimator and the bootstrap estimator goes to zero when the sample grows-up. Different notions of distance can be used : sup-norm, Mallow's distance.... Sup-norm is the commonly used. The notations used for one dimension data must be to panel data, in order to be more formal. Because of multiple asymptotic distributions, there are several consistency definitions. The presentation takes into account percentile confidence interval. Similar definitions can be formulated for percentile-t confidence interval.

A bootstrap method is said *consistent* for β if :

$$\sup_{x\in\mathbb{R}^K}\left|P^*\left(\sqrt{NT}\left(\widehat{\beta}^*-\widehat{\beta}\right)\leq x\right)-P\left(\sqrt{NT}\left(\widehat{\beta}-\beta\right)\leq x\right)\right|\underset{NT\to\infty}{\overset{P}{\to}}0 \tag{4.13}$$

or

$$\sup_{x\in\mathbb{R}^K}\left|P^*\left(\sqrt{N}\left(\widehat{\beta}^*-\widehat{\beta}\right)\leq x\right)-P\left(\sqrt{N}\left(\widehat{\beta}-\beta\right)\leq x\right)\right|\underset{NT\to\infty}{\overset{P}{\to}}0 \tag{4.14}$$

or

$$\sup_{x\in\mathbb{R}^K}\left|P^*\left(\sqrt{T}\left(\widehat{\beta}^*-\widehat{\beta}\right)\leq x\right)-P\left(\sqrt{T}\left(\widehat{\beta}-\beta\right)\leq x\right)\right|\underset{NT\to\infty}{\overset{P}{\to}}0 \tag{4.15}$$

Definitions 4.12, 4.13 and 4.14 are given with convergence in probability ($\overset{P}{\to}$). This case implies a *weak consistency.* The case of almost surely ($a.s.$) convergence provides a *strong consistency.* These definitions of consistency does not require that the bootstrap estimator or the classical estimator has asymptotic distribution. The idea behind it, is the mimic analysis : when the sample grows, the bootstrap estimator *mimics* very well the behavior of the classical estimator. In the special when the sample mean asymptotic distribution is available, consistency can be established by showing that bootstrap-sample mean has the same distribution. The next proposition expresses this idea.

Proposition 1

Assume $\sqrt{NT}\left(\widehat{\beta}-\beta\right)\Longrightarrow L$ *and* $\sqrt{NT}\left(\widehat{\beta}^*-\widehat{\beta}\right)\overset{*}{\Longrightarrow}L^*$*. If* L^**and* L *are identical and continuous, then*

$$\sup_{x\in\mathbb{R}^K}\left|P^*\left(\sqrt{NT}\left(\widehat{\beta}^*-\widehat{\beta}\right)\leq x\right)-P\left(\sqrt{NT}\left(\widehat{\beta}-\beta\right)\leq x\right)\right|\underset{NT\to\infty}{\overset{P}{\to}}0$$

Proof. *The fact that* $\widehat{\beta}$ *and* $\widehat{\beta}^*$ *have the same asymptotic distribution, implies that* $|P^*(..)-P(..)|$ *converges to zero. Under continuity assumption, the uniform convergence is given by Pólya theorem (Pólya (1920) or Serfling (1980), p. 18)* ■

Similar propositions similar can be formulated for definitions *4.13* and *4.14*. Using Proposition 1, the methodology adopted in this document is, for each resampling method, to find the asymptotic distribution of the bootstrap-estimator $\widehat{\beta}^*$. Comparing theses distributions with those in Table 1, permits to find consistent and inconsistent bootstrap resampling methods for each panel model. Consistent resampling methods can be used to build confidence intervals.

Remarks

1 - The definitions of consistency are given for the vector of parameters, but confidence intervals are given for each component of this vector.

2 - The given definitions of consistency are appropriate to build percentile confidence interval. Similar definitions can be given using t_k^{b*} and its classical counterpart. In this case, for each parameter β_k, the percentile-t confidence interval is valid if on the definition given for consistency,.we must have :

$$\widehat{V}ar^*\left(\widehat{\beta}_k^{b*}\right) \xrightarrow{P^*} Var^*\left(\widehat{\beta}_k^{b*}\right) \tag{4.16}$$

3 - Consistency in an asymptotic property. It must be taken in mind that bootstrap procedure has been originally designed for small samples and its validity depend of the fact that the bootstrap data *mimic* as well as possible the behavior of the original data. In this paper, this approach will be called *mimic analysis*. Residual-based bootstrap estimator *mimics* very well the behavior the classical asymptotic estimator if the bootstraped residuals mimic very well the behavior of the original error terms.

In the following for each bootstrap resampling method, the mimic analysis is presented followed by the consistency analysis.

I.i.d. Bootstrap

In a mimic analysis, it can be said that iid resampling method does not take into account the structure of dependence of the error terms. That leads to inconsistent estimator for all the panel model specifications.

