NICEP Working Paper: 2018-06

Dynamic Legislative Policy Making under Adverse Selection

Vincent Anesi

Nottingham Interdisciplinary Centre for Economic and Political Research
https://nicep.nottingham.ac.uk/
School of Politics, The University of Nottingham, Law & Social Sciences Building,
University Park, Nottingham, NG7 2RD
ISSN 2397-9771

Dynamic Legislative Policy Making under Adverse Selection

Vincent Anesi

NICEP Working Paper Series 2018-06
August 2018
ISSN 2397-9771

Vincent Anesi
University of Nottingham
Vincent.Anesi@nottingham.ac.uk

Dynamic Legislative Policy Making under Adverse Selection

Vincent Anesi*

August 3, 2018

Abstract

This paper develops a dynamic model of legislative policy making with evolving, privately observed policy preferences. Our goal is to find conditions under which decision rules, which assign feasible policies based on the legislators' preferences, are sustainable in the long run. We show that under some mild conditions, every decision rule that would be implementable with monetary transfers can be approximately sustained in a perfect Bayesian equilibrium of the dynamic model. In this equilibrium, the legislators receive payoffs arbitrarily close to those they would obtain if they could commit ex ante to truthfully apply the decision rule in every period. An application of our result yields a dynamic issue-by-issue median voter theorem in the vein of Baron's (1996) for a spatial framework with incomplete information.

JEL classification: D71; D72; D78

Keywords: Committee voting; Information; Legislative bargaining; Sustainability

*School of Economics, University of Nottingham. Email: `vincent.anesi@nottingham.ac.uk`. I would like to thank Wiola Dziuda, Tasos Kalandrakis, Daniel Seidmann, and participants at various seminars and conferences for very helpful comments and discussions.

1 Introduction

Since the seminal work of Baron (1996), dynamic bargaining games with an endogenous status quo have taken on increasing prominence in the analysis of legislative policy making. As pointed out by Kalandrakis (2004), in many cases, "legislation remains in effect after its promulgation only until or unless the legislature passes a new law. Thus it appears natural to study dynamic bargaining games where (a) policy decisions can be reached in any period, and (b) in the absence of agreement among the bargaining parties in any given period the status quo policy prevails." In that vein of research, the notion of dynamic stability has received particular attention — e.g., Diermeier and Fong (2011, 2012), Acemoglu et al. (2012), Richter (2014), Anesi and Seidmann (2015), Baron and Bowen (2015), Anesi and Duggan (2017), and Baron (forthcoming). A feasible policy is dynamically stable if, when legislators are sufficiently patient, it is an equilibrium absorbing state of the bargaining game; i.e., there is an equilibrium in which this policy is implemented and never amended. The literature has focused its attention so far on the characterization of the dynamically stable policies in settings where the legislators' preferences are publicly observable.

The main objective of this paper is to contribute further to this research program by developing a theory of dynamic stability in general legislative-bargaining environments with evolving, privately observed policy preferences. In such environments, bargaining outcomes in every period can be described by outcome functions or, to use the language of collective choice theory, "decision rules" that map the realizations of the legislators' privately observed types to policies. Our main goal is to identify sufficient conditions for such a decision rule to be sustainable in the long run, in the sense that the policy implemented by the legislature in (almost) every period in equilibrium coincides with the outcome prescribed by the decision rule.

To provide a basic intuition for the approach taken in this paper to dynamic stability, suppose a legislature has (informally) agreed on some decision rule to be used for making compromises among its members' preferences in every period. Two main obstacles must be overcome for this rule to be successfully sustained in the long run. First, its application is the responsibility of the legislators themselves and, in most cases, they cannot commit *ex ante* to implement it. Second, an important feature of legislative decision making is that legislators' preferences are not publicly observable. This private information creates incentives for legislators to dissemble and, as a result, there is no guarantee that the application of the decision rule will generate the intended outcome (e.g., Austen-Smith and Banks, 2005). This is the well-known adverse selection problem. One standard mode of attack on this problem in conventional mechanism-design settings is to use "compensatory transfers" among agents to induce them to truthfully reveal their preferences (e.g., Laffont and Maskin, 1982). The idea is to couple the decision rule with preference-dependent monetary transfers among agents so as to obtain a social choice function that is strategy-proof, in the sense that strategic behavior leads to the outcomes that would have been obtained were preferences publicly observable. Such transfers, however, are typically unavailable in legislative bargaining environments, either because utilities are nontransferable, and/or because the resources the legislature can distribute among its members are insufficient to overcome incentive constraints. Our main result shows that in many cases of interest, this is a nonissue.

More precisely, we develop a model in which each period begins with a status quo policy inherited from the previous period. The legislators first receive private information determining their types, and then communicate. Types are distributed according to Markov chains, independent across players. As is common in the literature (e.g. Austen-Smith 1990a,b), we model communication as cheap-talk messages between the legislators. Following the communication stage, legislators are given the opportunity to propose amendments to the ongoing status quo in a random order; the first proposal that is voted up is implemented in that period and becomes the next period's status quo; if all proposals are voted down, then the status quo is implemented and remains in place until the next period. This process continues indefinitely. We find that under a weak gradient condition on the legislators' utilities, if a decision rule is implementable with compensatory transfers, then it is approximately sustainable in the dynamic bargaining game, in the sense that if legislators are sufficiently patient, then there is an equilibrium in which policies arbitrarily close to those prescribed by the rule are implemented arbitrarily often. In this equilibrium, legislators receive payoffs arbitrarily close to those they would get if they could commit to truthfully implement the decision rule in every period. As an application of this result, we prove a "dynamic issue-by-issue median voter theorem" in the vein of Baron's (1996) for a spatial setting with incomplete information.

The gradient condition holds generically outside a set of decision rules with measure zero if the policy space is sufficiently high dimensional. However, we also show that it can be dispensed with in two variants of our baseline model. In the first variant, the legislature can also distribute a given amount of a *limited* and divisible resource among its members. In the second, the legislators can use a public randomization device in every period.

Existing models that combine legislative policy making and asymmetric information typically focus on situations in which the legislature makes a single decision — e.g., Austen-Smith and Riker (1987), Meirowitz (2007), Tsai and Yang (2010), and Chen and Eraslan (2013, 2014). Our aim is to study how decision rules can be sustained through repeated interaction despite informational asymmetries. This calls for a model in which legislators make a sequence of decisions.

As mentioned above, there is a the large and growing body of literature which, like this paper, studies infinitely repeated legislative interaction in settings with an endogenous status quo.[1] But this entire literature has focused on complete-information environments. To the best of our knowledge, this paper constitutes the first attempt at introducing incomplete information into the dynamic legislative bargaining process with endogenous status quo. The Markovian evolution of types is however a feature that we share with Kalandrakis' (2009) model of repeated elections, in which the two parties' privately observed preferences change with higher probability following defeat in elections.

The idea of using continuation payoffs as substitutes to monetary transfers to overcome incentive constraints in dynamic environments is certainly not new. In particular, various approaches of doing so can be found in the literature on dynamic Bayesian games — see

[1]See, in addition to the references mentioned above, Baron and Herron (2003), Kalandrakis (2010, 2016), Battaglini and Palfrey (2012), Bowen and Zahran (2012), Duggan and Kalandrakis (2012), Bowen et al. (2014), Nunnari (2014), Anesi and Seidmann (2015), Baron and Bowen (2015), Piguillem and Riboni (2015), Dziuda and Loeper (2016), Zápal (2016), and Bowen et al. (2017) to cite a few.

Hörner et al. (2015) and the references therein for a recent account. Our approach, as well as our goal and framework, are different. Moreover, in contrast to most existing work in this area, our main results allow for infinite choice sets, do not require a public randomization device, and our focus is on policy outcomes rather than payoff vectors. In particular, our main result allows us to sustain (without a randomization device) decision rules of interest, which may be Pareto inefficient (such as the issue-by-issue rule).[2]

The next section introduces the dynamic bargaining framework. Section 3 presents our main approximation result, followed by an application and a sketch of its proof. Finally, Section 4 contains variants on the main result that do not require high-dimensional policy spaces. The appendix and supplementary appendix contain proofs omitted from the body of the paper.

2 Framework

Legislative bargaining game. Let $X \subseteq \mathbb{R}^d$ be a set of alternatives with nonempty interior; and let $N \equiv \{1, \ldots, n\}$ be a finite set of players, or "legislators," who must choose an alternative from X in each of an infinite number of discrete periods, indexed $t = 1, 2, \ldots$.

The timing of legislative interaction is as follows in each period t. Each legislator $i \in N$ first privately learns her type, θ_i^t, which is drawn from a finite set Θ_i according to an autonomous Markov chain (λ_i, P_i), where λ_i is the initial distribution and P_i is the transition matrix. We assume that the Markov chains (λ_i, P_i), $i \in N$, are independent. Let $\Theta \equiv \prod_{i \in N} \Theta_i$, and let (λ, P) be the joint type process. We further assume that (λ, P) is irreducible. Its invariant distribution, π, can be expressed as $\pi = \pi_1 \times \cdots \times \pi_n$, where π_i denotes the invariant distribution for (λ_i, P_i).

After learning their types, the legislators simultaneously send messages to the other members of the legislature, with legislator i's message $m_i^t \in \Theta_i$ being publicly observable. Finally, they collectively choose a single policy x^t from X as follows. There is a status

[2]Although to our knowledge our arguments are novel, it is worth emphasizing that they are predicated on a statistical test Escobar and Toikka (2013) developed to approximate efficient payoff vectors in dynamic Bayesian games — see Subsection 3.3.

quo policy x^{t-1}, inherited from the previous period. An order of proposers $(\rho_1, \dots, \rho_n)$ is randomly selected from the set Π of all permutations of N, with each permutation in Π having a positive probability of being selected. Proposer ρ_1 then makes the first proposal $y \in X$; once the proposal is made, legislators vote sequentially (in an arbitrary order) over whether to accept it. The proposal is accepted if at least q legislators vote to accept, and it is rejected otherwise, where $n/2 < q < n$. If the proposal is accepted, then it is implemented, payoffs (which we elaborate on below) accrue and the game transitions to the next period, where the new status quo is $x^t = y$; otherwise, proposer ρ_2 is called upon to make a proposal and the same process is repeated. If all legislators make unsuccessful proposals, then the status quo x^{t-1} is implemented and remains the status quo in period $t+1$. The game begins with some exogenously given status quo x^0, and each of the proposal rounds takes a negligible amount of time.

The payoff structure is one of private values: each player's preferences over X only depend on her own type. More precisely, at the end of period t, each player i receives a payoff $u_i(x^t, \theta_i^t)$ where, for every $\theta_i \in \Theta_i$, $u_i(\cdot, \theta_i)$ is a bounded and continuously differentiable utility function on X.

The above process is repeated *ad infinitum.* Each player discounts her per-period utilities with a discount factor $\delta \in [0, 1)$, and seeks to maximize her expected average discounted sum of per-period utilities. Players are allowed to use mixed strategies, and the equilibrium concept is that of a perfect Bayesian equilibrium (PBE), defined in the usual fashion.

