Saddlepoint Approximations for the Unit Root Power Envelope

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Abstract

The unit root power envelope is used both as a benchmark and as a mechanism for generating feasible unit root tests, via quasi-differencing. This paper derives an explicit representation for the envelope, via direct saddlepoint expansions for the inversion formulae for both the null and alternative distributions of the set of point optimal tests. It is shown to be both more accurate and computationally efficient than the current partial sum based approximations to limiting representations in terms of stochastic integrals. Accuracy is demonstrated through a sequence of experiments and efficiency via application to find the efficient detrending parameter in models with broken trends.

1 Introduction

The unit root power envelope, i.e. the set of powers of point optimal tests, serves two vital roles. First, in the absence of a uniformly most powerful test of a unit root every proposed feasible test must have its power characteristics compared with the envelope. Second, in the presence of deterministic components, the Quasi-Differenced (QD) Dickey-Fuller tests, introduced in Elliott, Rothenberg and Stock (1996) rely on our ability to calculate the point at which the envelope reaches 0.5. For sample size T, this paper provides a formal $O(T^{-1})$ saddlepoint approximation for both the size and power of point optimal unit root tests.

The power of unit root tests is known to be sensitive to the precise specification of any deterministic component in the model, see for example Durlauf and Phillips (1988), Perron (1989), Zivot and Andrews (1992), Leybourne, Mills and Newbold (1998) and Phillips and Xiao (1998, §4), Perron and Rodríguez (2003) and Harris, Harvey, Leybourne and Taylor (2007). In the presence of such model heterogeneity, to both objectively assess the performance of feasible tests and provide new QD-based tests requires a method which is both efficient and accurate. Below the Saddlepoint approximation is seen to achieve both.

Ever since the pioneering work of Phillips (1987a, 1987b) the *modus operandi* of papers in this area has been, for any given test, to characterize the asymptotic representation for the statistic itself. This is usually in terms of functionals of Brownian motion and under alternatives local to the null hypothesis. Evaluation of power is then achieved via Monte Carlo simulation either of partial sum approximations to stochastic integrals or by direct simulation of the statistic. Exceptions are the numerical inversions of asymptotic inversion formulae by Nabeya and Tanaka (1990) and Juhl and Xiao (2003) or the closed form results of Abadir (1993) and approximations of Larsson (1998). The latter two are available only in the special case of no deterministic component, while the former work applies only for the cases of a linear trend and/or constant. Here, the approximation may be used for any deterministic specification, unlike any of these methods.

This paper returns to the original characterization of unit root tests in the seminal paper of Dickey and Fuller (1979). There the "Dickey-Fuller" statistics were represented via an infinite sum of weighted chi-square random variables. Utilizing that approach here we are able to both write down inversion formulae for both the size and the power of any point optimal test and give an explicit asymptotic representation, via a Saddlepoint expansion. Using methods developed in Phillips (1978), Lugannani and Rice (1980), Daniels (1987), Jensen (1992), Lieberman (1994) and Marsh (1998), asymptotic size and power can be expressed as Gaussian probabilities at a quantity which may be instantly computed. Numerical evidence establishes that the proposed characterizations are much more accurate than relying upon partial sum approximations to limiting stochastic integrals and therefore, by implication, other approximations of those probabilities. Moreover, to illustrate computational efficiency, the approximation is used to evaluate the efficient QD parameter in set of broken trend models of Harris, Harvey, Leyboune and Taylor (2007). This was accomplished in a fraction of the time required for the equivalent Monte Carlo experiment, while none of the other techniques are available for these models.

The plan for the paper is as follows. The next section derives explicit inversions for the size and power of point optimal tests. Section 3 derives Saddlepoint approximations to these and compares the resulting asymptotic approximation with that based on partial sum approximations. Following the conclusions and references an appendix contains all proofs as well as tables used in the numerical evaluations.

2 Inversion Formulae for the Power Envelope

In common with the vast majority of the literature, the unit root problem is considered within the context of the following equations which specify the generation of data $(y_t)_{t=1}^T$, via

$$\begin{aligned} i) \quad &: \quad y_t = x_t'\beta + u_t \\ ii) \quad &: \quad u_t = \rho u_{t-1} + \eta_t \\ iii) \quad &: \quad \eta_t = \phi(l)\varepsilon_t \quad . \end{aligned} \tag{1}$$

In (1) x_t is a $k \times 1$ deterministic regressor, β a $k \times 1$ unknown parameter, $\phi(l)$ a lag polynomial, of degree m, with roots lying outside the unit circle and an error process

$(\varepsilon_t)_{t=1}^T$, which here will be assumed to be independent and identically distributed with zero mean and variance σ^2. To proceed define the following vectors and matrices; $y = (y_1, .., y_T)'$ and $\varepsilon = (\varepsilon_1, .., \varepsilon_T)'$, and let $X = (x_1, x_2, ..., x_T)'$ and $\beta = (\beta_1, ..., \beta_k)'$. Now let $L^{(j)}$ define a lower triangular matrix with $1's$ on the j^{th} lower diagonal and $0's$ elsewhere, so that we can construct the matrices

$$\Delta_\rho = I - \rho L^{(1)} \quad \text{and} \quad K_\phi = I + \sum_{j=1}^{m} L^{(j)} \phi_j,$$

then, when $y_0 = 0$, the equations in (1) define the following generalized linear regression model,

$$y = X\beta + \Delta_\rho^{-1} K_\phi \varepsilon. \tag{2}$$

The focus is upon tests on hypothesized values of ρ, specifically

$$H_0 : \rho = 1 \quad vs. \quad H_1 : |\rho| < 1. \tag{3}$$

Notice that neither the null nor the alternative in (3) change if the data y are transformed according to the following group of transformations $G = (a, g)$, with $a \in \mathbb{R}$ and $g \in \mathbb{R}^k$ and with action

$$y \to ay + Xg\,,$$

as a consequence the meaningful power envelope will be that for the class of invariant, under G, tests. Invariant tests can be found, most generally, under the following assumption on the distribution of y:

Assumption 1 *(i) Let the density of* y, given X, *be* $f(y; \beta, \rho, \sigma^2\Omega \,|\, X) = f(y) \in \mathcal{F}$, *the elliptically symmetric family, with*

$$\mathcal{F} = \left\{ f : f(y; \beta, \rho, \sigma^2\Omega) = \frac{q\left[(y - X\beta)'(\sigma^2\Sigma_\rho(\Omega))^{-1}(y - X\beta)\right]}{\left|\sigma^2\Sigma_\rho(\Omega)\right|^{1/2}} \right\},$$

where X *and* β *are defined above,* σ^2 *is a scalar and* $\Omega = K_\phi K_\phi'$ *a* $T \times T$ *matrix with* $\Sigma_\rho(\Omega) = \Delta_\rho^{-1}\Omega(\Delta_\rho^{-1})'$. *Furthermore, we assume* $q[.]$ *is a nonincreasing convex function on* $[0, \infty)$.