$Model$	$Asymptotic\ distribution$	$Variance\,(\Omega)$	$Consistency$
$Individual$ $One-way$ ECM	$\sqrt{NT}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{N\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\left(\sigma_\mu^2 + \sigma_\varepsilon^2\right)Q^{-1}$	No
	$\sqrt{NT}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\left(\sigma_\mu^2 + \sigma_\varepsilon^2\right)Q^{-1}$	
$Temporal$ $One-way$ ECM	$\sqrt{NT}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\left(\sigma_f^2 + \sigma_\varepsilon^2\right)Q^{-1}$	No
	$\sqrt{NT}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\left(\sigma_f^2 + \sigma_\varepsilon^2\right)Q^{-1}$	
$Two-way$ ECM	$\sqrt{NT}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\left(\sigma_\mu^2 + \sigma_f^2 + \sigma_\varepsilon^2\right)Q^{-1}$	No
$Factor$ $model$	$\sqrt{NT}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\left(\sigma_\mu^2 + \sigma_\lambda^2\sigma_F^2 + \sigma_\varepsilon^2\right)Q^{-1}$	No

Table 2 : Asymptotic distributions with i.i.d. bootstrap

Individual Bootstrap

$$U = \widehat{\mu} + \widehat{f} + \widehat{\lambda}^{/}\widehat{F} + \widehat{\varepsilon} \quad ; \quad U^{*}_{ind} = \widehat{\mu}^{*} + \widehat{f} + \widehat{\lambda}^{*/}\widehat{F} + [\widehat{\varepsilon}_{\text{col}}]^{*}$$

$$U^{*}_{ind} - E^{*}(U^{*}) = (\widehat{\mu}^{*} - \widehat{\mu}) + \left(\widehat{\lambda}^{*/} - \widehat{\lambda}^{/}\right)\widehat{F} + ([\widehat{\varepsilon}_{\text{row}}]^{*} - \widehat{\varepsilon}) \qquad (4.17)$$

Model	*Asymptotic distribution*	*Variance* (Ω)	*Consistency*
Individual	$\sqrt{N}\left(\widehat{\beta}^{*} - \widehat{\beta}\right) \underset{N\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\sigma^2_\mu\left(Q^{-1}\overline{Q}Q^{-1}\right) + \frac{\sigma^2_\varepsilon}{T}Q^{-1}$	*Yes*
One − way ECM	$\sqrt{N}\left(\widehat{\beta}^{*} - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\sigma^2_\mu\left(Q^{-1}\overline{Q}Q^{-1}\right)$	
Temporal	$\sqrt{T}\left(\widehat{\beta}^{*} - \widehat{\beta}\right) \underset{T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\sigma^2_f\left(Q^{-1}\underline{Q}Q^{-1}\right) + \frac{\sigma^2_\varepsilon}{N}Q^{-1}$	*No*
One − way ECM	$\sqrt{T}\left(\widehat{\beta}^{*} - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\sigma^2_f\left(Q^{-1}\underline{Q}Q^{-1}\right)$	
Two − way	$\sqrt{N}\left(\widehat{\beta}^{*} - \widehat{\beta}\right) \underset{N\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\sigma^2_\mu\left(Q^{-1}\overline{Q}Q^{-1}\right) + \frac{\sigma^2_\varepsilon}{T}Q^{-1}$	*No*
ECM	$\sqrt{N}\left(\widehat{\beta}^{*} - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\sigma^2_\mu\left(Q^{-1}\overline{Q}Q^{-1}\right)$	
Factor model	$\sqrt{N}\left(\widehat{\beta}^{*} - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\sigma^2_\mu\left(Q^{-1}\overline{Q}Q^{-1}\right)$	*Yes*

Table 3 : Asymptotic distribution with individual bootstrap

Temporal Bootstrap

$$U^*_{temp} - E^*\left(U^*\right) = \left(\widehat{f}^* - \widehat{f}\right) + \widehat{\lambda}^{/}\left(\widehat{F}^* - \widehat{F}\right) + \left([\widehat{\varepsilon}_{\text{col}}]^* - \widehat{\varepsilon}\right) \qquad (4.18)$$

Model	*Asymptotic distribution*	*Variance* (Ω)	*Consistency*
Individual One − way ECM	$\sqrt{NT}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$	$\sigma^2_{\varepsilon}Q^{-1}$	*No*
	$\sqrt{NT}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$	$\sigma^2_{\varepsilon}Q^{-1}$	
Temporal One − way ECM	$\sqrt{NT}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$	$\sigma^2_f\left(Q^{-1}\underline{Q}Q^{-1}\right) + \frac{\sigma^2_{\varepsilon}}{N}Q^{-1}$	*No if f_t is correlated*
	$\sqrt{NT}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$	$\sigma^2_f\left(Q^{-1}\underline{Q}Q^{-1}\right)$	
Two − way ECM	$\sqrt{T}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$	$\sigma^2_f\left(Q^{-1}\underline{Q}Q^{-1}\right) + \frac{\sigma^2_{\varepsilon}}{N}Q^{-1}$	*No*
	$\sqrt{T}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$	$\sigma^2_f\left(Q^{-1}\overline{Q}Q^{-1}\right)$	
Factor model	$\sqrt{T}\left(\widehat{\beta}^* - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{m.s.*}{\to}} 0$		*No*

Table 4 : Asymptotic distributions with temporal bootstrap

Block Bootstrap

$$U_{bl}^{*}-E^{*}\left(U_{bl}^{*}\right)=\left(\widehat{f}_{bl}^{*}-\widehat{f}\right)+\widehat{\lambda}^{/}\left(\widehat{F}_{bl}^{*}-\widehat{F}\right)+\left([\widehat{\varepsilon}_{\text{col}}]_{bl}^{*}-\widehat{\varepsilon}\right)$$