Implementable decision rules and sustainability. A *decision rule* is a mapping from the type space into the space of alternatives, $\chi\colon \Theta \to X$. In what follows, we view the set of decision rules as $X^{|\Theta|} \subseteq \mathbb{R}^{d|\Theta|}$, and refer to any element of $\text{int}X^{|\Theta|}$ as an *interior decision rule.*

A decision rule is said to be *implementable (with compensatory transfers)* if there exists a transfer function $\psi = (\psi_1, \dots, \psi_n)\colon \Theta \to \mathbb{R}^n$ such that

$$u_i\big(\chi(\theta_i, \theta_{-i}), \theta_i\big) + \psi_i(\theta_i, \theta_{-i}) \geq u_i\big(\chi(\theta_i', \theta_{-i}), \theta_i\big) + \psi_i(\theta_i', \theta_{-i}) \ ,$$

for all $i \in N$, $\theta_{-i} \in \Theta_{-i}$ and $\theta_i, \theta_i' \in \Theta_i$,[3] or to use the language of social choice theory, such that the social choice function $\big(\chi(\cdot), \psi(\cdot)\big)$ is strategy-proof. In other words, implementable decision rules provide no opportunities for any legislator to manipulate the outcome profitably by misrepresenting her preferences, given an appropriately chosen transfer function.

As mentioned earlier, the political economy literature on dynamic collective choice in complete-information environments has devoted considerable attention to dynamically stable policies, i.e., policies that can be supported as absorbing points of equilibria for dynamic bargaining games when players are sufficiently patient. The following definition introduces a weaker version of dynamic stability for settings with adverse selection.

Definition 1. *A decision rule χ is* approximately sustainable *if for every $\varepsilon > 0$, there exists a strategy profile σ for the legislative bargaining game such that:*

(i) there is $\bar{\delta} < 1$ such that, for all $\delta \in (\bar{\delta}, 1)$, σ is a PBE; and

(ii) denoting by $\{\tilde{x}^t(\sigma)\}$ the policy sequence induced by σ, we have

$$\limsup \Pr_\sigma \left\{ \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}_{\left\{|\tilde{x}^t(\sigma) - \chi(\tilde{\theta}^t)| > \varepsilon\right\}} > \varepsilon \right\} < \varepsilon$$

(where the $\limsup$ *is over $T \in \mathbb{N}$).*

In other words, a decision rule χ is approximately sustainable if, when players are sufficiently patient, there exist PBEs in which policies arbitrarily close to those prescribed by χ are implemented arbitrarily often, in the sense that the empirical frequency of deviations from that rule can be made arbitrarily small with a probability arbitrarily close to one. An important implication of approximate sustainability is that the payoff vector induced by the decision rule χ can be approximated arbitrarily closely in a PBE as $\delta \to 1$.

[3] Rochet (1987) and Bikhchandani et al. (2006) provide general characterizations of implementable decision rules.

3 Sustainability in High-Dimensional Policy Spaces

3.1 Main Result

In this section, we present a simple condition under which decision rules that are implementable with transfers can be approximately sustained in equilibria of the legislative bargaining game (without transfers). Consider first the complete-information case in which $|\Theta_i| = 1$ for all $i \in N$, so that an (implementable) decision rule is simply a policy in X. A sufficient condition for the approximation of a given policy x is that in every open neighborhood of x, there is a policy that is implemented in every period in some equilibrium. It follows from Anesi and Duggan's (2018) analysis of a complete-information version of the legislative bargaining game above that, given a policy space of sufficiently high dimension and sufficiently patient legislators, this condition holds generically outside a set of policies with measure zero. Our goal is to extend their result to non-trivial decision rules in incomplete-information environments.

To begin, we must establish some notation. For all $x = (x_1, \ldots, x_d) \in X$, $i \in N$ and $\theta_i \in \Theta_i$, let $\nabla_1 u_i(x, \theta_i)$ denote the gradient of $u_i(\cdot, \theta_i)$ at x, i.e.,

$$\nabla_1 u_i(x, \theta_i) \equiv \left(\frac{\partial u_i}{\partial x_1}(x, \theta_i), \ldots, \frac{\partial u_i}{\partial x_d}(x, \theta_i) \right) .$$

We say that an interior decision rule χ satisfies condition (C) if the following holds:

(C) $\{\nabla_1 u_i(\chi(m), \theta_i) \colon i \in N \;\&\; \theta_i \in \Theta_i\}$ is linearly independent for every $m \in \Theta$.

Intuitively, when χ satisfies condition (C), we can obtain all values of the utilities $(u_i(\cdot, \theta_i))_{i \in N, \theta_i \in \Theta_i}$ in some open neighborhood of $(u_i(\chi(m), \theta_i))_{i \in N, \theta_i \in \Theta_i}$ by arbitrarily small variations of $\chi(m)$, for each $m \in \Theta$ — see Lemma 1 below. How restrictive is condition (C)? In sufficiently high dimensional settings with $d \geq \sum_{i \in N} |\Theta_i| \equiv k$, Schofield's (1980) Singularity Theorem A implies that for generic profiles of utility functions $(u_i(\cdot, \theta_i))_{i \in N, \theta_i \in \Theta_i}$, condition (C) generically holds outside a closed set of decision rules with measure zero.[4] We conclude that, in sufficiently high dimensional policy spaces, "almost all" decision rules satisfy this condition.

[4]Recall that we view the set of decision rules as a subset of $\mathbb{R}^{d|\Theta|}$. Schofield's (1980) result holds if

Theorem 1. *Let χ be an interior decision rule that satisfies condition (C). If χ is implementable with compensatory transfers, then it is approximately sustainable.*

A sketch of the proof is provided in Subsection 3.3, and the complete proof can be found in the appendix. Theorem 1 establishes that, coupled with condition (C), implementability of a decision rule guarantees that it can be approximately sustained in equilibrium. An immediate implication of this result is that, for every implementable decision rule χ that satisfies (C), there is a PBE in which all legislators receive approximately the same payoffs as if they were able to commit to truthfully implement χ in every period.

As explained above, condition (C) is mild in high-dimensional policy spaces. Choice sets in legislative bargaining contexts are more often than not highly dimensional. Not only do legislators bargain over bundles of policies (e.g., Câmara and Eguia 2017), but most policy issues are themselves multidimensional in nature (e.g., Baumgartner et al. 2000). Nevertheless, policy spaces only have a small number of dimensions in some applications of interest. We show in Section 4 that if the legislature can either distribute some benefit among its members or use a public randomization device, then the above results can be extended to a very general class of policy spaces, including finite sets.

3.2 Application: Dynamic (Issue-by-issue) Median Voter Theorem

Before turning to the proof of Theorem 1, we consider an example that illustrates how it can be applied. A common approach for studying majority voting outcomes in multidimensional settings is to use the issue-by-issue rule (e.g., Austen-Smith and Banks 2005), which selects the median of the legislators' ideal points in each dimension, or "issue," of the policy space. Can it be sustained as a long-run equilibrium outcome of the dynamic legislative bargaining game? Baron (1996) and Zápal (2016) establish "dynamic median voter theorems" for the one-dimensional case under complete information, providing conditions under which the median-ranked ideal policy (or Condorcet winner) is an absorbing

we give the space of twice continuously differentiable utility profiles the Whitney topology. We obtain the claim by setting $w = d$ and $z = k$ in Schofield's theorem (and from the fact that the product of measure-zero sets is null).

point of stationary equilibria. In this subsection, we apply our weaker notion of dynamic stability to the issue-by-issue rule in an incomplete-information version of the standard multidimensional spatial model with Euclidean preferences. (The one-dimensional case will be studied in Section 4.)

Let the policy space be $X = [-B, B]^d$, where $B > 0$ is arbitrarily large and $d \geq k$; let $q = (n+1)/2$; and, for each $i \in N$, let $\Theta_i \subset \mathbb{R}$. For convenience here, assume there is an odd number of legislators and that policy preferences are of the form:

$$u_i(x, \theta_i) = -\sum_{\ell=1}^{d} \alpha_{i,\ell} \big[x_\ell - \hat{x}_{i,\ell}(\theta_i)\big]^2 \ ,$$

for all i and $x = (x_1, \ldots, x_d)$, where $\hat{x}_{i,\ell}(\cdot)$ is an increasing, continuously differentiable real function, and $\alpha_{i,\ell}$ is a positive number for each ℓ. Thus, the *issue-by-issue rule* χ^* is defined simply by

$$\chi^*(\theta) \equiv \Big(\text{med}\big\{\hat{x}_{1,1}(\theta_1), \ldots \hat{x}_{n,1}(\theta_n)\big\}, \ldots, \text{med}\big\{\hat{x}_{1,d}(\theta_1), \ldots \hat{x}_{n,d}(\theta_n)\big\}\Big) \ ,$$

for all $\theta = (\theta_1, \ldots, \theta_n) \in \Theta$; so that, for each profile of types, it selects the corresponding issue-by-issue core (Kramer, 1972). We have

$$\nabla_1 u_i\big(\chi^*(m), \theta_i\big) = -2 \begin{pmatrix} \alpha_{i,1}\big[\chi_1^*(m) - \hat{x}_{i,1}(\theta_i)\big] \\ \vdots \\ \alpha_{i,d}\big[\chi_d^*(m) - \hat{x}_{i,d}(\theta_i)\big] \end{pmatrix} \ ,$$

so that condition (C) is simply the weak condition that, for every $m \in \Theta$, the set $\big\{\big(\alpha_{i,\ell}[\chi_\ell^*(m) - \hat{x}_{i,\ell}(\theta_i)]\big)_{\ell=1,\ldots,d} \colon i \in N \ \&\ \theta_i \in \Theta_i\big\}$ be linearly independent.

It is well known that the issue-by-issue rule may be manipulable (and Pareto inefficient). Nevertheless, it is implementable with compensatory transfers. To see this, consider the transfer function ψ^* defined by

$$\psi_i^*(m) \equiv \sum_{\ell=1}^{d} \alpha_{i,\ell} \left[\big[\chi_\ell^*(m) - \hat{x}_{i,\ell}(m_i)\big]^2 + 2 \int_{m_i}^{1+\max\Theta_i} \big[\hat{x}_{i,\ell}(z) - \chi_\ell^*(z, m_{-i})\big] \hat{x}'_{i,\ell}(z) dz \right] ,$$

for all $i \in N$ and $m = (m_1, \ldots, m_n) \in \Theta$. When confronted with the social choice function (χ^*, ψ^*), the type-θ_i individual i chooses a message m_i to maximize $u_i\big(\chi^*(m_i, m_{-i}), \theta_i\big) +$

$\psi_i^*(m_i, m_{-i})$. The left- and right-derivatives if this function with respect to $m_i \in \mathbb{R}$ are

$$\sum_{\ell=1}^{d} 2\alpha_{i,\ell}\left[\hat{x}_{i,\ell}(\theta_i) - \hat{x}_{i,\ell}(m_i)\right]\frac{\partial \chi_\ell^*}{\partial m_i}(m_i, m_{-i})_-$$

and

$$\sum_{\ell=1}^{d} 2\alpha_{i,\ell}\left[\hat{x}_{i,\ell}(\theta_i) - \hat{x}_{i,\ell}(m_i)\right]\frac{\partial \chi_\ell^*}{\partial m_i}(m_i, m_{-i})_+ \ ,$$

respectively, where $\frac{\partial \chi_\ell^*}{\partial m_i}(m_i, m_{-i})_+ = -\frac{\partial \chi_\ell^*}{\partial m_i}(m_i, m_{-i})_- \in \left\{0, \hat{x}'_{i,\ell}(m_i)\right\}$. It is follows that truthfully reporting her type (i.e., choosing $m_i = \theta_i$) is always optimal for individual i. Coupled with Theorem 1, this observation yields the following result.

Corollary 1. *In the spatial model described above, if* $\left\{\left(\alpha_{i,\ell}[\chi_\ell^*(m) - \hat{x}_{i,\ell}(\theta_i)]\right)_{\ell=1,\ldots,d} : i \in N \text{ and } \theta_i \in \Theta_i\right\}$ *is linearly independent for each* $m \in \Theta$*, then the issue-by-issue rule* χ^* *is approximately sustainable.*

In other words, under generic conditions, there exists an equilibrium of the dynamic bargaining game in which, despite incomplete information, the legislature implements a policy close to the issue-by-issue median policy arbitrarily frequently.