(ii) $||\Omega||_1 = \sup_j \sum_{i=1}^T |\Omega_{i,j}| < M < \infty$, *for all* T. ■

Notice that implicit in Assumption 1 as applied to the regression in (2) is that any value y_0 is taken to be zero. A non-zero, exogenously determined, observed initial condition may be incorporated into the regressor set X (as the column $\bar{x}_1 = (y_0, 0..0)'$) as detailed in Marsh (2007). By so doing it is possible to abstract the results that follow from any influence caused by uncertainty over the initial condition. In any case, all asymptotic results will apply for any initial value satisfying $T^{-1/2}y_0 = o_p(1)$. Part (ii) of Assumption 1 is satisfied, asymptotically, for the standard innovation assumption that

$$\lim_{m\to\infty} \sum_{j=1}^{m} j\left|\phi_j\right| < \infty, \tag{4}$$

as in Condition A of Elliott, Rothenberg and Stock (1996).

Defining $W = \Delta_1 X$ and applying standard results, see King (1980) and Dufour and King (1991), the maximal invariant for testing the hypotheses in (3) is given by

$$v = \frac{w}{|w|} = \frac{C'\Delta_1 y}{\sqrt{y'\Delta_1' M_W \Delta_1 y}},$$

where the symmetric idempotent is,

$$M_W = I - W\left(W'W\right)^{-1} W',$$

and has singular value decomposition,

$$CC' = M_W \quad ; \quad C'C = I_{T-k}.$$

Following King (1980), the density of v (with respect to normalized Haar measure on the surface of the Unit sphere in $n = T - k$ dimensions) is given by

$$pdf\ (v) = |A|^{-1/2}\left(v'A^{-1}v\right)^{-\frac{n}{2}}, \tag{5}$$

where

$$A = C'\Delta_1'\Delta_\rho^{-1}\Omega\left(\Delta_\rho^{-1}\right)'\Delta_1 C.$$

Applying the Neyman-Pearson Lemma, we have the most powerful test of H_0 against a fixed alternative $|\rho| < 1$, (i.e. the Point Optimal (PO) test) is given by:

$$\text{reject } H_0 \text{ if } PO = \frac{pdf\ (v|H_1)}{pdf\ (v|H_0)} > k_\alpha,$$

where k is chosen so that the size is fixed, at α say. Consequently, the PO test is,

$$\text{reject } H_0 \text{ if } PO = \frac{v' A^{-1} v}{v' \left(C'\Omega C\right)^{-1} v} < k_\alpha, \tag{6}$$

where the critical value k_α in (6) is chosen so that the size of the test is α, with

$$\alpha = \Pr\left[\frac{v' A^{-1} v}{v' \left(C'\Omega C\right)^{-1} v} < k_\alpha | H_0\right]. \tag{7}$$

The power envelope is then the set of powers of each PO test at size α, given by

$$\Pi_\alpha = \Pi_\alpha(\rho) = \Pr\left[\frac{v' A^{-1} v}{v' \left(C'\Omega C\right)^{-1} v} < k_\alpha | H_1\right]. \tag{8}$$

In the wider family $\mathcal{F}$, we explicitly require tests invariant to scale, i.e. to σ. By so doing we obtain results analogous to those presented in Elliott, Rothenberg and Stock (1996), for the narrower Gaussian case. To characterize those, assume σ^2 is known and that the ε_t are iid Gaussian, then the likelihood for y is (up to constants and other quantities not depending on ρ),

$$L = -\frac{1}{2}\left(y - X\beta\right)' \Delta_\rho^{-1} \Omega^{-1} \left(\Delta_\rho^{-1}\right)' \left(y - X\beta\right).$$

Noting that only invariance with respect to $y \to y + X\beta$ (or equivalently $\Delta_\rho y \to \Delta_\rho y + \Delta_\rho X\beta$, provided that $y_0 = 0$) is required, then the PO test involves rejecting for small values of

$$L^*(\Omega) = y_\rho' M_{W_\rho} \Omega^{-1} M_{W_\rho} y_\rho - y_1' M_{W_1} \Omega^{-1} M_{W_1} y_1, \tag{9}$$

where $y_\rho = \Delta_\rho y$ and $W_\rho = \Delta_\rho X$, and M_{W_ρ} is defined analogously to M_W. As Francke and de Vos (2007) point out, $L^*(\Omega)$ is the likelihood ratio in the marginalized likelihood. This approach is essentially identical to constructing the maximal invariant, which is in this case $\bar{y} = C'\Delta_1 y$, where C is as defined above, giving point optimal tests,

$$PO^* = y'\Delta_1' C A^{-1} C' \Delta_1 y - y' \Delta_1 M \Delta_1 y,$$

where A and M are defined above. Consequently, it is easily seen that previous characterizations of the unit root power envelope are either identical to or are special cases of that given here.

To characterize the power envelope, first consider

$$\tilde{A} = \left(C'\Omega C\right)^{-1/2} A \left(C'\Omega C\right)^{-1/2},$$

and its ordered eigenvalues $(\lambda_i)_{i=1}^n$, and the associated eigenvectors, r_i, with

$$\tilde{A} r_i = \lambda_i r_i \quad ; \quad r_i' r_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}.$$

Elliott, Rothenberg and Stock (1996) characterize the asymptotic power envelope generated by the set of tests defined in (9) in terms of probabilities of certain functionals of Brownian motion. Here, as with the characterizations of Dickey and Fuller (1979), it instead involves probabilities for certain weighted averages of Chi-square distributed random variables as in the following lemma.