Model	*Asymptotic distribution*	*Variance* (Ω)	*Consistency*
Individual One − way ECM	$\sqrt{NT}\left(\widehat{\beta}^{*}-\widehat{\beta}\right)\underset{T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\Omega\right)$	$\sigma_{\varepsilon}^{2}Q^{-1}$	*No*
	$\sqrt{NT}\left(\widehat{\beta}^{*}-\widehat{\beta}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\Omega\right)$	$\sigma_{\varepsilon}^{2}Q^{-1}$	
Temporal One − way ECM	$\sqrt{NT}\left(\widehat{\beta}^{*}-\widehat{\beta}\right)\underset{T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\Omega\right)$	$\Sigma_{f}^{\infty}+\frac{\sigma_{\varepsilon}^{2}}{N}Q^{-1}$	*Yes*
	$\sqrt{NT}\left(\widehat{\beta}^{*}-\widehat{\beta}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\Omega\right)$	Σ_{f}^{∞}	
Two − way ECM	$\sqrt{T}\left(\widehat{\beta}^{*}-\widehat{\beta}\right)\underset{T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\Omega\right)$	$\Sigma_{f}^{\infty}+\frac{\sigma_{\varepsilon}^{2}}{N}Q^{-1}$	*No*
	$\sqrt{T}\left(\widehat{\beta}^{*}-\widehat{\beta}\right)\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}N\left(0,\Omega\right)$	Σ_{f}^{∞}	
Factor model	$\sqrt{T}\left(\widehat{\beta}^{*}-\widehat{\beta}\right)\underset{N,T\to\infty}{\overset{m.s.*}{\to}}0$		*No*

Table 4 : Asymptotic distributions with block bootstrap

Double Resampling Bootstrap

$$U^{**} = \widehat{\mu}^{*} + \widehat{f}^{*} + \widehat{\lambda}^{*/}\widehat{F}^{*} + \widehat{\varepsilon}^{**}$$

$$U^{**} - E^{**}\left(U^{**}\right) = \left(\widehat{\mu}^{*} - \widehat{\mu}\right) + \left(\widehat{f}^{*} - \widehat{f}\right) + \left(\widehat{\lambda}^{*/} - \widehat{\lambda}^{/}\right)\left(\widehat{F}^{*} - \widehat{F}\right) + \left(\widehat{\varepsilon}^{**} - \widehat{\varepsilon}\right)$$

Model	*Asymptotic distribution*	*Variance* (Ω)	*Consistency*
Individual One − way ECM	$\sqrt{N}\left(\widehat{\beta}^{**} - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$	$\sigma_{\mu}^{2}\left(Q^{-1}\overline{Q}Q^{-1}\right)$	*Yes*
Temporal One − way ECM	$\sqrt{N}\left(\widehat{\beta}^{**} - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$	$\sigma_{f}^{2}\left(Q^{-1}\underline{Q}Q^{-1}\right)$	*Yes*
Two − way	$\sqrt{N}\left(\widehat{\beta}^{**} - \widehat{\beta}\right) \underset{\substack{N,T\to\infty \\ \frac{N}{T}\to\delta\in[0,\infty)}}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$	$\sigma_{\mu}^{2}\left(Q^{-1}\overline{Q}Q^{-1}\right) + \delta\sigma_{f}^{2}\left(Q^{-1}\underline{Q}Q^{-1}\right)$	*No if f_t is correlated*
ECM	$\sqrt{T}\left(\widehat{\beta}^{**} - \widehat{\beta}\right) \underset{\substack{N,T\to\infty \\ \frac{N}{T}\to\infty}}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$	$\sigma_{f}^{2}\left(Q^{-1}\underline{Q}Q^{-1}\right)$	
Factor model	$\sqrt{N}\left(\widehat{\beta}^{**} - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$	$\sigma_{\mu}^{2}\left(Q^{-1}\overline{Q}Q^{-1}\right)$	*Yes*

Table 6 : Asymptotic distributions with double resampling bootstrap : *scheme 1*

$$U_{bl}^{**} - E^{**}\left(U_{bl}^{**}\right) = \left(\widehat{\mu}^{*} - \widehat{\mu}\right) + \left(\widehat{f}_{bl}^{*} - \widehat{f}\right) + \left(\widehat{\lambda}^{*/} - \widehat{\lambda}^{/}\right)\left(\widehat{F}_{bl}^{*} - \widehat{F}\right) + \left(\widehat{\varepsilon}_{bl}^{**} - \widehat{\varepsilon}\right) \qquad (4.21)$$

Model	*Asymptotic distribution*	*Variance* (Ω)	*Consistency*
Individual One − way ECM	$\sqrt{N}\left(\widehat{\beta}^{**} - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\sigma_{\mu}^{2}\left(Q^{-1}\overline{Q}Q^{-1}\right)$	*Yes*
Temporal One − way ECM	$\sqrt{N}\left(\widehat{\beta}^{**} - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	Σ_{f}^{∞}	*Yes*
Two − way	$\sqrt{N}\left(\widehat{\beta}^{**} - \widehat{\beta}\right) \underset{\frac{N}{T}\to\delta\in[0,\infty)}{\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}} N(0,\Omega)$	$\sigma_{\mu}^{2}\left(Q^{-1}\overline{Q}Q^{-1}\right) + \delta\Sigma_{f}^{\infty}$	*Yes*
ECM	$\sqrt{T}\left(\widehat{\beta}^{**} - \widehat{\beta}\right) \underset{\frac{N}{T}\to\infty}{\underset{N,T\to\infty}{\overset{*}{\Longrightarrow}}} N(0,\Omega)$	Σ_{f}^{∞}	
Factor model	$\sqrt{N}\left(\widehat{\beta}^{**} - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N(0,\Omega)$	$\sigma_{\mu}^{2}\left(Q^{-1}\overline{Q}Q^{-1}\right)$	*Yes*