3.3 Sketch of the Proof of Theorem 1

A complete proof of Theorem 1 can be found in the appendix; here we only provide a detailed overview. An important advantage of unbounded compensatory transfers (were they available) is that they would allow the legislature to freely increase or decrease the utility of any of its members without affecting the others' utilities. Such freedom to adjust stage-game utilities is unavailable in the legislative bargaining game. We must therefore find an alternative approach to ensure that the desired payoff vectors (or approximations of those) will be induced by the legislators' messages. A main ingredient of the proof is the use of "simulated" compensatory transfers obtained by perturbing stage utilities across periods.

Lemma 1. *Let* χ *be an interior decision rule at which (C) is satisfied. Then there is* $\gamma > 0$ *such that, for all* $\phi \in [-2\gamma, 2\gamma]^n$*, there exists a decision rule* χ^ϕ *that satisfies:*

$$u_i(\chi^\phi(m), \theta_i) = u_i(\chi(m), \theta_i) + \phi_i \ ,$$

for all $i \in N$, $m \in \Theta$ *and* $\theta_i \in \Theta_i$.

Lemma 1 shows that any decision rule χ satisfying (C) can be perturbed to another decision rule χ^{ϕ} so as to vary the legislators' utilities independently: an application of the local submersion theorem (e.g., Guillemin and Pollack, 1974) gives a vector of independent, small transfers $(\phi_1, \ldots, \phi_n)$. The purpose of the lemma is twofold: (i) simulated transfers ϕ_i in $[-\gamma, \gamma]$ will permit to exploit the implementability of χ and deal with legislator i's "on-schedule deviations" in stages where she has to report her type; and (ii) simulated transfers in $[-2\gamma, -\gamma) \cup (\gamma, 2\gamma]$ will be used to preclude "off-schedule deviations" by legislator i in policy-making stages.

Given a profile of reports $(m_1^t, \ldots, m_n^t)$ in period t and a transfer function $\psi = (\psi_1, \ldots, \psi_n)$, it is thus possible to simulate the transfer $\psi_i(m_1^t, \ldots, m_n^t)$ for each (patient) legislator i by giving her $\phi_i = \psi_i(m_1^t, \ldots, m_n^t)/T \in [-\gamma, \gamma]$ in each of a sufficiently large number of periods T. There is an obvious difficulty with this approach: it is not one but infinitely many transfers that must be simulated for each player i (i.e., one for each period), and there is no guarantee that $(1/T)\sum_{t=1}^{T} \psi_i(m_1^t, \ldots, m_n^t)$ converges to a value in $[-\gamma, \gamma]$. Our first step in addressing this difficulty is to change ψ_i to another transfer function $\bar{\psi}_i$, defined by $\bar{\psi}_i(\theta) \equiv \psi_i(\theta) - \mathbb{E}_{\pi}\big[\psi_i(\tilde{\theta})\big]$ for all $i \in N$ and all $\theta \in \Theta$ — observe that $\bar{\psi} = (\bar{\psi}_1, \ldots, \bar{\psi}_n)$ also implements χ. Next we divide time into successive T-period blocks, $T \in \mathbb{N}$. Given a sequence of report profiles $(m^{(b-1)T+1}, \ldots, m^{bT})$ in the bth block, legislators are prescribed to implement a policy according to decision rule χ^{ϕ^b} in each period of the $(b+1)$th block, where

$$\phi_i^b \equiv \begin{cases} \frac{1}{T}\sum_{t=(b-1)T+1}^{bT} \bar{\psi}_i(m^t) & \text{if } \left|\frac{1}{T}\sum_{t=(b-1)T+1}^{bT} \bar{\psi}_i(m^t)\right| \le \gamma \ , \\ -\gamma & \text{otherwise,} \end{cases}$$

for every $i \in N$. It follows from a law of large numbers for Markov processes (e.g., Stokey and Lucas, 1989) that the probability of the event $\left\{\left|\frac{1}{T}\sum_{t=(b-1)T+1}^{bT} \bar{\psi}_i(\tilde{\theta}^t)\right| \le \gamma\right\}$ can be made arbitrarily close to one by picking a sufficiently large T (irrespective of $\theta^{(b-1)T}$). Therefore, if arbitrarily patient legislators truthfully reported their types in each of the T periods of the bth block, then they would receive simulated transfers arbitrarily close to those prescribed by $\bar{\psi}$ with a probability arbitrarily close to one in the $(b+1)$th block —

as illustrated in Figure 1 for the first block.

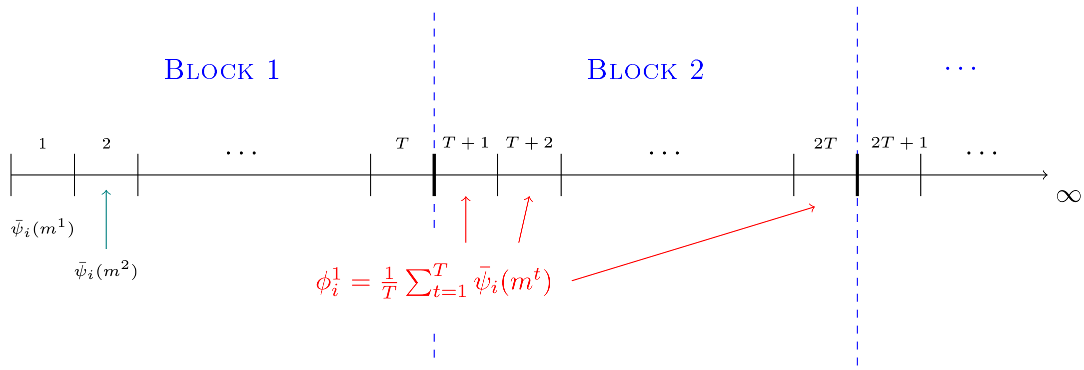

Figure 1: Simulating compensatory transfers.

However, there are two major difficulties here. First, there is no guarantee that ϕ_i^b converges to zero if legislator i misreports her types in block b. Second, while being in the middle of the bth block and observing the reports in previous periods, legislator i may realize that ϕ_i^b will be equal to $-\gamma$ with a very high probability, irrespective of her behavior in the remaining periods of the block. This may create incentives for her to misreport her types. The key to resolving these difficulties is provided by a statistical test, due to Escobar and Toikka (2013).

In each block, the messages sent by the legislators are submitted to an Escobar-Toikka test (see Section A.2 of the supplementary appendix for a formal definition) and in each period of the block, the decision rule χ^{ϕ^b} is only applied to the messages that are "credible," i.e., those that pass the test. If a legislator's message fails the test in a given period, then her future messages are then ignored and replaced by random reports generated from the Markov chain until the end of the block. Escobar and Toikka (2013) show that their test can be designed in such a way that (i) a player who truthfully reports her type in every period of the block is highly likely to pass it, and (ii) the empirical distribution of the sequence of message profiles generated by the test converges to the invariant distribution π as $T \to \infty$, irrespective of the legislators' actual messages. This allows us to prove that if T and δ are sufficiently large, then: (i) every legislator i can obtain a payoff arbitrarily close to $\mathbb{E}_\pi\big[u_i\big(\chi(\tilde{\theta}), \tilde{\theta}_i\big)\big]$ by truthfully reporting her type in every period, irrespective of the other legislators' messages; and (ii) her equilibrium payoffs have an upper bound arbitrarily close to $\mathbb{E}_\pi\big[u_i\big(\chi(\tilde{\theta}), \tilde{\theta}_i\big)\big]$. It follows that her equilibrium payoff must also be arbitrarily close to $\mathbb{E}_\pi\big[u_i\big(\chi(\tilde{\theta}), \tilde{\theta}_i\big)\big]$. (Note that this result holds *even if the decision rule is inefficient,* as in the example above.)

To complete the proof of the theorem, it remains to establish that in such an equilibrium, the empirical frequency of deviations from the prescribed decision rules can be made arbitrarily small by taking sufficiently large T and δ. We show in the appendix that an implementable decision rule that satisfies condition (C) can be perturbed to an arbitrarily close decision rule that also satisfies (C) and with the property that each player i's cost of misreporting her type is bounded away from zero. It follows that if she did misreport her type too often in the PBE constructed above, then her payoff would be bounded away from $\mathbb{E}_\pi\big[u_i\big(\chi(\tilde{\theta}), \tilde{\theta}_i\big)\big]$, yielding a contradiction. This completes the proof of the theorem.

4 Discussion: General Policy Spaces

We saw in Section 3 that, in our baseline model, any implementable decision rule that satisfies the gradient condition (C) is approximately sustainable. Although this condition typically holds if the policy space X is a sufficiently high-dimensional subset of some Euclidean space, there are cases of interest in which such a restriction does not hold. We conclude this paper by presenting two variants of the model to which the arguments in the previous subsection can be adapted without imposing any restriction on the dimensionality of X. It may be a finite set, a set of functions, or any other set.

Variant 1: Redistributive dimension. One simple way in which the approximation result can be obtained without imposing restrictions on X is by assuming that the legislature can distribute a given amount of a limited and divisible resource among its members, in addition to choosing a policy from X. Such an assumption is common in the political-economy literature where, in addition to policy (or "ideological") issues, choice sets may also include a distributive dimension — e.g, Austen-Smith and Banks (1988), Diermeier and Merlo (2000), Jackson and Moselle (2002), and Che and Eraslan (2013, 2014). Suppose we modify our baseline model as follows. The legislature can now allocate shares of a *fixed total* $r > 0$ of a divisible resource among its members, so that the policy space is now of the form $\widehat{X} \equiv Z \times X$, where $Z \equiv \{(z_1, \dots, z_n) \in [0, r]^n \colon \sum_{i=1}^n z_i \leq r\}$ and X can be any set. Note that r can be arbitrarily small, possibly smaller than the compensatory transfers needed to implement the decision rule one may seek to approximate in equilibrium. Assume further that each player i's utility function is of the form $\hat{u}_i(\hat{x}, \theta_i) \equiv g_i(z_i) + u_i(x, \theta_i)$ for all $\hat{x} = (z, x) \in \widehat{X}$, where $g_i \colon [0, r] \to \mathbb{R}$ is continuous and strictly increasing. In this case, the statement of Lemma 1 trivially holds for *any* interior decision rule and the rest of the argument in Subsection 3.3 is completely analogous.

Variant 2: Public randomization. Another approach is to allow the legislature to use a public randomization device. More precisely, suppose that in every period t, legislators first report their types; then a publicly observable realization of a random variable is drawn from the uniform distribution on $[0, 1]$; and then a policy is chosen from X as in the baseline model. Henceforth, we assume that X is any separable policy space and impose the following mild condition on the legislators' utilities:

(C*) There exists a finite subset $Y = \{y_1, \dots, y_L\}$ of X such that

$$\big\{\big(u_i(y_1, \theta_i) - u_i(y_L, \theta_i), \dots, u_i(y_{L-1}, \theta_i) - u_i(y_L, \theta_i)\big) \colon i \in N, \theta_i \in \Theta_i\big\}$$

is linearly independent.

This requires that policy preferences be in some sense heterogeneous across players and types of players over a finite subset of alternatives.