Lemma 1 *Let z_i be a sequence of independent standard normal random variables, then under Assumption 1, the size (α) and power (Π_α) of the Point Optimal test in (6) are defined by;*

$$\begin{aligned} \alpha &= \Pr\left[\sum_{i=1}^{n}\left(\frac{1}{k_\alpha \lambda_i} - 1\right) z_i^2 < 0\right], \quad \textit{and} \\ \Pi_\alpha &= \Pr\left[\sum_{i=1}^{n}\left(1 - k_\alpha \lambda_i\right) z_i^2 < 0\right]. \quad \blacksquare \end{aligned} \tag{10}$$

Lemma 1 provides unresolved expressions for the size and power in terms of weighted sums of Chi-square random variables. Unlike related unresolved expressions involving probabilities of functionals of Brownian motion, these may be explicitly evaluated or approximated. For example, given a matrix $\tilde{A}$ and its eigenvalues $(\lambda_i)_{i=1}^n$, Imhof's (1961) procedure will yield accurate numerical approximations, although at the expense of any analytic information concerning the dependence of the probabilities in (10) on the model features, via those eigenvalues.

3 An Explicit Representation for the Asymptotic Power Envelope

In this section an asymptotic representation of the power envelope is derived and then analyzed in a series of Monte Carlo experiments. From Theorem 2 of Elliott, Rothenberg and Stock (1996), the asymptotic power envelope depends neither on σ^2, nor the coefficients in (4). Moreover, Jansson (2008) proves that the asymptotic envelope is identical for every Locally Asymptotically Normal likelihood. Given also that here we will impose invariance to scale, then the envelope is also identical for every member of the elliptically symmetric family. Consequently, the envelope obtained by imposing $\Omega = I$, with point optimal tests,

$$\text{reject } H_0 \text{ if } v'A^{-1}v < k_\alpha, \quad A = C'\Delta_1\Delta_\rho^{-1}\left(\Delta_\rho^{-1}\right)'\Delta_1 C, \tag{11}$$

is the benchmark against which all current procedures in the literature should be measured.

3.1 Main Result

Lemma 1 gives the size and power of the point optimal tests in terms of weighted sums of chi-square random variables. To derive expressions for asymptotic power, let $\rho = 1 - c/T$, where $c > 0$. Returning to the definitions in (7) and (8), the power envelope for unit root tests can be written in terms of probabilities associated with the two random variables,

$$p_0 = \sum_{i=1}^{n}\left(\lambda_i^{-1} - k_\alpha\right) z_i^2 \quad \text{and} \quad p_1 = \sum_{i=1}^{n}\left(1 - k_\alpha\lambda_i\right) z_i^2, \tag{12}$$

where $z_i \sim N(0,1)$ and the $(\lambda_i)_1^n$ are the ordered eigenvalues of A, as in (11), with $\lambda_1 \leq \lambda_n$. Size and power are then, $\alpha = \Pr[p_0 < 0]$ and $\Pi_\alpha = \Pr[p_1 < 0]$.

Now let $\lambda_i^0 = \lambda_i/\lambda_1$, and $\lambda_i^1 = \lambda_i/\lambda_n$, so that both $\left(\lambda_i^0\right)^{-1}$ and λ_i^1 are bounded between 0 and 1. Consequently, we can define

$$q_0 = \sum_{i=1}^{n}\left(\left(\lambda_i^0\right)^{-1} - \bar{k}_\alpha\right) z_i^2 \quad \text{and} \quad q_1 = \sum_{i=1}^{n}\left(\bar{\lambda} - \bar{k}_\alpha\lambda_i^1\right) z_i^2, \tag{13}$$

where $\bar{k}_\alpha = \lambda_1 k_\alpha$ and $\bar{\lambda} = \lambda_1/\lambda_n$, so that limiting size and power have the representation,

$$\alpha = \lim_{n\to\infty} \Pr[q_0 < 0] \quad \text{and} \quad \lim_{n\to\infty} \Pi_\alpha = \Pr[q_1 < 0].$$

What we require here is an explicit representation for both the size α, which may then be inverted to obtain the critical value $\bar{k}_\alpha$, and also the resulting power, Π_α. To proceed, note that the characteristic functions of q_0 and q_1 are,

$$\psi_0(i\omega) = E\left[\exp\left\{i\omega\sum_{i=1}^{n}\left(\left(\lambda_i^0\right)^{-1} - \bar{k}_\alpha\right)z_i^2\right\}\right] = \prod_{i=1}^{n}\left[1 - 2i\omega\left(\left(\lambda_i^0\right)^{-1} - \bar{k}_\alpha\right)\right]^{-1/2},$$

and

$$\psi_1(i\omega) = E\left[\exp\left\{i\omega\sum_{i=1}^{n}\left(\bar{\lambda} - \bar{k}_\alpha\lambda_i^1\right)\right\}\right] = \prod_{i=1}^{n}\left[1 - 2i\omega\left(\bar{\lambda} - \bar{k}_\alpha\lambda_i^1\right)\right]^{-1/2}.$$

Consequently, transforming with $\theta = i\omega$ and applying the closed curve theorem, the respective densities of q_0 and q_1 can be obtained, in principle, from the Inverse Fourier Transforms,

$$f_j(q) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^{-q\theta}\psi_j(\theta)d\theta, \quad j = 0, 1.$$

Immediately, the distributions of q_0 and q_1 are given by,

$$F_j(q) = \int_0^q f_j(k)dk, \qquad j = 0, 1,$$

and then the asymptotic size and power of point optimal tests are,

$$\alpha = \lim_{n\to\infty} F_0(q)\big|_{q=0} \quad \text{and} \quad \Pi_\alpha = \lim_{n\to\infty} F_1(q)\big|_{q=0}.$$

As is evident from the development above, in order to characterize the power envelope under Assumption 1, what is actually required is the distribution of weighted sums of Chi-squared random variables. Currently the literature does contain exact representations for such distributions, for example see Hillier (2000) or Forchini (2002). However, as yet no exact representation has been found which is at all suitable for actually providing usable critical values, nor indications of power; the current given expressions being far too complicated for such purposes. Instead here we utilize a form of the Saddlepoint approximation, developed through Phillips (1978), Lugannani and Rice (1980), Daniels (1987), Lieberman (1994) and Marsh (1998) to give an explicit representation for the asymptotic power envelope in terms only of simple Gaussian probabilities. Before proceeding define the following quantities,

$$\begin{aligned} R_j(\theta) &= \frac{1}{n}\ln\left[\psi_j(\theta)\right] && (14) \\ R_j(\theta) &= \operatorname{sign}[\theta]\sqrt{-2nR_j(\theta)} \quad \text{and} \quad \delta_j(\theta) = \theta\sqrt{n\frac{d^2R_j(\theta)}{d\theta^2}}, \end{aligned}$$

for $j = 0, 1$, then asymptotic representations for the size and power of point optimal unit root tests, and their orders of error, are as given in the following theorem, again proved in the appendix.