Table 7 : Asymptotic distributions with double resampling bootstrap : *scheme 2*

5 Simulations

Data Generating Process for errors is the following : $\mu_i \sim i.i.d.N(0,1)$, $\lambda_i \sim i.i.d.N(0,1)$, $f_t \sim i.i.d.N(0,1)$, $\varepsilon_{it} \sim i.i.d.N(0,1)$, $F_t = \rho F_{t-1} + \eta_t$, $\eta_t \sim i.i.d.N\left(0,\left(1-\rho^2\right)\right)$ $\rho = 0$ and $\rho = 0.5$. Data Generating Process for data for regressors is the following : $\theta = 1$, $U_i \sim i.i.d.N(1,1)$, $W_t \sim i.i.d.N(1,1)$, $X_{it} \sim i.i.d.N(1,1)$. For each bootstrap resampling method, *999 Replications and 1000 Simulations* are used. Six sample sizes are considered : $(N,T) = (30,30)$, $(50,50)$, $(100,100)$, $(50,10)$ and $(10,50)$. Tables 7, 8, 9, 10 and 11 give rejection rates, for theoretical level $\alpha = 5\%$.

		I.i.d.	Ind.	Temp.	Block	2Res-1	2Res-2
	θ	0.480	**0.076**	0.620	0.676	0.070	**0.065**
Individual	τ	0.656	**0.077**	0.740	0.774	0.006	**0.064**
One-way	γ	0.007	**0.059**	0.067	0.098	0.069	**0.015**
ECM	ζ	0.046	**0.053**	0.161	0.420	0.049	**0.059**
Temporal	θ	0.619	0.716	0.197	0.146	0.186	0.123
One-way	τ	0.007	0.080	**0.063**	0.210	0.009	0.003
ECM	γ	0.619	0.733	**0.065**	0.080	**0.060**	**0.058**
$\rho = 0.5$	ζ	0.059	0.165	**0.063**	0.058	**0.064**	**0.053**
	θ	0.554	0.183	0.202	0.277	**0.062**	0.096
Two-way	τ	0.527	0.077	0.705	0.770	**0.065**	**0.053**
ECM	γ	0.548	0.737	0.069	0.095	**0.068**	**0.078**
$\rho = 0$	ζ	0.051	0.118	0.128	0.292	**0.057**	**0.065**
	θ	0.562	0.184	0.185	0.377	**0.046**	0.180
Two-way	τ	0.560	0.074	0.748	0.724	**0.069**	**0.058**
ECM	γ	0.535	0.721	0.071	0.079	**0.067**	**0.068**
$\rho = 0.5$	ζ	0.053	0.115	0.107	0.295	**0.060**	**0.065**
	θ	0.430	**0.063**	0.543	0.588	**0.048**	**0.046**
Factor	τ	0.574	**0.063**	0.668	0.664	**0.064**	**0.041**
model	γ	0.021	**0.060**	0.071	0.086	**0.009**	**0.014**
	ζ	0.056	**0.060**	0.127	0.287	**0.059**	**0.060**

Table 7 : Simulations with $(N;T)$=(30;30)

		I.i.d.	Ind.	Temp.	Block	2Res-1	2Res-2
	θ	0.569	**0.069**	0.685	0.739	**0.062**	**0.042**
Individual	τ	0.690	**0.067**	0.775	0.798	**0.068**	**0.063**
One-way	γ	0.008	**0.057**	0.061	0.077	**0.009**	**0.010**
ECM	ζ	0.060	**0.063**	0.184	0.391	**0.061**	**0.048**
	θ	0.696	0.781	0.168	0.130	0.160	0.098
Temporal	τ	0.010	0.076	0.072	0.203	*0.013*	*0.007*
One-way	γ	0.687	0.773	**0.057**	**0.070**	**0.053**	**0.055**
ECM	ζ	0.050	0.160	**0.051**	**0.066**	**0.050**	**0.058**
	θ	0.681	0.196	0.202	0.242	**0.056**	0.099
Two-way	τ	0.609	0.066	0.761	0.794	**0.058**	**0.069**
ECM	γ	0.636	0.781	0.060	0.072	**0.059**	**0.068**
$\rho = 0$	ζ	0.060	0.112	0.113	0.282	**0.060**	**0.064**
	θ	0.636	0.174	0.167	0.343	**0.048**	0.160
Two-way	τ	0.640	0.060	0.777	0.797	**0.058**	**0.066**
ECM	γ	0.653	0.794	0.051	0.078	**0.051**	**0.068**
$\rho = 0.5$	ζ	0.050	0.136	0.122	0.278	**0.058**	**0.050**
	θ	0.513	0.061	0.603	0.618	**0.057**	**0.052**
Factor	τ	0.670	0.060	0.745	0.731	**0.057**	**0.055**
model	γ	0.022	0.066	0.062	0.074	**0.012**	**0.011**
	ζ	0.058	0.057	0.128	0.290	**0.058**	**0.061**