Our next task is to develop the counterpart for legislative bargaining games with public randomization of the method for simulating compensatory transfers in games without randomization devices introduced in Subsection 3.3. For every $\nu \in \Delta(X)$, let $v_i(\nu, \theta_i) \equiv \mathbb{E}_\nu\big[u_i(\tilde{x}, \theta_i)\big]$ denote the corresponding expected utility of the type-θ_i player i. We define a *stochastic decision rule* as a mapping $\chi \colon \Theta \to \Delta(X)$, where $\Delta(X)$ is equipped with the Prokhorov metric; and we say that χ is *implementable with transfers* if there exists a transfer function $\psi = (\psi_1, \ldots, \psi_n)\colon \Theta \to \mathbb{R}^n$ such that

$$v_i\big(\chi(\theta_i, \theta_{-i}), \theta_i\big) + \psi_i(\theta_i, \theta_{-i}) \geq v_i\big(\chi(\theta_i', \theta_{-i}), \theta_i\big) + \psi_i(\theta_i', \theta_{-i}) \ ,$$

for all $i \in N$, $\theta_{-i} \in \Theta_{-i}$ and $\theta_i, \theta_i' \in \Theta_i$.

The following lemma is the analogue of Lemma 1 for the model without randomization.

Lemma 2. *Let $\widehat{\chi}$ be a stochastic decision rule that is implementable with transfers. For all $\varepsilon > 0$, there is a stochastic decision rule χ within ε of $\widehat{\chi}$ such that*
(i) χ is also implementable with transfers; and
(ii) there is $\gamma > 0$ such that, for all $\phi \in [-2\gamma, 2\gamma]^n$, there exists $\chi^\phi \in \Delta(X)^\Theta$ such that

$$v_i\big(\chi^\phi(m), \theta_i\big) = v_i\big(\chi(m), \theta_i\big) + \phi_i \ ,$$

for all $i \in N$, $\theta_i \in \Theta_i$ and $m \in \Theta$.

With Lemma 2 in hand, it is straightforward to show that all of the arguments presented in Subsection 3.3 can be applied here to approximately sustain any stochastic decision rule that is implementable with transfers. We thus have the following analogue to Theorem 1.

Theorem 1'. *If a stochastic decision rule χ is implementable with compensatory transfers then it is approximately sustainable in the dynamic bargaining game with public randomization.*

As in the previous variant, this result does not impose any restriction on the dimension of the policy space. Returning to the example of the issue-by-issue rule discussed in Subsection 3.2, we can use Theorem 1' to obtain a version of Corollary 1 for the one-dimensional case where $d = 1$. Except in pathological cases (e.g., two types and/or players having exactly the same preferences), one can easily verify that (C*) holds in this setting by taking a sufficiently large set Y: since there is a continuum of feasible policies (here, X is an interval), one can pick an arbitrarily large number L of policies to obtain linear independence. Moreover, as the issue-by-issue rule is a degenerate stochastic decision rule, Theorem 1' can be applied to obtain a dynamic median voter theorem under incomplete information: the issue-by-issue rule, which here is the decision rule that selects the Condorcet winner at every type profile, is approximately sustainable in the dynamic bargaining game with public randomization.

Appendix

A Proof of Theorem 1

A.1 Proof of Lemma 1

Let χ be an interior decision rule. As the players' type sets are finite, we can write each Θ_i as $\{\theta_{i,1}, \ldots, \theta_{i,k_i}\}$, $k_i \in \mathbb{N}$, for each i. Let $k \equiv \sum_{i=1}^{n} k_i$, and define the mapping $f \colon X \to \mathbb{R}^k$ by

$$f(x) \equiv \begin{pmatrix} u_1(x, \theta_{1,1}) \\ \vdots \\ u_1(x, \theta_{1,k_1}) \\ \vdots \\ u_n(x, \theta_{n,1}) \\ \vdots \\ u_n(x, \theta_{n,k_n}) \end{pmatrix},$$

for all $x \in X$. The derivative of f at arbitrary $x \in X$ is the $k \times d$ matrix

$$Df(x) = \begin{pmatrix} Du_1(x, \theta_{1,1}) \\ \vdots \\ Du_1(x, \theta_{1,k_1}) \\ \vdots \\ Du_n(x, \theta_{n,1}) \\ \vdots \\ Du_n(x, \theta_{n,k_n}) \end{pmatrix},$$

where we view each $Du_i(x, \theta_{i,k_\ell})$ as a $1 \times d$ matrix — i.e., the transpose of $\nabla_1 u_i(x, \theta_{i,k_\ell})$. It follows from condition (C) that $Df\big(\chi(m)\big)$ has full row rank, for all $m \in \Theta$. By the local submersion theorem (e.g., Guillemin and Pollack, 1974), this implies that, for each $m \in \Theta$, we can choose an arbitrarily small open set U^m containing $\chi(m)$ such that the image $V^m \equiv f(U^m)$ is an open set containing $f\big(\chi(m)\big)$. Therefore, there exists a sufficiently small $\gamma^m > 0$ such that the k-dimensional closed rectangle $\prod_{\ell=1}^{k} \big[f_\ell\big(\chi(m)\big) - \gamma^m, f_\ell\big(\chi(m)\big) + \gamma^m\big]$ is contained in V^m. For all $\phi = (\phi_1, \ldots, \phi_n) \in [-\gamma, \gamma]^n$, where $\gamma \equiv \min\{\gamma^m \colon m \in \Theta\}$, there must consequently be a a decision rule $\chi^\phi \in \prod_{m\in\Theta} U^m$ such that

$$
f\big(\chi^\phi(m)\big) \equiv \begin{pmatrix} u_1\big(\chi^\phi(m), \theta_{1,1}\big) \\ \vdots \\ u_1\big(\chi^\phi(m), \theta_{1,k_1}\big) \\ \vdots \\ u_n\big(\chi^\phi(m), \theta_{n,1}\big) \\ \vdots \\ u_n\big(\chi^\phi(m), \theta_{n,k_n}\big) \end{pmatrix} = \begin{pmatrix} u_1\big(\chi(m), \theta_{1,1}\big) + \phi_1 \\ \vdots \\ u_1\big(\chi(m), \theta_{1,k_1}\big) + \phi_1 \\ \vdots \\ u_n\big(\chi(m), \theta_{n,1}\big) + \phi_n \\ \vdots \\ u_n\big(\chi(m), \theta_{n,k_n}\big) + \phi_n \end{pmatrix},
$$

for all $m \in \Theta$, as desired.

A.2 Proof of the Main Theorem

From imlementability to strict implementability. We begin with a useful property of implementable decision rules that satisfy condition (C). Let χ be implementable with transfer function ψ. We say that decision rule χ is *strictly implementable* with transfer function ψ if the incentive constraints in the definition of implementability hold with strict inequalities; i.e., given the transfer function ψ, all legislators are strictly better off truthfully reporting their types.

Lemma A1. *Let χ be an interior decision rule that satisfies condition (C). If χ is implementable with the transfer function ψ then, for every $\varepsilon > 0$, there is decision rule within ε of χ that is strictly implementable with ψ.*

We prove Lemma A1 in the supplementary appendix. In what follows, we focus on the case where χ is strictly implementable, keeping in mind that we might actually be referring to an arbitrarily close approximation of the (possibly only) implementable χ.

Auxiliary games. To establish Theorem 1, it is useful to consider first a class of finite-horizon auxiliary games, in which the exogenous decision rule χ is used to select a policy on behalf of the legislature after every history of message profiles $(m^1, \ldots, m^t) \in \Theta^t$. We will first show that in any PBE of any such auxiliary game, each legislator i's expected payoff is within ε of $\mathbb{E}_\pi\big[u_i\big(\chi(\tilde{\theta}), \tilde{\theta}_i\big)\big]$, and we will then use this result to show that the same is true for the dynamic legislative bargaining game.

The auxiliary game Γ has $T \in \mathbb{N}$ periods. During the play of this game, the messages sent by the legislators are submitted to an Escobar-Toikka test (see the supplementary appendix for a formal definition) and in each period, the decision rule χ only takes into consideration the messages that are "credible," i.e., those that pass the test. If a legislator's message fails the test in a given period, then the rule will ignore her future messages and replace them by messages generated from the Markov chain (λ_i, P_i). More precisely, Γ is constructed as follows. At the start of the auxiliary game Γ, each player i holds some beliefs about her opponents types in the first period. Then, in each period $t = 1, \ldots, T$:

(i) All legislators simultaneously send messages $(m_1^t, \ldots, m_n^t) \in \Theta$.

(ii) Each legislator i's message m_i^t is submitted to the Escobar-Toikka test and a profile of messages $(\mu_1^t, \ldots, \mu_n^t)$ is generated: if all of i's previous reports passed the test, then $\mu_i^t = m_i^t$; otherwise, μ_i^t is obtained from the theoretical distribution of i's types.

(iii) Alternative $\chi(\mu_1^t, \ldots, \mu_n^t)$ is selected.

Given a strategy profile ς for Γ, the (normalized) payoff to player i is

$$\mathcal{U}_i^T(\varsigma) \equiv \frac{1-\delta}{1-\delta^T}\mathbb{E}_\varsigma\left[\sum_{t=1}^{T} \delta^{t-1} u_i\big(\chi(\tilde{\mu}_1^t, \ldots, \tilde{\mu}_n^t), \tilde{\theta}_i^t\big) + \frac{\delta^T(1-\delta^T)}{1-\delta}\tilde{\phi}_i\right] ,$$

where $\phi = (\phi_1, \ldots, \phi_n) \in \mathbb{R}^n$ is an exogenously given vector of transfers defined by:

$$\phi_i \equiv \begin{cases} \frac{1}{T}\sum_{t=1}^{T} \bar{\psi}_i(\mu^t) & \text{if } \left|\frac{1}{T}\sum_{t=1}^{T} \bar{\psi}_i(\mu^t)\right| \leq \gamma \ , \\ -\gamma & \text{otherwise,} \end{cases}$$

with $\bar{\psi}(\cdot)$ defined as in Subsection 3.3. Thus, the exogenous data of the auxiliary game — which will vary when we turn to the dynamic version of the game — is the horizon T, the discount factor $\delta \in [0,1)$, the initial beliefs about the first-period types, the decision rule χ, and the vector of transfers ϕ.

The next lemma establishes that, for sufficiently large T and δ, the payoff to player i in any Nash equilibrium of any auxiliary game defined above approximates $\mathbb{E}_\pi\left[u_i\left(\chi(\tilde{\theta}), \tilde{\theta}_i\right)\right]$. A proof can be found in the supplementary appendix.

Lemma A2. *Let $\varepsilon > 0$, and let χ be a decision rule that is implementable with compensatory transfers. There exists $\widehat{T} \in \mathbb{N}$ such that the following holds for all $T \geq \widehat{T}$: there exists $\bar{\delta}(T) < 1$ such that, for all $\delta \in \left(\bar{\delta}(T), 1\right)$, each agent i's payoff in any PBE of any auxiliary game is within ε of $\mathbb{E}_\pi\left[u_i\left(\chi(\tilde{\theta}), \tilde{\theta}_i\right)\right]$.*

The dynamic auxiliary game. Our next step is to characterize the equilibrium payoffs of a dynamic version of the auxiliary game above. The dynamic auxiliary game divides time into blocks of T periods, $T \in \mathbb{N}$, each corresponding to an auxiliary game. More precisely, each block $b = 1, 2, \ldots$ begins with beliefs and a profile of simulated transfers $\phi^{b-1} = (\phi_1^{b-1}, \ldots, \phi_n^{b-1})$ inherited from the previous block — we set $\phi^0 \equiv (0, \ldots, 0)$. Then, in each period $t = (b-1)T + 1, \ldots, bT$:

(i) All legislators simultaneously send messages $(m_1^t, \ldots, m_n^t) \in \Theta$.