Theorem 1 *Suppose that Assumption 1 holds, then:*
(i) Asymptotically the size of the point optimal tests (6) for the unit root hypothesis in (3) satisfies,

$$\alpha = \Phi(\hat{\gamma}_0) - \phi(\hat{\gamma}_0)\left(\frac{1}{\hat{\delta}_0} - \frac{1}{\hat{\gamma}_0}\right) + O(n^{-1}), \tag{15}$$

where $\Phi[.]$ *is the standard normal CDF,*

$$\hat{\gamma}_0 = \gamma_0(\hat{\theta}_0) \quad \textit{and} \quad \hat{\delta}_0 = \delta_0(\hat{\theta}_0),$$

and the saddlepoint $\hat{\theta}_0$ *is the unique solution to,*

$$\sum_{i=1}^{n} \frac{\left(\left(\lambda_i^0\right)^{-1} - \bar{k}_\alpha\right)}{1 - 2\hat{\theta}_0\left(\left(\lambda_i^0\right)^{-1} - \bar{k}_\alpha\right)} = 0 \quad ; \quad \frac{1}{2\left(\left(\lambda_i^0\right)^{-1} - \bar{k}_\alpha\right)} \leq \hat{\theta}_0 \leq \frac{1}{2\left(\left(\lambda_i^0\right)^{-1} - \bar{k}_\alpha\right)}.$$

(ii) Letting k_α *be the unique solution to* $\lim_{n\to\infty} \Phi^{-1}[\alpha] = \hat{r}_0$, *then the asymptotic power of each point optimal test (6) satisfies,*

$$\Pi_\alpha = \lim_{n\to\infty} \Phi(\hat{\gamma}_1) - \phi(\hat{\gamma}_1)\left(\frac{1}{\hat{\delta}_1} - \frac{1}{\hat{\gamma}_1}\right) + O(n^{-1}), \tag{16}$$

where $\hat{\gamma}_1$ *and* $\hat{\delta}_1$ *are defined analogously, while in this case the saddlepoint* $\hat{\theta}_1$ *satisfies,*

$$\sum_{i=1}^{n} \frac{\left(\bar{\lambda} - \bar{k}_\alpha \lambda_i^1\right)}{1 - 2\hat{\theta}_1\left(\bar{\lambda} - \bar{k}_\alpha \lambda_i^1\right)} = 0 \quad ; \quad \frac{1}{2\left(\bar{\lambda} - \bar{k}_\alpha \lambda_i^1\right)} \leq \hat{\theta}_1 \leq \frac{1}{2\left(\bar{\lambda} - \bar{k}_\alpha \lambda_i^1\right)}. \quad \blacksquare$$

As in Jing and Robinson (1994), we may exploit the transformation detailed in Lemma 2.1 of Jensen (1992) to provide an even more computationally convenient representation for the asymptotic power envelope. Specifically, in terms of the approximate distributions above, that if, for q_j it is true that,

$$\Pr[q_j < 0] = \Phi(\hat{\gamma}_j) - \phi(\hat{\gamma}_j)\left(\frac{1}{\hat{\delta}_j} - \frac{1}{\hat{\gamma}_j}\right)(1 + o(1)),$$

then also,

$$\Pr[q_j < 0] = \Phi(r_j^*)(1 + o(1)), \tag{17}$$

where

$$r_j^* = r_j^*(q) = \hat{\gamma}_j + \frac{1}{\hat{\gamma}_j} \ln\left(\frac{\hat{\delta}_j}{\hat{\gamma}_j}\right).$$

Consequently, we can more easily find asymptotic critical values $\bar{k}_\alpha$, of size α,via

$$r_0^*(\bar{k}_\alpha) = \Phi^{-1}(\alpha), \tag{18}$$

and the asymptotic power associated with those critical values is then,

$$\Pi_\alpha = \Phi\left(\hat{\gamma}_1 + \frac{1}{\hat{\gamma}_1} \ln\left(\frac{\hat{\delta}_1}{\hat{\gamma}_1}\right)\right).$$

In the numerical analysis to follow the representations based on (17) were employed, rather than those given in Theorem 1, since inversion of (18) is far more straight forward than that of (15).

4 Numerical Analysis

Theorem 1 provides an $O(T^{-1}) = O(n^{-1})$ representation for the asymptotic power envelope. For a given critical value k_α the approximation of the power envelope, for finite T, at any point ρ can be calculated essentially instantaneously. This contrasts sharply with the often lengthy Monte Carlo simulation of partial sum approximations to stochastic integrals, the approach most often employed in the literature.

Computational efficiency is, of itself, a worthless virtue if what is being calculated does not accurately approximate what can be expected in finite samples. Consequently here we will numerically compare the accuracy of the formulae given in Theorem 1 with those partial sum approximations for the envelope of Elliott, Rothenberg and Stock (1996), as given by the point optimal tests in (9).

We will consider the two simple models most commonly employed,

$$M_a \;:\; y_t = \beta_1 + u_t \quad ; \quad u_t = (1 - c/T)u_{t-1} + \varepsilon_t \tag{19}$$

$$M_b \;:\; y_t = \beta_1 + \beta_2 t + u_t \quad ; \quad u_t = (1 - c/T)u_{t-1} + \varepsilon_t, \tag{20}$$

as well as two specifications for the errors,

$$E_1 : \varepsilon_t \sim iidN(0,1), \quad ; \quad E_2 : \varepsilon_t \sim \frac{iid\chi^2(1) - 1}{\sqrt{2}}.$$

The latter specification allows some limited analysis of the robustness of both methods of approximation to the specification of the underlying distribution. All of the numerical analysis to follow was performed using the symbolic package Mathematica and all experiments are based upon 20000 replications.