Table 8 : Simulations with (N;T)=(50;50)

		I.i.d.	Ind.	Temp.	Block	2Res-1	2Res-2
Individual One-way ECM	θ	0.709	**0.057**	0.794	0.801	**0.049**	**0.047**
	τ	0.782	**0.064**	0.843	0.861	**0.063**	**0.045**
	γ	0.006	**0.045**	0.049	0.063	**0.005**	**0.005**
	ζ	0.041	**0.046**	0.160	0.377	**0.042**	**0.043**
Temporal One-way ECM $\rho = 0.5$	θ	0.796	0.849	0.196	0.299	0.188	0.278
	τ	0.009	0.051	**0.048**	0.453	0.009	0.002
	γ	0.794	0.861	**0.051**	**0.062**	**0.042**	**0.062**
	ζ	0.057	0.160	**0.058**	**0.056**	**0.055**	**0.053**
Two-way ECM $\rho = 0$	θ	0.737	0.147	0.154	0.315	**0.047**	0.142
	τ	0.735	0.057	0.850	0.875	**0.054**	**0.049**
	γ	0.745	0.852	0.062	0.075	**0.061**	**0.067**
	ζ	0.047	0.112	0.112	0.277	**0.048**	**0.058**
Two-way ECM $\rho = 0.5$	θ	0.789	0.247	0.267	0.459	0.109	0.295
	τ	0.751	0.070	0.852	0.887	0.073	**0.043**
	γ	0.712	0.830	0.048	0.072	**0.047**	**0.056**
	ζ	0.040	0.103	0.107	0.294	**0.049**	**0.057**
Factor model	θ	0.649	**0.051**	0.726	0.719	**0.043**	**0.050**
	τ	0.744	**0.052**	0.796	0.800	**0.049**	**0.045**
	γ	0.031	**0.056**	0.056	0.066	**0.009**	**0.010**
	ζ	0.046	**0.050**	0.107	0.267	**0.052**	**0.056**

Table 9 : Simulations with $(N;T)$=(100;100)

		I.i.d.	Ind.	Temp.	Block	2Res-1	2Res-2
Individual One-way ECM	θ	0.242	**0.054**	0.458	0.586	**0.035**	**0.044**
	τ	0.402	**0.051**	0.584	0.684	**0.039**	**0.040**
	γ	0.008	**0.059**	0.109	0.186	**0.013**	**0.021**
	ζ	0.043	**0.041**	0.163	0.424	**0.044**	**0.061**
Temporal One-way ECM	θ	0.713	0.777	0.283	0.420	0.266	0.395
	τ	0.011	0.062	0.089	0.310	0.008	0.018
	γ	0.675	0.748	0.120	0.243	0.110	0.225
	ζ	0.054	0.133	0.079	0.145	0.062	0.090
Two-way ECM $\rho = 0$	θ	0.561	0.410	0.145	0.295	0.091	0.195
	τ	0.329	0.062	0.572	0.688	0.048	0.034
	γ	0.667	0.781	0.116	0.190	0.117	0.179
	ζ	0.037	0.106	0.148	0.372	0.045	0.067
Two-way ECM $\rho = 0.5$	θ	0.610	0.483	0.267	0.420	0.192	0.286
	τ	0.332	0.058	0.597	0.669	0.053	0.048
	γ	0.642	0.777	0.107	0.191	0.099	0.179
	ζ	0.056	0.097	0.161	0.351	0.059	0.063
Factor model	θ	0.215	**0.055**	0.404	0.554	**0.039**	**0.031**
	τ	0.363	**0.053**	0.543	0.639	**0.041**	**0.045**
	γ	0.030	**0.068**	0.114	0.185	**0.015**	**0.023**
	ζ	0.049	**0.051**	0.140	0.406	**0.046**	**0.051**

Table 10 : Simulations with $(N;T)$=(50;10)

		I.i.d.	Ind.	Temp.	Block	2Res-1	2Res-2
Individual One-way ECM	θ	0.600	0.111	0.708	0.731	0.106	0.099
	τ	0.730	0.104	0.791	0.789	0.102	0.102
	γ	0.008	0.093	0.065	0.054	0.008	0.006
	ζ	0.056	0.076	0.137	0.345	0.060	0.065
Temporal One-way ECM	θ	0.386	0.573	0.152	0.134	0.130	0.084
	τ	0.012	0.130	0.064	0.186	**0.014**	0.006
	γ	0.395	0.578	**0.060**	**0.065**	**0.048**	**0.060**
	ζ	0.038	0.178	**0.043**	**0.042**	**0.041**	**0.041**
Two-way ECM $\rho = 0$	θ	0.573	0.140	0.421	0.440	0.078	0.093
	τ	0.652	0.109	0.781	0.793	0.108	0.101
	γ	0.340	0.578	0.057	0.078	0.051	0.069
	ζ	0.062	0.169	0.117	0.222	0.061	0.060
Two-way ECM $\rho = 0.5$	θ	0.601	0.173	0.458	0.473	0.116	0.133
	τ	0.673	0.126	0.805	0.805	0.124	0.115
	γ	0.326	0.575	0.058	0.081	0.044	0.062
	ζ	0.054	0.149	0.098	0.230	0.064	0.055
Factor model	θ	0.529	0.126	0.612	0.627	0.105	0.090
	τ	0.647	0.122	0.705	0.734	0.107	0.105
	γ	0.032	0.095	0.054	0.074	0.005	0.012
	ζ	0.046	0.061	0.090	0.249	0.053	0.060