(ii) Each legislator i's message m_i^t is submitted to the Escobar-Toikka test and a profile of messages $(\mu_1^t, \ldots, \mu_n^t)$ is generated: if all of i's previous reports (in the current block) passed the test, then $\mu_i^t = m_i^t$; otherwise, μ_i^t is obtained from the theoretical distribution of i's types.

(iii) Alternative $\chi^{\phi^{b-1}}(\mu_1^t, \ldots, \mu_n^t)$, as defined in Lemma 1, is selected.

Then, block $b+1$ begins with simulated transfers $\phi^b = (\phi_1^b, \ldots, \phi_n^b)$, defined by

$$\phi_i^b \equiv \begin{cases} \frac{1}{T}\sum_{t=(b-1)T+1}^{bT} \bar{\psi}_i(\mu^t) & \text{if } \left|\frac{1}{T}\sum_{t=(b-1)T+1}^{bT} \bar{\psi}_i(\mu^t)\right| \leq \gamma \ , \\ -\gamma & \text{otherwise,} \end{cases}$$

and the same process as above is repeated. Payoffs are defined as in the original legislative bargaining game.

Let ς^b be a strategy profile for the auxiliary game corresponding to the bth block. Observe that by concatenating the ς^b's, we obtain a strategy profile for the dynamic auxiliary game, which yields a payoff of $(1-\delta^T)\sum_{b=1}^{\infty}\delta^{(b-1)T}\mathcal{U}_i^T(\varsigma^b)$ to each player i. By Lemma A2, each legislator i's PBE payoff in the dynamic auxiliary game can be made arbitrarily close to $\mathbb{E}_\pi\big[u_i\big(\chi(\tilde{\theta}),\tilde{\theta}_i\big)\big]$ by picking sufficiently large T and δ. Moreover, as each period of the dynamic auxiliary game has a finite action set, it follows from Fudenberg and Levine's (1983) existence result that a PBE exists.

As χ is strictly implementable, each player i's cost of misreporting her type is bounded away from zero. This allows us to establish the following lemma, whose proof can be found in the supplementary appendix.

Lemma A3. *Let $\varepsilon > 0$, and let χ be a decision rule that is strictly implementable with compensatory transfers. There exists $\overline{T} \in \mathbb{N}$ such that the following holds for all $T \geq \overline{T}$: there is $\bar{\delta}(T) < 1$ such that, for all $\delta \in \big(\bar{\delta}(T), 1\big)$, the dynamic auxiliary game has a PBE σ such that*

$$\limsup \Pr_\sigma\left\{\frac{1}{T}\sum_{t=1}^{T}\mathbf{1}_{\left\{|\tilde{x}^t(\sigma)-\chi(\tilde{\theta}^t)|>\varepsilon\right\}} > \varepsilon\right\} < \varepsilon\ .$$

The PBE of the dynamic bargaining game. We now return to the original legislative bargaining game, in which policies are collectively chosen by the legislators. It follows from Lemma A3 that, to complete the proof of Theorem 1, it suffices to ensure that legislators will spontaneously implement the policies prescribed by the exogenous rule χ^{ϕ^b} in each block b of the dynamic auxiliary game. The incentives to do so are created by punishing the deviating players with negative transfers; Lemma 1 ensures that it is possible to do so.

More precisely, consider a strategy profile σ that prescribes the following behavior in each period t:

• After learning their types, all legislators send the same messages as in the equilibrium of the dynamic auxiliary game described above.

• Given the history of message profiles, let $\chi^{\phi^{b-1}}(\mu_1^t,\ldots,\mu_n^t)$ be the policy that would have been chosen by the exogenous decision rule in the dynamic auxiliary game. In each proposal stage, if there has been no (off-schedule) deviation from σ in the previous periods, then the selected proposer offers $\chi^{\phi^{b-1}}(\mu_1^t, \ldots, \mu_n^t)$; if there has been a deviation and legislator i was the last deviator, then the proposer offers $\chi^{\phi'}(\mu_1^t, \ldots, \mu_n^t)$, where $\phi_i' \equiv \phi_i^{b-1} - \gamma$, and $\phi_j' \equiv \phi_j^{b-1}$ for all $j \neq i$.

- In every voting stage, the legislator accepts the proposal if and only if the proposer acted as prescribed by σ.

It follows from the analysis of the dynamic auxiliary game that on-schedule deviations from σ must be unprofitable. Moreover, off-schedule deviations are punished by decreasing the stage-game payoffs of any deviating legislator by γ in all future periods. Although γ is small, the cost of the deviation becomes arbitrarily large (relative to its benefit) as $\delta \to 1$. This punishment is implemented by the other legislators as they would otherwise be punished themselves in the same way.

References

Acemoglu, D., G. Egorov, and K. Sonin (2012) "Dynamics and Stability of Constitutions, Coalitions, and Clubs," *American Economic Review* 102, 1446-1476.

Anesi, V., and J. Duggan (2017) "Dynamic Bargaining and Stability with Veto Players," *Games and Economic Behavior* 103, 30-40.

Anesi, V., and J. Duggan (2018) "Existence and Indeterminacy of Markovian Equilibria in Dynamic Bargaining Games," *Theoretical Economics* 13, 505-526.

Anesi, V., and D.J. Seidmann (2015) "Bargaining in Standing Committees with an Endogenous Default," *Review of Economic Studies* 82, 825-867.

Austen-Smith, D. (1990a) "Credible Debate Equilibria," *Social Choice and Welfare* 7, 75-93.

Austen-Smith, D. (1990b) "Information Transmission in Debate," *American Journal of Political Science* 34, 124-152.

Austen-Smith, D., and J.S. Banks (1988) "Elections, Coalitions, and Legislative Outcomes," *American Political Science Review* 82, 405-422.

Austen-Smith, D., and J.S. Banks (2005) *Positive Political Theory II: Strategy and Structure.* Ann Arbor: University of Michigan Press.

Austen-Smith, D., and W.H. Riker (1987) "Asymmetric Information and the Coherence of Legislation," *American Political Science Review* 81, 97-918.

Baron, D.P. (1996) "A Dynamic Theory of Collective Goods Programs," *American Political Science Review* 90, 316-330.

Baron, D.P. (forthcoming) "Simple Dynamics of Legislative Bargaining: Coalitions and Proposal Power," *Economic Theory.*

Baron, D.P., and T.R. Bowen (2015) "Dynamic Coalitions," working paper.

Baron, D.P., and M. Herron (2003) "A Dynamic Model of Multidimensional Collective Choice." In: *Computational Models in Political Economy,* K. Kollman, J.H. Miller, and S.E. Page, eds. Cambridge, MA: The MIT Press.

Battaglini, M., and T.R. Palfrey (2012) "The Dynamics of Distributive Politics," *Economic Theory* 49, 739-777.

Baumgartner, F.R., B.D. Jones, and M.C. MacLeod (2000) "The Evolution of Legislative Jurisdiction," *Journal of Politics* 62, 321-349.

Bikhchandani, S., S. Chatterji, R. Lavi, A. Mu'alem, N. Nisan and A. Sen (2006) "Weak Monotonicity Characterizes Deterministic Dominant-Strategy Implementation," *Econometrica* 74, 1109-1132.

Bowen, T. R., Y. Chen, and H. Eraslan (2014) "Mandatory versus Discretionary Spending: The Status Quo Effect," *American Economic Review* 104, 2941-2974.

Bowen, T. R., Y. Chen, H. Eraslan, and Jan Zápal (2017) "Efficiency of Flexible Budgetary Institutions," *Journal of Economic Theory* 167, 148-176.

Bowen, T. R., and Z. Zahran (2012) "On Dynamic Compromise," *Games and Economic Behavior* 76, 391-419.

Câmara, O., and J.X. Eguia (2017) "Slicing and Bundling," *Journal of Politics* 79, 1460-1464.

Chen, Y., and H. Eraslan (2013) "Informational Loss in Bundled Bargaining," *Journal of Theoretical Politics* 25, 338-362.

Chen, Y., and H. Eraslan (2014) "Rhetoric in Legislative Bargaining with Asymmetric Information," *Theoretical Economics* 9, 483-513.

Diermeier, D., and P. Fong (2011) "Legislative Bargaining with Reconsideration," *Quarterly Journal of Economics* 126, 947-985.

Diermeier, D., and P. Fong (2012) "Characterization of the von Neumann-Morgenstern Stable Set in a Non-Cooperative Model of Dynamic Policy-Making with a Persistent Agenda Setter," *Games and Economic Behavior* 76, 349-353.

Diermeier, D., and A. Merlo (2000) "Government Turnover in Parliamentary Democracies," *Journal of Economic Theory* 94, 46-79.

Duggan, J. and T. Kalandrakis (2012) "Dynamic Legislative Policy Making," *Journal of Economic Theory* 147, 1653-1688.

Dziuda, W., and A. Loeper (2016) "Dynamic Collective Choice with Endogenous Status Quo," *Journal of Political Economy* 124, 1148-1186.

Escobar, J.F., and J. Toikka (2013) "Efficiency in Games with Markovian Private Information," *Econometrica* 81, 1887-1934.

Fudenberg, D., and D. Levine (1983) "Subgame Perfect Equilibria of Finite and Infinite Horizon Games," *Journal of Economic Theory* 31, 251-268.

Guillemin, V., and A. Pollack (1974) *Differential Topology,* Prentice Hall, Englewood Cliffs, New Jersey.

Hörner, J., S. Takahashi, and N. Vieille (2015) "Truthful Equilibria in Dynamic Bayesian Games," *Econometrica* 83, 1795-1848.

Jackson, M.O., and B. Moselle (2002) "Coalition and Party Formation in a Legislative Voting Game," *Journal of Economic Theory* 103, 49-87.

Kalandrakis, T. (2004) "A Three-Player Dynamic Majoritarian Bargaining Game," *Journal of Economic Theory* 116, 294-322.

Kalandrakis, T. (2009) "A Reputational Theory of Two-Party Competition," *Quarterly Journal of Political Science* 4, 343-378.

Kalandrakis, T. (2010) "Minimum Winning Coalitions and Endogenous Status Quo," *International Journal of Game Theory* 39, 617-643.

Kalandrakis, T. (2016) "Pareto Efficiency in the Dynamic One-Dimensional Bargaining Model," *Journal of Theoretical Politics* 28,525-536.

Kramer, G.H. (1972) "Sophisticated Voting over Multidimensional Choice Spaces," *Journal of Mathematical Sociology* 2, 165-180.

Laffont, J.-J., and E. Maskin (1982) "The Theory of Incentives: An Overview." In: *Advances in Economic Theory,* W. Hildenbrand, ed. Cambridge: Cambridge University Press.

Meirowitz, A. (2007) "Communication and Bargaining in the Spatial Model," *International Journal of Game Theory* 35, 251-266.

Nunnari, S. (2014) "Dynamic Legislative Bargaining with Veto Power: Theory and Experiments," working paper.

Piguillem, F., and A. Riboni (2015) "Spending-Biased Legislators: Discipline Through Disagreement," *Quarterly Journal of Economics* 130, 901-949.

Richter, M. (2014) "Fully Absorbing Dynamic Compromise," *Journal of Economic Theory* 152, 92-104.

ROCHET, J.-C. (1987) "A Necessary and Sufficient Condition for Rationalizability in a Quasi-Linear Context," *Journal of Mathematical Economics* 16, 191-200.

SCHOFIELD, N. (1980) "Generic Properties of Simple Bergson-Samuelson Welfare Functions," *Journal of Mathematical Economics* 7, 175-192.

STOKEY, N.L., AND R.E. LUCAS JR. (1989) *Recursive Methods in Economic Dynamics,* Harvard University Press, Cambridge, MA.