Tables 1a) and 1b) give the finite sample rejection frequencies of critical values, for two sizes, for the point optimal tests in (6) obtained from inverting (15). Sample sizes 25 through 150 and different values of the local parameter, $c = T(1-\rho)$, are considered for both models M_a and M_b, respectively. Tables 2a) and 2b) give those for critical values obtained through partial sum approximations of the stochastic integrals,

$$\begin{aligned} M_a \quad &: \quad L^*(I) + c \Rightarrow c^2 \int_0^1 W_0^2 + cW_0^2(1) \\ M_b \quad &: \quad L^*(I) + c \Rightarrow c^2 \int_0^1 V_0^2(t,-c) + (1+c)V_0^2(1,-c) \end{aligned} \tag{21}$$

with W and V defined as in Theorem 1 of Elliott, Rothenberg and Stock (1996), and noting the slightly different definitions of c. For both models, when the errors are Gaussian, the Saddlepoint approximation provides critical values which have true size almost uniformly equal to nominal. For critical values obtained by partial summation this is not so, particularly for larger values of c, although as the sample size grows the accuracy rapidly becomes much more acceptable.

Tables 3a) and 3b), as well as 4a) and 4b) report the outcomes of a repetition of the experiment but with errors which are proportional to a Chi-square. In this case the accuracy of the Saddlepoint approximation is maintained, particularly for sample sizes of 100 and more, while clearly the use of partial sum approximations does not deliver critical values with the same level of robustness, at least in the small samples given here. Although no author explicitly recommends use of such approximations in sample sizes this small it is worth noting that the range of sample sizes in the (updated) Nelson and Plosser (1982) data set falls within that considered here.

The primary purpose of this paper is to characterize the asymptotic power envelope and so the next set of experiments seek to analyze the accuracy of (16) as an approximation to finite sample power. Here for models M_a and M_b finite sample power was simulated for the same combinations of sample sizes and local parameters for the single size, 5% and only for the case of Gaussian errors. Note that the power envelope is no longer actually that if the errors are not Gaussian. Recorded in tables 5a) and 5b) under the heading MC are the simulated powers, minus the size 5%. Under the heading SA is the Saddlepoint approximation for the power of the point optimal test (6) based on approximate critical values obtained from (15), minus the size of those critical values. Once again the Saddlepoint approximation is seen to be highly accurate.

At each point the approximation is available essentially instantaneously. A Monte Carlo study, however, requires simulation of the point optimal test, or an approximation to its limiting form, both under the null and alternative. Given also its high accuracy then this method seems particularly appropriate for providing envelopes for the task of assessing the properties of feasible tests, in light of increased model heterogeneity. Moreover, the effect of any particular specification of the deterministics may be efficiently and accurately measured. To illustrate this efficacy, consider two models having breaks in their trends, viz.

$$\begin{aligned} M_1 &: \; y_t = \beta_1 + \beta_2(t - \tau T)I_t(\tau) + u_t \\ M_2 &: \; y_t = \beta_1 + \beta_2 t + \beta_3(t - \tau T)I_t(\tau) + u_t, \end{aligned} \tag{22}$$

where $u_t = (1 - c/T)u_{t-1} + \varepsilon_t$, $I_t(\tau)$ is the indicator function taking values 1 if $t \geq \tau T$, and 0 otherwise. Thus M_1 has a trend which only begins a fraction τ into the series while M_2 has a segmented trend, at the fraction τ. Even this rather simple extension, introduced in the work of Perron and Rodríguez (2003) and Harris, Harvey, Leybourne and Taylor (2007), implies computational complexities necessitating the use of explicit distributional formulae, as given here.

To apply the suggested procedure of Elliott, Rothenberg and Stock (1996), in full, requires calculation of the point $\bar{c}$ at which the power envelope is 0.5. Then $\bar{c}$ is used to Quasi-difference the data to ultimately form the t-test in the detrended data. Notionally, therefore, this value must be calculated for every τ, and moreover, for every chosen significance level. Table 6, in the appendix, gives values of $\bar{c}$ obtained from the expressions given in Theorem 1, for values of $\alpha = .01$, .05 and .10, and for fractions, $\tau = 0.1, 0.2, .., 0.9$ and based on a sample size of $T = 250$ (increasing the sample size beyond 250 did not alter the outcomes to the given accuracy). The computational time (for a 3.0 Ghz Pentium D PC running Mathematica 4.0) was just over 15 minutes per model. Notice also that the values of $\bar{c}$ for M_2 are almost identical to those derived by a partial summation approximation and given in Table 1 of Harris, Harvey, Leybourne and Taylor (2007), but are available at a fraction of the computational cost.

5 Conclusions

This paper has derived an explicit asymptotic representation for the invariant power envelope based upon Saddlepoint expansions of inversion formulae for the distributions of the set of point optimal tests. Recent innovations in the literature have greater heterogeneity in the model specification when unit root testing. The representations derived here have the advantages of computational efficiency and numerical accuracy over the more usual Monte Carlo partial summation approach as well as more generality than the approaches of Abadir (1993) or Nabeya and Tanaka (1990). Together these facts indicate the importance of these results in objectively assessing the performance of feasible tests in ever more general cases.

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Appendix I

(1) Proof of Lemma 1

For tests of size α, the critical value k_α is chosen according to,

$$\alpha = \Pr\left[\frac{v'A^{-1}v}{v'\left(C'\Omega C\right)^{-1}v} < k_\alpha \mid H_0\right] = \Pr\left[\frac{w'A^{-1}w}{w'\left(C'\Omega C\right)^{-1}w} < k_\alpha \mid H_0\right],$$

where $w = C'\Delta_1 y$. Then if we let $z = \left(C'\Omega C\right)^{1/2} w$ and $\tilde{A} = \left(C'\Omega C\right)^{-1/2} A \left(C'\Omega C\right)^{-1/2}$, we have

$$\begin{aligned}\alpha &= \Pr\left[\frac{z'\tilde{A}^{-1}z}{z'z} < k_\alpha \mid H_0\right] = \Pr\left[z'\tilde{A}^{-1}z < \left(w'w\right)k_\alpha \mid H_0\right] \\ &= \Pr\left[z'\left(\tilde{A}^{-1} - k_\alpha I_n\right)z < 0 \mid H_0\right].\end{aligned}$$

Since α does not depend upon the particular member of the elliptically symmetric family generating the data y, we can, without loss of generality, assume $z \sim N(0, I_n)$.