Table 11 : Simulations with $(N;T)$=(10;50)

6 Conclusion

This paper considers the issue of bootstrap methods for panel data models. Four specifications of panel data have been considered, namely, individual one-way error component model, temporal one-way error component model, two-way error component model and lastly factor model. Five bootstrap methods are explored in order to make inference about vector of parameters.It is demonstrated that simple resampling methods (i.i.d., individual only or temporal only) are not valid in simple cases of interest, while double resampling that combines resampling in both individual and temporal dimensions is valid in these situations. Simulations confirm these results. A logical follow-up of this paper is to extend the results to dymamic panel models.

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7 APPENDIX

Appendix 1 :
Matrix Notations

Error terms :

$$\underset{(N,T)}{U} = \begin{pmatrix} U_{11} & U_{12} & ... & ... & U_{1T} \\ U_{21} & U_{22} & ... & ... & U_{2T} \\ ... & ... & ... & ... & ... \\ ... & ... & ... & .. & ... \\ U_{N1} & U_{N2} & .. & ... & U_{NT} \end{pmatrix} = \begin{pmatrix} U_{(1)} \\ U_{(2)} \\ ... \\ ... \\ U_{(N)} \end{pmatrix}$$

$$\underset{(NT,1)}{\widetilde{U}} = \begin{pmatrix} U'_{(1)} & U'_{(2)} & ... & ... & U'_{(N)} \end{pmatrix}'$$

Regressors :

$$\underset{(NT,K)}{\widetilde{Z}} = \begin{pmatrix} Z_{(11)} \\ Z_{(12)} \\ ... \\ ... \\ Z_{(NT)} \end{pmatrix}$$

$$\underset{(1,K)}{Z_{(i,t)}} = \begin{pmatrix} Z^{(1)}_{(i,t)} & Z^{(2)}_{(i,t)} & .... & Z^{(K)}_{(i,t)} \end{pmatrix}$$

$$\underset{(T,K)}{Z_{(i)}} = \begin{pmatrix} Z_{(i,1)} & Z_{(i,2)} & ... & Z_{(i,T)} \end{pmatrix}' ; \ \underset{(N,K)}{Z_{(t)}} = \begin{pmatrix} Z_{(1,t)} & Z_{(2,t)} & ... & Z_{(N,t)} \end{pmatrix}'$$

Subbar is put for aggeegation in individual dimension. Upbar refers to aggregation in temporal dimension.

$$\underset{(1,K)}{\overline{Z}_{(i)}} = \frac{1}{T}\sum_{t=1}^{T} Z_{(it)} \quad , \ \underset{(1,K)}{\underline{Z}_{(t)}} = \frac{1}{N}\sum_{i=1}^{N} Z_{(it)}$$

$$\underset{(N,K)}{\overline{Z}} = \begin{pmatrix} \overline{Z}_{(1)} & \overline{Z}_{(2)} & ... & \overline{Z}_{(N)} \end{pmatrix}' \quad ; \quad \underset{(T,K)}{\underline{Z}} = \begin{pmatrix} \underline{Z}_{(1)} & \underline{Z}_{(2)} & ... & \underline{Z}_{(T)} \end{pmatrix}' ;$$

Mixing Process

Definition

Let $\{f_t\}_{t\in\mathbb{Z}}$ be a sequence of random variables. The strong mixing or α-mixing coefficient of $\{f_t\}_{t\in\mathbb{Z}}$ is defined as :

$$\begin{aligned} \alpha(j) &= \sup\{|P(A\cap B) - P(A)P(B)|\},\ j\in\mathbb{N} \\ with\ A &\in \sigma\langle\{f_t : t\leq k\}\rangle, B\in\sigma\langle\{f_t : t\geq k+j+1\}\rangle,\ k\in\mathbb{Z} \end{aligned}$$

$\{f_t\}_{t\in\mathbb{Z}}$ is called strongly mixing (or α-mixing) if $\alpha(j)\to 0$ as $j\to\infty$.

Appendix 2 : Classical Asymptotic Theory

Individual One-way ECM

$$\eta_i = \left(\frac{1}{T}\sum_{t=1}^{T} Z^{/}_{(it)} E_{it}\right) = \frac{1}{T} Z^{/}_{(i)} E_{(i)}$$

$$E(\eta_i) = 0$$

$$Var(\eta_i) = \frac{1}{T^2} Z^{/}_{(i)} \left[Var\left(E^{/}_{(i)}\right)\right] Z_{(i)}$$

$$\left[Var\left(E^{/}_{(i)}\right)\right] = \sigma^2_{\varepsilon} I_T + \sigma^2_{\mu} J_T$$

where I_T is $(T*T)$ identity matrix, and J_T is $(T*T)$ matrix with each element equal to one. η_i are independent. Lindeberg condition can be written :