TSAI, T.-S., AND C.C. YANG (2010) "On Majoritarian Bargaining with Incomplete Information," *International Economic Review* 51, 959-979.

ZÁPAL, J. (2016) "Markovian Equilibria in Dynamic Spatial Legislative Bargaining: Existence with Three Players," *Games and Economic Behavior* 98, 235-242.

Supplementary Appendix:

Proofs Omitted from Text

A Proofs of Lemmata A1-A3

A.1 Proof of Lemma A1

Let χ be an interior decision rule that satisfies condition (C); and suppose it is implementable with compensatory transfers $\psi = (\psi_1, \ldots, \psi_n)$. Fix $\varepsilon > 0$, and let $\eta \equiv \varepsilon / \sqrt{|\Theta|}$. By the same logic as in the proof of Lemma 1, for each $\theta \in \Theta$, we can choose an arbitrarily small open set U_θ containing $\chi(\theta)$ and itself contained in the η-neighborhood of $\chi(\theta)$, such that the image $V_\theta = f(U_\theta)$ is an open set containing $f\big(\chi(\theta)\big)$. Let $\gamma \equiv \min_{\theta \in \Theta} \mathrm{diam} V_\theta > 0$.

By definition of γ, for all $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_n) \in \Theta$, there exists a policy $x_{\hat{\theta}}$ in the η-neighborhood of $\chi(\hat{\theta})$ such that

$$u_i(x_{\hat{\theta}}, \theta_i) = \begin{cases} u_i\big(\chi(\hat{\theta}), \theta_i\big) + \gamma/2 & \text{if } \theta_i = \hat{\theta}_i \ , \\ u_i\big(\chi(\hat{\theta}), \theta_i\big) & \text{if } \theta_i \neq \hat{\theta}_i \ , \end{cases}$$

for all $i \in N$ and all $\theta_i \in \Theta_i$. (Indeed, $f(x_{\hat{\theta}}) \in V_{\hat{\theta}}$.) For each $\hat{\theta} \in \Theta$, we define the decision rule $\chi' \colon \Theta \to X$ by $\chi'(\hat{\theta}) \equiv x_{\hat{\theta}}$, for all $\hat{\theta} \in \Theta$. By construction, χ' is within ε of χ. Moreover, under the social choice function (χ', ψ), each player i receives the same utility as in (χ, ψ), plus an extra $\gamma/2 > 0$ if and only if she reports her type truthfully. As (χ, ψ) is strategy proof, this implies that χ' is strictly implementable with transfer function ψ.

A.2 Proof of Lemma A2

The construction of the auxiliary game uses a statistical test developed by Escobar and Toikka (2013). Given a sequence of messages $(\mu^1, \ldots, \mu^t) \in \Theta^t$, $t \in \{1, \ldots, T\}$, let $\tau^t(\theta, \theta') \equiv \Big|\big\{2 \leq s \leq t \colon (\mu^{s-1}, \mu^s) = (\theta, \theta')\big\}\Big|$, $\tau_i^t(\theta, \theta'_{-i}) \equiv \sum_{\theta'_i \in \Theta_i} \tau^t(\theta, \theta')$, and

$$P_i^t(\theta'_i \mid \theta, \theta'_{-i}) \equiv \frac{\tau^t(\theta, \theta')}{\tau_i^t(\theta, \theta'_{-i})} \ ,$$

for all $(\theta, \theta') \in \Theta^2$ and $i \in N$ (where we set $0/0 = 0$). An *Esbobar-Toikka test* is a sequence $\{b_k\} \in \mathbb{R}_+^\infty$ that converges to zero. Legislator i *passes the test at* $(\mu^1, \ldots, \mu^t)$ if, for all $(\theta, \theta'_{-i}) \in \Theta \times \Theta_{-i}$,

$$\sup_{\theta'_i} \left| P_i(\theta'_i \mid \theta_i) - P_i^t(\theta'_i \mid \theta, \theta'_{-i}) \right| < b_{\tau_i^t(\theta, \theta'_{-i})} \ .$$

In words, she passes the test if, for all (θ, θ'_{-i}), the distribution of the μ_i^s's over the periods s where $(\mu^{s-1}, \mu^s) = (\theta, \theta')$ is within $b_{\tau_i^t(\theta, \theta'_{-i})}$ of her true conditional distribution $P_i(\cdot \mid \theta_i)$ (in the sup-norm).

Now let $\xi \colon \Theta \to \Theta$ be such that $\xi(\theta)$ is distributed on Θ according to the true conditional distribution $P(\cdot \mid \theta)$, for all $\theta \in \Theta$. Given a sequence of reports $(m^1, \ldots, m^T)$ made by the legislators in the auxiliary game, the decision rule will generate its own sequence $(\mu^1, \ldots, \mu^T)$ as follows:

$$\mu_i^t = \begin{cases} m_i^t & \text{if } i \text{ passes } \{b_k\} \text{ at } (\mu^1, \ldots, \mu^s) \text{ for all } 1 \leq s < t \ , \\ \xi_i(\mu^{t-1}) & \text{otherwise,} \end{cases}$$

for all $t \in \{1, \ldots, T$ and all $i \in N$.

Legislator i's truthful strategy, which prescribes her to truthfully report her type at every history of the auxiliary game, is denoted by ς_i^*. Escobar and Toikka (2013) establish the following result.

Lemma A4. *Let $\epsilon > 0$. There exists a test $\{b_k\}$ that satisfies the following conditions:*

(i) For every $T \in \mathbb{N}$, we have

$$\Pr_{\varsigma_i^*, \varsigma_{-i}} \big\{ i \textit{ passes } \{b_k\} \textit{ at } (\tilde{\mu}^1, \ldots, \tilde{\mu}^t) \textit{ for all } t \in \{1, \ldots, T\} \big\} \geq 1 - \epsilon \ ,$$

for all i, ς_{-i} and λ.

(ii) There exists $\overline{T} < \infty$ such that the following holds for all $T > \overline{T}$: for all ς and all λ, the empirical distribution of $(\tilde{\mu}^1, \ldots, \tilde{\mu}^T)$, denoted $\tilde{\pi}^T$, satisfies

$$\Pr_\varsigma \big\{ \|\tilde{\pi}^T - \pi\| < \epsilon \big\} \geq 1 - \epsilon \ .$$

We are now in a position to prove Lemma A2. Let $\varepsilon > 0$; let χ be a decision rule that is implementable with transfers; and let $\varphi \colon \Theta \to \mathbb{R}^n$ be a transfer function that implements

χ. Observe that $\psi\colon \Theta \to \mathbb{R}^n$, defined by $\psi \equiv \varphi - \mathbb{E}_\pi\left[\varphi_i(\tilde{\theta})\right]$, also implements χ; that is,

$$u_i\big(\chi(\theta_i, \theta_{-i}), \theta_i\big) + \psi_i(\theta_i, \theta_{-i}) \geq u_i\big(\chi(\theta_i', \theta_{-i}), \theta_i\big) + \psi_i(\theta_i', \theta_{-i}) \ , \tag{A1}$$

for all $i \in N$, $\theta_{-i} \in \Theta_{-i}$ and $\theta_i, \theta_i' \in \Theta_i$.

Now consider any auxiliary game Γ. The expected payoff to legislator i from a given strategy profile ς is

$$\mathcal{U}_i^T(\varsigma) \equiv \frac{1-\delta}{1-\delta^T}\left\{\mathbb{E}_\varsigma\left[\sum_{t=1}^T \delta^{t-1} u_i\big(\chi(\tilde{\mu}^t), \tilde{\theta}_i^t\big)\right] \ldots\right.$$
$$\left.\ldots + \delta^T \sum_{s=1}^T \delta^{s-1}\left(\Pr_\varsigma\{\tilde{\mu} \in \Lambda\}\mathbb{E}_\varsigma\left[\frac{\sum_{t=1}^T \psi_i(\tilde{\mu}^t)}{T} \mid \tilde{\mu} \in \Lambda\right] - \Pr_\varsigma\{\tilde{\mu} \in \Lambda^c\}\gamma\right)\right\},$$

where $\Lambda \equiv \left\{\mu = (\mu^1, \ldots, \mu^T)\colon \left|\frac{\sum_{t=1}^T \psi_i(\mu^t)}{T}\right| \leq \gamma\right\}$. Observe that

$$\Pr_\varsigma\{\tilde{\mu} \in \Lambda\} = \Pr_\varsigma\left\{\left|\frac{\sum_{t=1}^T \psi_i(\tilde{\mu}^t)}{T}\right| \leq \gamma\right\} = \Pr_\varsigma\left\{\left|\mathbb{E}_{\tilde{\pi}^T}\left[\psi_i(\tilde{\mu}^t)\right]\right| \leq \gamma\right\}$$
$$= \Pr_\varsigma\left\{\left|\mathbb{E}_{\tilde{\pi}^T}\left[\psi_i(\tilde{\mu}^t)\right] - \mathbb{E}_\pi\left[\psi_i(\tilde{\mu}^t)\right]\right| \leq \gamma\right\} .$$

It follows from Lemma A4(ii) (and continuity of the expectation functional) that, for any $\epsilon > 0$, there is a sufficiently large $T_\epsilon \in \mathbb{N}$ such that, for all $T > T_\epsilon$ and all strategy profiles ς, $\Pr_\varsigma\{\tilde{\mu} \in \Lambda\} > 1 - \epsilon$. Hence, there is a sufficiently large $\overline{T}_1 \in \mathbb{N}$ such that, for all $T \geq \overline{T}_1$,

$$\mathcal{U}_i^T(\varsigma) \leq \frac{1-\delta}{1-\delta^T}\left\{\mathbb{E}_\varsigma\left[\sum_{t=1}^T \delta^{t-1} u_i\big(\chi(\tilde{\mu}^t), \tilde{\theta}_i^t\big)\right] + \delta^T \sum_{s=1}^T \delta^{s-1}\mathbb{E}_\varsigma\left[\frac{\sum_{t=1}^T \psi_i(\tilde{\mu}^t)}{T}\right]\right\} + \frac{\varepsilon}{4}$$
$$= \frac{1-\delta}{1-\delta^T}\left\{\mathbb{E}_\varsigma\left[\sum_{t=1}^T \delta^{t-1}\left(u_i\big(\chi(\tilde{\mu}^t), \tilde{\theta}_i^t\big) + \frac{\delta^{T-t+1}(1-\delta^T)}{(1-\delta)T}\psi_i(\tilde{\mu}^t)\right)\right]\right\} + \frac{\varepsilon}{4} \ ,$$

for all $i \in N$ all all strategy profiles ς. Let $\underline{u} \equiv \inf\left\{u_i(x, \theta_i)\colon i \in N, \theta_i \in \Theta_i, x \in A\right\} > -\infty$ and $\overline{u} \equiv \sup\left\{u_i(x, \theta_i)\colon i \in N, \theta_i \in \Theta_i, x \in A\right\} < \infty$. It is readily checked that, for all $T \in \mathbb{N}$, there is $\bar{\delta}_0(T)$ such that, for all $\delta > \bar{\delta}_0(T)$,

$$\sup\left\{\left|\frac{1}{T}\sum_{t=1}^T v^t - \frac{1-\delta}{1-\delta^T}\sum_{t=1}^T \delta^{t-1} v^t\right| : (v^1, \ldots, v^T) \in [\underline{u}, \overline{u}]^T\right\} < \frac{\varepsilon}{4} \ ;$$

so that

$$\mathcal{U}_i^T(\varsigma) < \mathbb{E}_\varsigma\left[\frac{1}{T}\sum_{t=1}^T\left(u_i\big(\chi(\tilde{\mu}^t), \tilde{\theta}_i^t\big) + \frac{\delta^{T-t+1}(1-\delta^T)}{(1-\delta)T}\psi_i(\tilde{\mu}^t)\right)\right] + \frac{\varepsilon}{2} \ ,$$