Consider the spectral decomposition of $\tilde{A}$,

$$\tilde{A} = U'\Lambda U,$$

with $U'U = I$ and $\Lambda = diag\left\{\lambda_1, \lambda_2, .., \lambda_{T-k}\right\}$, with λ_i the ordered eigenvalues of $\tilde{A}$, then also

$$\tilde{A}^{-1} = U\Lambda^{-1}U'.$$

Consequently, for $z_i \sim iidN(0,1)$, the critical value may instead be found from

$$\alpha = \Pr\left[\sum_{i=1}^{n}\left(\lambda_i^{-1} - k_\alpha\right)z_i^2 < 0\right] = \Pr\left[\sum_{i=1}^{n}\left(\frac{1}{k_\alpha\lambda_i} - 1\right)z_i^2 < 0\right].$$

Given the critical value, we can then calculate power,

$$\Pi_\alpha = \Pr\left[\frac{v'A^{-1}v}{v'\left(C'\Omega C\right)^{-1}v} < k_\alpha \mid H_1\right] = \Pr\left[\frac{w'A^{-1}w}{w'\left(C'\Omega C\right)^{-1}w} < k_\alpha \mid H_1\right].$$

Now define

$$\zeta = \sigma^{-1}A^{-1/2}w \sim N(0, I_n),$$

so that

$$\Pi_\alpha = \Pr\left[\frac{\zeta'\zeta}{\zeta' A^{1/2}\left(C'\Omega C\right)^{-1}A^{1/2}\zeta} < k_\alpha\right] = \Pr\left[\zeta'\left(I - k_\alpha A^{1/2}\left(C'\Omega C\right)^{-1}A^{1/2}\zeta\right)\zeta < k_\alpha\right],$$

and since the eigenvalues of $A^{1/2}\left(C'\Omega C\right)^{-1}A^{1/2}$ are identical to those of $\tilde{A}$, then, as required,

$$\Pi_\alpha = \Pr\left[\sum_{i=1}^{n}\left(1-k_\alpha\lambda_i\right)z_i^2<0\right]. \quad \blacksquare$$

(2) Proof of Theorem 1

In order to find both asymptotic critical values and thence the asymptotic power envelopes we require approximations for the distributions of q_0 and q_1. To proceed, define the distribution of q_j by

$$\begin{aligned} F_j(q) &= \Pr[q_j \leq q] = \int_0^q f_j(z)dz \\ &= 1-\int_q^\infty f_j(z)dz = 1-\int_q^\infty \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^{-z\theta}\psi_j(\theta)d\theta dz \\ &= 1-\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\frac{\exp\{-q\theta\}}{\theta}\psi_j(\theta)d\theta, \quad j=0,1. \end{aligned} \tag{23}$$

The integrals in (23) are precisely of the type considered in Lugannani and Rice (1980) and exploited to give tail probability approximations for ratios of quadratic forms in Lieberman (1994) and Marsh (1998). While derivation of the leading term approximation is quite routine (see for example Lieberman (1994)), here we focus on the asymptotic character of the approximation, i.e. that size and power can be explicitly approximated to order $O(n^{-1})$.

To proceed, let

$$\xi_{0,i} = \left(\left(\lambda_i^0\right)^{-1}-\bar{k}_\alpha\right) \quad \text{and} \quad \xi_{1,i} = \left(\bar{\lambda}-\bar{k}_\alpha\lambda_i^1\right),$$

where $\bar{k}_\alpha$ and $\bar{\lambda}$ are defined under (13), and write

$$\bar{\psi}_j(\theta) = \prod_{i=1}^{n}\left(1-2\theta\xi_{j,i}\right)^{-\frac{1}{2n}},$$

so that the integrals (23) can be written as

$$F_j(q) = 1-\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}\frac{\exp\{-q\theta\}}{\theta}\left(\bar{\psi}_j(\theta)\right)^n d\theta, \quad j=0,1$$

which is of the same form as the inversion in Lugannani and Rice (1980, equation (2)). Define

$$R_j(\theta) = \log\left[\bar{\psi}_j(\theta)\right] = -\frac{1}{2n}\sum_{i=1}^{n}\log\left(1-2\theta\xi_{j,i}\right),$$

so that

$$F_j(q) = 1 - \frac{1}{2\pi i} \int_{\tau - i\infty}^{\tau + i\infty} \frac{\exp\{n(R_j(\theta) - x\theta)\}}{\theta} d\theta, \quad j = 0, 1, \tag{24}$$

where $x = q/n$. Then for each x, a valid asymptotic expansion in powers of n^{-1} and derivatives of $R_j(\theta)$ of (23) exists if the two conditions in Section 4 of Lugannani and Rice (1980) hold and if the derivatives of $R_j^n(\theta)$ are $O(1)$.