$$\lim_{N\to\infty} \max_{1\leq i\leq N} \overline{Z}^{/}_{(i)} \overline{Z}_{(i)} \left(\overline{Z}^{/}\overline{Z}\right)^{-1} = 0$$

$$\begin{aligned} \frac{1}{N}\sum_{i=1}^{N} Var(\eta_i) &= \frac{1}{N}\sum_{i=1}^{N} \frac{1}{T^2} Z^{/}_{(i)} \left[Var\left(E^{/}_{(i)}\right)\right] Z_{(i)} = \sigma^2_{\varepsilon} \frac{\widetilde{Z}^{/}\widetilde{Z}}{N}\frac{1}{T^2} + \sigma^2_{\mu}\frac{\overline{Z}^{/}\overline{Z}}{N} \\ &\underset{N}{\overset{P}{\to}} \sigma^2_{\varepsilon}\frac{1}{T} Q + \sigma^2_{\mu}\overline{Q} \end{aligned}$$

Apply Lindberg-Feller CLT to $(\eta_1, \eta_2, \ldots\ldots, \eta_N)$.

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\eta_i \underset{NT\to\infty}{\Longrightarrow} N\left(0, \sigma^2_{\varepsilon}\frac{1}{T}Q + \sigma^2_{\mu}\overline{Q}\right)$$

$$\sqrt{N}\left(\widehat{\beta} - \beta\right) \underset{N\to\infty}{\Longrightarrow} N(0, \Omega)$$

with

$$\Omega = Q^{-1}\Lambda Q^{-1} = \sigma^2_{\mu}\left(Q^{-1}\overline{Q}Q^{-1}\right) + \frac{\sigma^2_{\varepsilon}}{T} Q^{-1}$$

Two-way ECM

a) $\frac{N}{T} \to \delta \in [0,\infty)$

$$\sqrt{N}\left(\widehat{\beta}-\beta\right)=\left(\frac{\widetilde{Z}^{/}\widetilde{Z}}{NT}\right)^{-1}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{Z}_{(i)}^{/}\mu_i+\frac{\sqrt{N}}{\sqrt{T}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\underline{Z}_{(t)}^{/}f_t+\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}Z_{(it)}^{/}\varepsilon_{it}\right)\right]$$

$$\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}Z_{(it)}^{/}\varepsilon_{it}\right)\underset{T\to\infty}{\overset{m.s.}{\to}}0$$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{Z}_{(i)}^{/}\mu_i\underset{N\to\infty}{\Longrightarrow}N\left(0,\sigma_{\mu}^{2}\overline{Q}\right)$$

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\underline{Z}_{(t)}^{/}f_t\underset{T\to\infty}{\Longrightarrow}N\left(0,\sigma_{f}^{2}\underline{Q}\right)$$

$$\sqrt{N}\left(\widehat{\beta}-\beta\right)\underset{N,T\to\infty}{\Longrightarrow}N\left(0,\sigma_{\mu}^{2}\left(Q^{-1}\overline{Q}Q^{-1}\right)+\delta\sigma_{f}^{2}\left(Q^{-1}\underline{Q}Q^{-1}\right)\right)$$

b) $\frac{N}{T} \to \infty$

$$\sqrt{T}\left(\widehat{\beta}-\beta\right)=\left(\frac{\widetilde{Z}^{/}\widetilde{Z}}{NT}\right)^{-1}\left[\frac{\sqrt{T}}{\sqrt{N}}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{Z}_{(i)}^{/}\mu_i\right)+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\underline{Z}_{(t)}^{/}f_t+\frac{1}{\sqrt{N}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}Z_{(it)}^{/}\varepsilon_{it}\right)\right]$$

$$\frac{1}{\sqrt{N}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}Z_{(it)}^{/}\varepsilon_{it}\right)\underset{N\to\infty}{\overset{m.s.}{\to}}0$$

$$\frac{\sqrt{T}}{\sqrt{N}}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{Z}_{(i)}^{/}\mu_i\right)\underset{N,T\to\infty}{\overset{m.s.}{\to}}0$$

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\underline{Z}_{(t)}^{/}f_t\underset{T\to\infty}{\Longrightarrow}N\left(0,\sigma_{f}^{2}\underline{Q}\right)$$

The result follows.

Factor model

$$\kappa_i = \frac{1}{T}\sum_{t=1}^{T} Z_{(it)}^{/} F_t = \frac{1}{T} Z_{(i)}^{/} F$$

$$\sqrt{N}\left(\widehat{\beta}-\beta\right) = \left(\frac{\widetilde{Z}^{/}\widetilde{Z}}{NT}\right)^{-1}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{Z}_{(i)}^{/}\mu_i + \left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\kappa_i\lambda_i\right) + \frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T} Z_{(it)}^{/}\varepsilon_{it}\right)\right]$$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\overline{Z}_{(i)}^{/}\mu_i \underset{N\to\infty}{\Longrightarrow} N\left(0, \sigma_\mu^2 \overline{Q}\right)$$

$$\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T} Z_{(it)}^{/}\varepsilon_{it}\right) \underset{T\to\infty}{\overset{m.s.}{\to}} 0$$

$$\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\kappa_i\lambda_i\right) \underset{T\to\infty}{\overset{m.s.}{\to}} 0$$

The result follows.