for all i and all ς. Coupled with (A1) — and the observation that $\lim_{\delta\to 1}(1-\delta^T)/(1-\delta) = T$, for all $T \in \mathbb{N}$ — this inequality implies that, for all $T \geq T_1$, there is $\bar{\delta}_1(T) \geq \bar{\delta}_0(T)$ such that, whenever $\delta > \bar{\delta}_1(T)$,

$$\begin{aligned}\mathcal{U}_i^T(\varsigma) &< \mathbb{E}_\varsigma\left[\frac{1}{T}\sum_{t=1}^{T}\left(u_i\big(\chi(\tilde{\theta}_i^t, \tilde{\mu}_{-i}^t), \theta_i^t\big) + \psi_i(\tilde{\theta}_i^t, \tilde{\mu}_{-i}^t)\right)\right] + \frac{3\varepsilon}{4} \\ &= \mathbb{E}_{\varsigma_i^*, \varsigma_{-i}}\left[\mathbb{E}_{\tilde{\pi}^T}\left[u_i\big(\chi(\tilde{\theta}_i, \tilde{\mu}_{-i}), \tilde{\theta}_i\big) + \psi_i(\tilde{\theta}_i, \tilde{\mu}_{-i})\right]\right] + \frac{3\varepsilon}{4}\ ,\end{aligned}$$

for all i and all ς. From Lemma A4, this in turn implies that there exists $\overline{T}_2 < \infty$ such that the following holds for all $T > \overline{T}_2$: there is $\bar{\delta}_2(T) \geq \bar{\delta}_1(T)$ such that, whenever $\delta > \bar{\delta}_2(T)$,

$$\mathcal{U}_i^T(\varsigma) < \mathbb{E}_\pi\left[u_i\big(\chi(\tilde{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_i\big) + \psi_i(\tilde{\theta}_i, \tilde{\theta}_{-i})\right] + \varepsilon = \mathbb{E}_\pi\left[u_i\big(\chi(\tilde{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_i\big)\right] + \varepsilon\ ,$$

for all i and all ς. Hence, each legislator i's equilibrium payoff must be bounded above by $\mathbb{E}_\pi\left[u_i\big(\chi(\tilde{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_i\big)\right]$.

To complete the proof of the lemma, it remains to establish that if T and δ are sufficiently large, then each legislator i's equilibrium payoff in Γ is strictly larger than $\mathbb{E}_\pi\left[u_i\big(\chi(\tilde{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_i\big)\right] - \varepsilon$. By the same logic as above, there exists $\overline{T}_3 \in \mathbb{N}$ such that the following holds for all $T > \overline{T}_3$: there is $\bar{\delta}_3(T)$ such that, whenever $\delta > \bar{\delta}_3(T)$,

$$\mathcal{U}_i^T(\varsigma) > \mathbb{E}_\varsigma\left[\frac{1}{T}\sum_{t=1}^{T}\left(u_i\big(\chi(\tilde{\mu}^t), \tilde{\theta}_i^t\big) + \psi_i(\tilde{\mu}^t)\right)\right] - \frac{3\varepsilon}{4}\ ,$$

for all i and all ς. In particular, if legislator i plays her truthful strategy ς_i^*, then she can secure a payoff of

$$\begin{aligned}\mathcal{U}_i^T(\varsigma_i^*, \varsigma_{-i}) &> \mathbb{E}_{\varsigma_i^*, \varsigma_{-i}}\left[\frac{1}{T}\sum_{t=1}^{T}\left(u_i\big(\chi(\tilde{\mu}^t), \tilde{\theta}_i^t\big) + \psi_i(\tilde{\mu}^t)\right)\right] - \frac{3\varepsilon}{4} \\ &= \mathbb{E}_{\varsigma_i^*, \varsigma_{-i}}\left[\frac{1}{T}\sum_{t=1}^{T}\left(u_i\big(\chi(\tilde{\theta}_i^t, \tilde{\mu}_{-i}^t), \tilde{\theta}_i^t\big) + \psi_i(\tilde{\theta}_i^t, \tilde{\mu}_{-i}^t)\right)\right] - \frac{3\varepsilon}{4}\ ,\end{aligned}$$

for all ς_{-i}. Applying again Lemma A4, we obtain that there exists $\overline{T}_4 \in \mathbb{N}$ such that the following holds for all $T > \overline{T}_4$: there is $\bar{\delta}_4(T)$ such that, whenever $\delta > \bar{\delta}_4(T)$,

$$\mathcal{U}_i^T(\varsigma_i^*, \varsigma_{-i}) > \mathbb{E}_\pi\left[u_i\big(\chi(\tilde{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_i\big) + \psi_i(\tilde{\theta}_i, \tilde{\theta}_{-i})\right] - \varepsilon = \mathbb{E}_\pi\left[u_i\big(\chi(\tilde{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_i\big)\right] - \varepsilon\ .$$

It follows that i's payoff in any equilibrium must exceed $\mathbb{E}_\pi\left[u_i\big(\chi(\tilde{\theta}), \tilde{\theta}_i\big)\right] - \varepsilon$.

We obtain the lemma by setting $\widehat{T} \equiv \max\{\overline{T}_2, \overline{T}_4\}$ and $\bar{\delta}(T) \equiv \max\left\{\bar{\delta}_2(T), \bar{\delta}_4(T)\right\}$.

A.3 Proof of Lemma A3

Fix a given legislator $i \in N$. As χ is strictly implementable with transfers $(\bar{\psi}_1, \ldots, \bar{\psi}_n)$,

$$\gamma \equiv \max_{\theta_i, \theta_i' \in \Theta_i \theta_{-i} \in \Theta_{-i}} u_i\big(\chi(\theta_i, \theta_{-i}), \theta_i\big) + \bar{\psi}_i(\theta_i, \theta_{-i}) - \Big[u_i\big(\chi(\theta_i', \theta_{-i}), \theta_i\big) + \bar{\psi}_i(\theta_i', \theta_{-i})\Big] > 0$$

Let σ_T be the PBE constructed using blocks of length T, and let ς_T^b be the strategy profile induced by σ_T in the bth block, $b \in \mathbb{N}$. Recall from the proof of Lemma A2 that for all $T \in \mathbb{N}$, there exists $\bar{\delta}_0(T)$ such that, for all $\delta \in \big(\bar{\delta}_0(T), 1\big)$,

$$\mathcal{U}_i^T(\varsigma_T^b) < \mathbb{E}_{\varsigma_T^b}\left[\frac{1}{T}\sum_{t=1}^{T}\left(u_i\big(\chi(\tilde{\mu}^t), \tilde{\theta}_i^t\big) + \frac{\delta^{T-t+1}(1-\delta^T)}{(1-\delta)T}\bar{\psi}_i(\tilde{\mu}^t)\right)\right] + \frac{\gamma\varepsilon^2}{8n} \ ,$$

for all $b \in \mathbb{N}$. Moreover, as $\bar{\psi}_i(\cdot)$ is bounded, there is $\bar{\delta}_1(T) \geq \bar{\delta}_0(T)$ such that, for all $\delta \in \big(\bar{\delta}_1(T), 1\big)$,

$$\mathcal{U}_i^T(\varsigma_T^b) < \mathbb{E}_{\varsigma_T^b}\left[\frac{1}{T}\sum_{t=1}^{T}\left(u_i\big(\chi(\tilde{\mu}^t), \tilde{\theta}_i^t\big) + \bar{\psi}_i(\tilde{\mu}^t)\right)\right] + \frac{\gamma\varepsilon^2}{4n} \ . \tag{A2}$$

Moreover, from Lemma A4(ii), there exists a sufficiently large $\overline{T}_1 \geq \overline{T}_0$ such that, for all $T > \overline{T}_1$,

$$\mathbb{E}_{\varsigma_T^b}\left[\frac{1}{T}\sum_{t=1}^{T}\left(u_i\big(\chi(\tilde{\theta}_i^t, \tilde{\mu}_{-i}^t), \tilde{\theta}_i^t\big) + \bar{\psi}_i(\tilde{\theta}_i^t, \tilde{\mu}_{-i}^t)\right)\right] \leq \mathbb{E}_\pi\Big[u_i\big(\chi(\tilde{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_i\big) + \bar{\psi}_i(\tilde{\theta}_i, \tilde{\theta}_{-i})\Big] + \frac{\gamma\varepsilon^2}{8n}.$$

In addition, from Lemma A2, there exists a sufficiently large $\overline{T}_2 \geq \overline{T}_1$ such that the following holds for all $T > \overline{T}_2$: there is $\bar{\delta}_2(T) < 1$ such that, for all $\delta \in \big(\bar{\delta}_2(T), 1\big)$,

$$\mathcal{U}_i^T(\varsigma_T^b) > \mathbb{E}_\pi\Big[u_i\big(\chi(\tilde{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_i\big)\Big] - \frac{\gamma\varepsilon^2}{8n} \ , \tag{A3}$$

for all $b \in \mathbb{N}$.

Now take an arbitrary $T > \max\{\overline{T}_1, \overline{T}_2\}$, and let $\delta > \max\big\{\bar{\delta}_1(T), \bar{\delta}_2(T)\big\}$; so that both (A2) and (A3) hold. For every integer $K \geq T/\varepsilon$, let $B_K \equiv \max\{B \in \mathbb{N} \colon BT \leq K\}$; so that $B_K T \geq K - T + 1$. From (A3), legislator i's average payoff over the first B_K blocks, $U_i^{B_K}(\sigma_T)$, satisfies

$$U_i^{B_K}(\sigma_T) \equiv \frac{1}{B_K}\sum_{b=1}^{B_K}\mathcal{U}_i^T(\varsigma_T^b) > \mathbb{E}_\pi\Big[u_i\big(\chi(\tilde{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_i\big)\Big] - \frac{\gamma\varepsilon^2}{8n} \ . \tag{A4}$$

Moreover, from (A2), we also have

$$
\begin{aligned}
U_i^{B_K}(\sigma_T) < & \frac{1}{B_K}\sum_{b=1}^{B_K}\Bigg\{\mathbb{E}_{\varsigma_T^b}\left[\frac{1}{T}\sum_{t=1}^{T}\Big(u_i\big(\chi(\tilde\theta_i^t,\tilde\mu_{-i}^t),\tilde\theta_i^t\big)+\bar\psi_i(\tilde\theta_i^t,\tilde\mu_{-i}^t)\Big)\right]+\frac{\gamma\varepsilon^2}{4n} \\
& +\mathbb{E}_{\varsigma_T^b}\left[\frac{1}{T}\sum_{t:\ \chi(\tilde\mu^t)\neq\chi(\tilde\theta^t)}\Big(u_i\big(\chi(\tilde\mu^t),\tilde\theta_i^t\big)+\bar\psi_i(\tilde\mu^t)-u_i\big(\chi(\tilde\theta_i^t,\tilde\mu_{-i}^t),\tilde\theta_i^t\big)-\bar\psi_i(\tilde\theta_i^t,\tilde\mu_{-i}^t)\Big)\right]\Bigg\} \\
& \leq \frac{1}{B_K}\sum_{b=1}^{B_K}\Bigg\{\mathbb{E}_{\varsigma_T^b}\left[\frac{1}{T}\sum_{t=1}^{T}\Big(u_i\big(\chi(\tilde\theta_i^t,\tilde\mu_{-i}^t),\tilde\theta_i^t\big)+\bar\psi_i(\tilde\theta_i^t,\tilde\mu_{-i}^t)\Big)\right]\Bigg\}+\frac{\gamma\varepsilon^2}{4n} \\
& -\frac{\gamma(\varepsilon K-T+1)}{B_K T}\Pr\{E_K^i\}\ ,
\end{aligned}
$$

where $E_K^i \equiv \left\{\frac{1}{K}\sum_{t=1}^{K}\mathbf{1}_{\{\tilde\mu_i^t\neq\tilde\theta_i^t\}} > \varepsilon/n\right\}$. Indeed, if event E_K^i occurs, then the minimum number of periods t in which legislator i misreports her type is $\lceil\varepsilon K\rceil-(T-1)$. We thus have

$$
U_i^{B_K}(\sigma_T) < \mathbb{E}_\pi\Big[u_i\big(\chi(\tilde\theta_i,\tilde\theta_{-i}),\tilde\theta_i\big)+\bar\psi_i(\tilde\theta_i,\tilde\theta_{-i})\Big]+\frac{3\gamma\varepsilon^2}{8n}-\frac{\gamma(\varepsilon K-T+1)}{K-T+1}\Pr\{E_K^i\}\ . \quad \text{(A5)}
$$