Notice that the sequences $(\xi_{0,i})_1^n$ and $(\xi_{1,i})_1^n$ are the eigenvalues of the matrices, $B_0 = (\lambda_1 A)^{-1} - \bar{k}_\alpha I_n$ and $B_1 = \bar{\lambda} I_n - \bar{k}_\alpha A$, respectively. Since both B_0 and B_1 are symmetric then $\xi_{0,i}$ and $\xi_{1,i}$ are real and so since,

$$\bar{\psi}_j(\theta) = \prod_{i=1}^{n} \left(1 - 2\theta\xi_{j,i}\right)^{-\frac{1}{2n}},$$

is continuous in θ, and $\lim_{\theta \to 0} \bar{\psi}_j(\theta)$ is bounded away from zero then continuity in θ ensures $\bar{\psi}_j(\theta) \neq 0$ in a strip of width 2ε around the imaginary axis, $\Theta_\varepsilon = \{\theta : -i\varepsilon \leq \theta \leq i\varepsilon\} - i\varepsilon \leq \theta \leq i\varepsilon$. As a consequence $\bar{\psi}_j(\theta)$ is analytic for all $\theta \in \Theta_\varepsilon$. Thus condition (i) holds. For condition (ii), note that

$$\left|\bar{\psi}_j(\theta)\right| = \prod_{i=1}^{n} \left|\left(1 - 2\theta\xi_{j,i}\right)\right|^{-\frac{1}{2n}} \leq \sup_i \left(1 - 2\theta\xi_{j,i}\right)^{-\frac{1}{2}}.$$

Since the $\xi_{j,i}$ are $O(1)$ for all i and j, then

$$\lim_{|\theta| \to \infty} \left|\bar{\psi}_j(\theta)\right| = 0,$$

and so condition (ii) is also satisfied.

To proceed we follow the analysis of Daniels (1987) and transform in the integral (24) according to

$$\frac{\gamma_j^2}{2} - \tilde{\gamma}_j \gamma_j = R_j(\theta) - \theta R_j'(\hat{\theta}),$$

where the saddlepoint $\hat{\theta}$ satisfies

$$R'(\hat{\theta}) = \left.\frac{dR_j(\theta)}{d\theta}\right|_{\theta = \hat{\theta}} = x,$$

and so

$$\tilde{\gamma}_j = sign\left[(\hat{\theta})\right] \left\{\left[2\left(\hat{\theta} R_j'(\hat{\theta}) - R_j(\hat{\theta})\right)\right]\right\}.$$

Thus (24) can be rewritten as

$$F_j(q) = 1 - \frac{1}{2\pi i} \int_{\tau - i\infty}^{\tau + i\infty} \exp\left\{n\left(\frac{1}{2}\gamma_j^2 - \tilde{\gamma}_j \gamma_j\right)\right\} \left(\frac{1}{\theta} \frac{d\theta}{d\gamma_j}\right) d\gamma_j,$$

from which the asymptotic expansion, identical to Daniels (1987, equation 4.6), then follows, i.e.

$$F_j(q) = \Phi\left(\sqrt{n}\tilde{\gamma}_j\right) - \phi\left(\sqrt{n}\tilde{\gamma}_j\right)\left\{\frac{1}{\sqrt{n}}\left(\frac{1}{\tilde{\delta}_j} - \frac{1}{\tilde{\gamma}_j} + \sum_{k=1}^{\infty} n^{-k} b_{k,j}\right)\right\}, \tag{25}$$

where $\tilde{\delta}_j = \hat{\theta}\left[R''(\hat{\theta})\right]^{1/2}$ and the b_k are functions only of the derivatives of $R_j(\theta)$, at $\theta = \hat{\theta}$, $R_j^{(r)}\left(\hat{\theta}\right)$. Letting,

$$\hat{\mu}_{(r)} = \frac{R_j^{(r)}\left(\hat{\theta}\right)}{\left[R''(\hat{\theta})\right]^{r/2}},$$

then the first correction is

$$b_{1,j} = \frac{1}{\tilde{\delta}_j}\left(\frac{1}{8}\hat{\mu}_{(r)} - \frac{5}{24}\left(\hat{\mu}_{(3)}^2\right) - \frac{\hat{\mu}_{(3)}}{2\tilde{\delta}_j^2} - \frac{1}{\tilde{\delta}_j^3} + \frac{1}{\tilde{\gamma}_j^3}\right).$$

For the unit root power envelope we only require the asymptotic expansion in (25) evaluated at $q = x = 0$, at which point the saddlepoints are the unique solutions to

$$R'(\hat{\theta}_j) = \frac{1}{n}\sum_{i=1}^{n} \frac{\xi_{j,i}}{1 - 2\hat{\theta}_1 \xi_{j,i}} = 0 \quad ; \quad \frac{1}{2\max_i \xi_{j,i}} \leq \hat{\theta}_1 \leq \frac{1}{2\min \xi_{j,i}}.$$

Moreover, the r^{th} derivative of $R_j(\theta)$ is then

$$R_j^{(r)}(\theta) = \frac{d^r R_n(\theta)}{d\theta^r} = \frac{(-1)^r (r-1)!}{n} \sum_{i=1}^{n} \left(\frac{2\xi_{j,i}}{1 - 2\theta\xi_{j,i}}\right)^r,$$

so that $R_j^{(r)}(\theta)$ is finite for all θ, and consequently noting that $R'(\hat{\theta}_j) = 0$, we have $\tilde{\delta}_j = O(1)$, $\hat{\mu}_{(r)} = O(1)$, for all r, and

$$\tilde{\gamma}_j = sign\left[(\hat{\theta})\right]\left\{\left[-2R_j(\hat{\theta})\right]\right\} = O(1).$$

Consequently the coefficients in (25) at $x = 0$ satisfy $b_{k,j} = O(1)$ for all k, and so letting $\hat{\gamma}_j = \sqrt{n}\tilde{\gamma}_j$ and $\hat{\delta}_j = \sqrt{n}\tilde{\delta}_j$, we have

$$F_j(q) = \Phi\left(\hat{\gamma}_j\right) - \phi\left(\hat{\gamma}_j\right)\left\{\left(\frac{1}{\hat{\delta}_j} - \frac{1}{\hat{\gamma}_j} + O(n^{-1})\right)\right\}, \quad j = 0, 1,$$

as required. ■

Appendix II Tables

Table 1a): Nominal size of critical values for (9) approximated by (15) in model (19), with $\varepsilon_t \sim iidN(0,1)$.

	T	25		50		100		150	
	α	.05	.10	.05	.10	.05	.10	.05	.10
	4	.054	.106	.053	.105	.051	.103	.051	.102
	8	.053	.107	.052	.104	.051	.103	.051	.103
c	12	.053	.106	.052	.103	.051	.102	.050	.100
	16	.053	.106	.053	.103	.051	.102	.049	.100
	20	.055	.107	.053	.104	.051	.102	.049	.100

Table 1b): Nominal size of critical values for (9) approximated by (15) in model (20), with $\varepsilon_t \sim iidN(0,1)$.