Appendix 3 : Iid Bootstrap

Proposition 2 *: CLT for i.i.d. bootstrap*

Under assumptions A and B, and appropriate Lindeberg conditions, if the bootstrap-variance of error term $Var^*\left(\widehat{E}_{it}^*\right) \overset{P}{\to} \sigma^2$*, then*

$$\sqrt{NT}\left(\widehat{\beta}_{iid}^* - \widehat{\beta}\right) \underset{N,T\to\infty}{\overset{*}{\Longrightarrow}} N\left(0, \sigma^2 Q^{-1}\right)$$

Proof of proposition 2. *Apply Lindeberg-Feller CLT to* $(\eta_{11}^*, \eta_{12}^*, ..., \eta_{NT}^*)$ *and correct the variance.*

$$E^*\left(\eta_{it}^*\right) = 0$$

$$\begin{aligned} Var^*\left(\eta_{it}^*\right) &\equiv Z_{(it)}^{/}\left[Var^*\left(\hat{E}_{it}^*\right)\right] Z_{(it)} \\ &= Z_{(it)}^{/} Z_{(it)} Var^*\left(\widehat{E}_{it}^*\right) \end{aligned}$$

$(\eta_{11}^*, \eta_{12}^*, ..., \eta_{NT}^*)$ *are independent with different variances.*

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} Var^*\left(\eta_{it}^*\right) = \frac{\widetilde{Z}^{/}\widetilde{Z}}{NT} Var^*\left(\widehat{E}_{it}^*\right) \underset{N,T\to\infty}{\overset{P}{\to}} Q\sigma^2$$

By Lindberg-Feller CLT,

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\eta_{it}^{*} \underset{NT\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\sigma^{2}Q\right)$$

then :

$$\sqrt{NT}\left(\widehat{\beta}_{iid}^{*}-\beta\right) \underset{NT\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\sigma^{2}Q^{-1}\right)$$

■

Appendix 4 : Individual Bootstrap

Proposition 3 *: CLT for individual bootstrap*

Under assumptions A and B, and appropriate Lindeberg conditions,

$$\sqrt{N}\left(\widehat{\beta}_{ind}^{*}-\widehat{\beta}\right) \underset{N\to\infty}{\overset{*}{\Longrightarrow}} N\left(0,\Omega\right)$$

with

$$\Omega = Q^{-1}\left[\underset{N\to\infty}{Plim}\left(\frac{1}{N}\sum_{i=1}^{N}Z_{(i)}^{/}\left[Var^{*}\left(\hat{E}_{(i)}^{*/}\right)\right]Z_{(i)}\right)\right]Q^{-1}$$

$$\sqrt{N}\left(\widehat{\beta}_{ind}^{*}-\widehat{\beta}\right) = \left(\frac{\widetilde{Z}^{/}\widetilde{Z}}{NT}\right)^{-1}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\eta_{i}^{*}\right]$$

$$\eta_{i}^{*} = \left(\frac{1}{T}\sum_{t=1}^{T}Z_{(it)}^{/}\hat{E}_{it}^{*}\right) = \frac{1}{T}Z_{(i)}^{/}\hat{E}_{(i)}^{*}$$

Proof of Proposition 3.

$$Var^{*}\left(\eta_{t}^{*}\right) = \frac{1}{T^{2}}Z_{(i)}^{/}\left[Var^{*}\left(\hat{E}_{(i)}^{*/}\right)\right]Z_{(i)}$$

$(\eta_{1}^{*},\eta_{i}^{*},......,\eta_{N}^{*})$ *are independent with different variances. Under assumptions G, apply Lindberg-Feller CLT to* $(\eta_{1}^{*},\eta_{i}^{*},......,\eta_{N}^{*})$.

$$\Omega = Q^{-1}\left[\underset{N\to\infty}{Lim}\left(\frac{1}{N}\sum_{i=1}^{N}Z_{(i)}^{/}\left[Var^{*}\left(\hat{E}_{(i)}^{*/}\right)\right]Z_{(i)}\right)\right]Q^{-1}$$

■

Appendix 5 : Temporal Bootstrap

$$\sqrt{T}\left(\widehat{\beta}_{temp}^{*}-\widehat{\beta}\right) = \left(\frac{\widetilde{Z}^{/}\widetilde{Z}}{NT}\right)^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\eta_{t}^{*}\right]$$

$$\eta_{t}^{*} = \left(\frac{1}{N}\sum_{i=1}^{N}Z_{(it)}^{/}\hat{E}_{it}^{*}\right) = \frac{1}{N}Z_{(t)}^{/}\hat{E}_{(t)}^{*}$$

Proposition 4 *: CLT for temporal bootstrap*

Under assumptions A and B, and appropriate Lindeberg conditions,

$$\sqrt{T}\left(\widehat{\beta}^{*}_{temp.} - \widehat{\beta}\right) \underset{T\rightarrow\infty}{\overset{*}{\Longrightarrow}} N\left(0, \Omega\right)$$

with

$$\Omega = Q^{-1}\left[\underset{T\rightarrow\infty}{Plim}\left(\frac{1}{T}\sum_{t=1}^{T} Z^{/}_{(t)}\left[Var^{*}\left(\hat{E}^{*/}_{(t)}\right)\right] Z_{(t)}\right)\right] Q^{-1}$$

Demonstrations are similar to individual bootstrap case.

Appendix 6 : Block Bootstrap

Appendix 7 : Double Resampling Bootstrap