Coupled (A4) and (A5) imply that, for all $K > T/\varepsilon$, we have

$$
\Pr\{E_K^i\} < \frac{\gamma\varepsilon^2}{2n}\frac{K-T+1}{\gamma(\varepsilon K-T+1)} = \frac{\varepsilon^2}{2n}\frac{1-\frac{T-1}{K}}{\varepsilon-\frac{T-1}{K}} \equiv \phi_K\ ,
$$

where $\lim_{T\to\infty}\phi_K = \varepsilon/(2n) < \varepsilon/n$. Hence, letting $\overline{E}_K^i$ denote the complement of E_K^i, we have

$$
\begin{aligned}
1-\Pr\left\{\frac{1}{K}\sum_{t=1}^{K}\mathbf{1}_{\{\chi(\tilde m^t)\neq\chi(\tilde\theta^t)\}} > \varepsilon\right\} &\geq \Pr\left\{\bigcap_{i=1}^{n}\overline{E}_K^i\right\} \\
&\geq \sum_{i=1}^{n}\Pr\{\overline{E}_K^i\}-n+1 \geq n(1-\phi_K)-n+1\ ,
\end{aligned}
$$

where the second inequality follows from the Bonferroni inequality. Taking the limit superior on both sides, we obtain the desired inequality.

B Proof of Lemma 2

Let $\widehat\chi$ be a stochastic decision rule that is implementable with transfers; and fix $\varepsilon > 0$.

Part (i). Let $\{\eta^m\}$ be a null sequence, and let $\{\chi^m\}$ be a sequence of stochastic decision rules, defined by:

$$\chi^m(\theta) \equiv (1-\eta^m)\widehat{\chi}(\theta) + \eta^m \lambda_Y \text{ , for all } \theta \in \Theta \text{ ,}$$

where λ_Y is the uniform distribution on Y: $\lambda_Y\big(\{x\}\big) = 1/L$, for all $x \in Y$. By assumption, there exists a transfer function $\psi = (\psi_1, \ldots, \psi_n) \colon \Theta \to \mathbb{R}^n$ such that

$$v_i\big(\widehat{\chi}(\theta_i, \theta_{-i}), \theta_i\big) + \psi_i(\theta_i, \theta_{-i}) \geq v_i\big(\widehat{\chi}(\theta_i', \theta_{-i}), \theta_i\big) + \psi_i(\theta_i', \theta_{-i}) \text{ ,}$$

for all $i \in N$, $\theta_{-i} \in \Theta_{-i}$ and $\theta_i, \theta_i' \in \Theta_i$. As the v_i's are affine on $\Delta(X)$, it follows from the above inequality that

$$\begin{aligned}
& v_i\big((1-\eta^m)\widehat{\chi}(\theta_i, \theta_{-i}) + \eta^m \lambda_Y, \theta_i\big) + (1-\eta^m)\psi_i(\theta_i, \theta_{-i}) \\
&= (1-\eta^m)\Big[v_i\big(\widehat{\chi}(\theta_i, \theta_{-i}), \theta_i\big) + \psi_i(\theta_i, \theta_{-i})\Big] + \eta^m v_i(\lambda_Y, \theta_i) \\
&\geq (1-\eta^m)\Big[v_i\big(\widehat{\chi}(\theta_i', \theta_{-i}), \theta_i\big) + \psi_i(\theta_i', \theta_{-i})\Big] + \eta^m v_i(\lambda_Y, \theta_i) \\
&= v_i\big((1-\eta^m)\widehat{\chi}(\theta_i', \theta_{-i}) + \eta^m \lambda_Y, \theta_i\big) + (1-\eta^m)\psi_i(\theta_i', \theta_{-i}) \text{ ,}
\end{aligned}$$

for all $m \in \mathbb{N}$, $i \in N$, $\theta_{-i} \in \Theta_{-i}$ and $\theta_i, \theta_i' \in \Theta_i$. Thus, the stochastic decision rule χ^m is implementable with transfers $(1-\eta^m)\psi_i$. Moreover, as $\big\{\chi^m(\theta)\big\}$ converges weakly to $\widehat{\chi}(\theta)$ for all $\theta \in \Theta$, there is a sufficiently large $M \in \mathbb{N}$ such that χ^M is within ε of $\widehat{\chi}$. We obtain the first part of the lemma by setting $\chi \equiv \chi^M$.

Part (ii). Define $h_{i,\theta_i} \colon B \equiv \big\{(\beta_1, \ldots, \beta_{L-1}) \in [0,1]^{L-1} \colon \sum_{\ell=1}^{L-1} \beta_\ell \leq 1\big\} \to \mathbb{R}$ by

$$h_{i,\theta_i}(\beta_1, \ldots, \beta_{L-1}) \equiv \left(1 - \sum_{\ell=1}^{L-1} \beta_j\right) u_i(y_L, \theta_i) + \sum_{\ell=1}^{L-1} \beta_\ell u_i(y_\ell, \theta_i) \text{ ;}$$

and let $H \colon B \to \mathbb{R}^{\sum_i |\Theta_i|}$ be defined by

$$H(\beta_1, \ldots, \beta_{L-1}) \equiv \big(h_{i,\theta_i}(\beta_1, \ldots, \beta_{L-1})\big)_{i \in N, \theta_i \in \Theta_i} \text{ .}$$

The derivative of H at arbitrary $(\beta_1, \ldots, \beta_{L-1})$ is the $\left(\sum_{i=1}^n |\Theta_i|\right) \times (L-1)$ matrix

$$DH(\beta_1, \ldots, \beta_{L-1}) = \begin{pmatrix} Dh_{1,\theta_{1,1}}(\beta_1, \ldots, \beta_{L-1}) \\ \vdots \\ Dh_{1,\theta_{1,|\Theta_1|}}(\beta_1, \ldots, \beta_{L-1}) \\ \vdots \\ Dh_{n,\theta_{n,1}}(\beta_1, \ldots, \beta_{L-1}) \\ \vdots \\ Dh_{n,\theta_{n,|\Theta_n|}}(\beta_1, \ldots, \beta_{L-1}) \end{pmatrix}$$

$$= \begin{pmatrix} u_1(y_1, \theta_{1,1}) - u_1(y_L, \theta_{1,1}) & \cdots & u_1(y_{L-1}, \theta_{1,1}) - u_1(y_L, \theta_{1,1}) \\ \vdots & \vdots & \vdots \\ u_1(y_1, \theta_{1,|\Theta_1|}) - u_1(y_L, \theta_{1,|\Theta_1|}) & \cdots & u_1(y_{L-1}, \theta_{1,|\Theta_1|}) - u_1(y_L, \theta_{1,|\Theta_1|}) \\ \vdots & & \\ u_n(y_1, \theta_{n,1}) - u_n(y_L, \theta_{n,1}) & \cdots & u_n(y_{L-1}, \theta_{n,1}) - u_n(y_L, \theta_{n,1}) \\ \vdots & & \\ u_n(y_1, \theta_{1,|\Theta_n|}) - u_n(y_L, \theta_{1,|\Theta_n|}) & \cdots & u_n(y_{L-1}, \theta_{1,|\Theta_n|}) - u_n(y_L, \theta_{1,|\Theta_n|}) \end{pmatrix}.$$

It follows from condition (C*) that this matrix has full row rank. By the local submersion theorem (e.g., Guillemin and Pollack, 1974), this implies that we can choose an arbitrarily small open set $U \subset B$ containing $(1/L, \ldots, 1/L)$ such that the image $V \equiv f(U)$ is an open set containing $H(1/L, \ldots, 1/L)$. Therefore, there exists a sufficiently small $\gamma > 0$ such that the $\left(\sum_{i=1}^n |\Theta_i|\right)$-dimensional closed rectangle $\prod_{\ell=1}^{\sum_{i=1}^n |\Theta_i|} \left[H_\ell(1/L, \ldots, 1/L) - 2\gamma/\eta^M, H_\ell(1/L, \ldots, L/k) + 2\gamma/\eta^M\right]$ is contained in V. For all $\phi = (\phi_1, \ldots, \phi_n) \in [-2\gamma, 2\gamma]^n$, there must consequently be a $\beta^\phi \in U$ such that

$$H(\beta^\phi) = \begin{pmatrix} h_{1,\theta_{1,1}}(1/L,\dots,1/L) + \frac{\phi_1}{\eta^M} \\ \vdots \\ h_{1,\theta_{1,|\Theta_1|}}(1/L,\dots,1/L) + \frac{\phi_1}{\eta^M} \\ \vdots \\ h_{n,\theta_{n,1}}(1/L,\dots,1/L) + \frac{\phi_n}{\eta^M} \\ \vdots \\ h_{n,\theta_{n,|\Theta_n|}}(1/L,\dots,1/L) + \frac{\phi_n}{\eta^M} \end{pmatrix} .$$

Now let $\lambda_Y^\phi \in \Delta(X)$ be defined by

$$\lambda_Y^\phi\big(\{y_\ell\}\big) \equiv \beta_\ell^\phi \text{ , for all } \ell = 1,\dots,L-1 \text{ , and } \lambda_Y^\phi\big(\{y_L\}\big) \equiv 1 - \sum_{\ell=1}^{L-1} \beta_\ell^\phi \ ;$$

and let the stochastic decision rule χ^ϕ be defined by $\chi^\phi(\theta) \equiv (1-\eta^M)\chi(\theta) + \eta^M \lambda_Y^\phi$, for all $\theta \in \Theta$. We thus have

$$\begin{aligned} v_i\big(\chi^\phi(m),\theta_i\big) &= v_i\big((1-\eta^M)\chi(m) + \eta^M\lambda_Y^\phi,\theta_i\big) \\ &= (1-\eta^M)v_i\big(\chi(m),\theta_i\big) + \eta^M v_i\big(\lambda_Y^\phi,\theta_i\big) \\ &= (1-\eta^M)v_i\big(\chi(m),\theta_i\big) + \eta^M h_{i,\theta_i}(\beta^\phi) \\ &= (1-\eta^M)v_i\big(\chi(m),\theta_i\big) + \eta^M \left[h_{i,\theta_i}(1/L,\dots,1/L) + \frac{\phi_i}{\eta^M}\right] \\ &= (1-\eta^M)v_i\big(\chi(m),\theta_i\big) + \eta^M v_i\big(\lambda_Y,\theta_i\big) + \phi_i \\ &= v_i\big((1-\eta^M)\chi(m) + \eta^M\lambda_Y,\theta_i\big) + \phi_i = v_i\big(\chi(m),\theta_i\big) + \phi_i \ , \end{aligned}$$

as desired.