	T	25		50		100		150	
	α	.05	.10	.05	.10	.05	.10	.05	.10
	4	.053	.106	.053	.104	.051	.103	.051	.103
	8	.053	.106	.052	.104	.052	.103	.051	.103
c	12	.053	.106	.051	.102	.050	.102	.051	.103
	16	.054	.105	.050	.102	.050	.102	.052	.104
	20	.053	.106	.051	.102	.051	.102	.051	.102

Table 2a): Nominal size of critical values for (9) approximated by Partial Sum in model (19), with $\varepsilon_t \sim iidN(0,1)$.

	T	25		50		100		150	
	α	.05	.10	.05	.10	.05	.10	.05	.10
	4	.054	.097	.050	.098	.050	.097	.048	.105
	8	.058	.148	.059	.143	.059	.117	.053	.114
c	12	.084	.194	.075	.167	.068	.134	.062	.119
	16	.087	.198	.087	.169	.078	.135	.064	.116
	20	.096	.214	.092	.180	.081	.134	.064	.121

Table 2b): Nominal size of critical values for (9) approximated by Partial Sum in model (20), with $\varepsilon_t \sim iidN(0,1)$.

	T	25		50		100		150	
	α	.05	.10	.05	.10	.05	.10	.05	.10
	4	.083	.111	.079	.109	.072	.111	.072	.117
	8	.045	.073	.054	.086	.049	.099	.047	.089
c	12	.043	.078	.047	.090	.046	.092	.047	.088
	16	.028	.074	.039	.088	.044	.104	.049	.103
	20	.042	.091	.055	.111	.052	.101	.061	.112

Table 3a): Nominal size of critical values for (9) approximated by (15) in model (19), with $\varepsilon_t \sim \frac{iid\chi^2(1)-1}{\sqrt{2}}$.

	T	25		50		100		150	
	α	.05	.10	.05	.10	.05	.10	.05	.10
	4	.064	.121	.058	.113	.056	.109	.055	.108
	8	.068	.125	.060	.114	.056	.110	.055	.108
c	12	.069	.126	.061	.115	.056	.109	.054	.108
	16	.069	.125	.061	.115	.056	.108	.054	.109
	20	.069	.125	.062	.117	.056	.109	.055	.109

Table 3b): Nominal size of critical values for (9) approximated by (15) in model (20), with $\varepsilon_t \sim \frac{iid\chi^2(1)-1}{\sqrt{2}}$.

	T	25		50		100		150	
	α	.05	.10	.05	.10	.05	.10	.05	.10
	4	.051	.101	.045	.093	.049	.102	.048	.098
	8	.051	.099	.046	.093	.050	.101	.048	.097
c	12	.050	.099	.045	.092	.049	.101	.049	.098
	16	.049	.097	.045	.094	.048	.101	.048	.098
	20	.048	.095	.046	.095	.050	.100	.048	.097

Table 4a): Nominal size of critical values for (9) approximated by Partial Sum in model (19), with $\varepsilon_t \sim \frac{iid\chi^2(1)-1}{\sqrt{2}}$.

	T	25		50		100		150	
	α	.05	.10	.05	.10	.05	.10	.05	.10
	4	.052	.084	.048	.085	.046	.087	.053	.099
	8	.054	.115	.048	.121	.054	.121	.052	.106
c	12	.063	.169	.066	.147	.056	.127	.060	.111
	16	.075	.193	.072	.166	.060	.128	.053	.120
	20	.081	.198	.083	.181	.067	.140	.064	.122

Table 4b): Nominal size of critical values for (9) approximated by Partial Sum in model (20), with $\varepsilon_t \sim \frac{iid\chi^2(1)-1}{\sqrt{2}}$.

	T	25		50		100		150	
	α	.05	.10	.05	.10	.05	.10	.05	.10
	4	.141	.158	.131	.148	.131	.152	.130	.162
	8	.075	.096	.075	.095	.073	.099	.063	.094
c	12	.054	.075	.055	.081	.053	.084	.046	.079
	16	.031	.050	.038	.068	.036	.067	.043	.078
	20	.025	.053	.041	.071	.037	.075	.047	.087

Table 5a): Monte Carlo (MC) and Saddlepoint Approximation (SA) to Power minus size in model (19), at the 5% level, with $\varepsilon_t \sim N(0,1)$.

	T	25		50		100		150	
		MC	SA	MC	SA	MC	SA	MC	SA
	4	.148	.153	.167	.168	.183	.184	.179	.175
	8	.403	.402	.449	.443	.477	.467	.503	.490
c	12	.687	.684	.700	.694	.743	.738	.754	.751
	16	.863	.865	.864	.867	.877	.884	.888	.895
	20	.930	.933	.928	.932	.930	.937	.932	.939

Table 5b): Monte Carlo (MC) and Saddlepoint Approximation (SA) to Power minus size in model (20), at the 5% level, with $\varepsilon_t \sim N(0,1)$.

	T	25		50		100		150	
		MC	SA	MC	SA	MC	SA	MC	SA
	4	.036	.043	.039	.041	.036	.039	.040	.044
	8	.152	.157	.157	.159	.158	.160	.160	.163
c	12	.360	.364	.352	.353	.348	.348	.360	.362
	16	.611	.612	.581	.581	.591	.591	.592	.596
	20	.811	.809	.773	.772	.774	.775	.778	.781

Table 6): Values of $\bar{c}$ satisfying $\Pi_\alpha(\bar{c}) = 0.5$ for the two models in (22), for different α and τ.

		M_1			M_2		
	α	.01	.05	.10	.01	.05	.10
	1	20.9	13.6	10.2	24.4	16.4	12.6
	2	20.8	13.3	9.87	25.6	17.4	13.4
	3	20.5	13.0	9.49	26.1	17.9	14.0
	4	20.1	12.5	9.04	26.3	18.1	14.2
τ	5	19.7	12.0	8.50	26.3	18.0	14.1
	6	19.0	11.4	7.88	25.9	17.9	13.9
	7	18.4	10.6	6.95	25.3	17.1	13.3
	8	17.5	9.34	6.30	24.4	16.1	12.3
	9	16.8	8.78	6.08	22.9	14.9	11.4