Preparatory questions for future students on MSc in Financial and Computational Mathematics

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Introduction

The MSc Financial and Computational Mathematics is designed for students with a first degree in mathematics or a related subject with substantial mathematical content (e.g. engineering and physics, computer science and economics). If you're taking this MSc you won't need a background in finance, just enthusiasm and a willingness to learn the subject.

- The following two books give an indication of **the level** of mathematics required:

    1. *All the Mathematics You Missed [But Need to Know for Graduate School] by T.A. Garrity*, published by CUP.

        A basic understanding of the content of chapters 1, 2, 3, 12, 13, 14, 15, and 16 would be advisable.

    2. *Scientific Computing: An Introduction to Numerical Analysis by E. Suli and D. Mayers*, also published by CUP.

        You should be able to understand, with some work (reading maths books is never easy!), chapters 1, 2, 6, 7, 11, 12.

Alternatively, you can use the textbook *Numerical Analysis by R.L. Burden and J.D. Faires* published by Brooks/Cole, where you can examine the basics of chapters 1-6, 10.

If you forgot some very basic material then you can use the following very basic textbook which covers most of the math topics required from you to know but at a low depth (we certainly expect that you have studied and know Math at a deeper level!): *Mathematical Techniques: An Introduction for the Engineering, Physical, and Mathematical Sciences by D. Jordan and P. Smith* published by OUP.

If you need to refresh your linear algebra knowledge, you can use, e.g. *Linear Algebra and Its Applications by D.C. Lay* published by Pearson, chapters 1-7.
If you need to refresh your advanced calculus knowledge, you can use, e.g. *The Calculus Lifesaver: All the Tools You Need to Excel at Calculus (Princeton Lifesaver Study Guides) by A. Banner* published by Princeton University Press.

- **Probability and Statistics**

  You should be familiar with elementary probability and statistics. The following are suggested as basic texts for revision purposes:

  - *Introductory Probability and Statistical Applications by P.L. Meyer*
  - *Probability by A.N. Shiryaev* - first two chapters
  - *A First Course in Probability by S.M. Ross*
  - *Probability and Math Statistics by L.D. Taylor.*

- **C++**

  We do not require any prior knowledge of C++, but some computational experience in any computer language is highly desirable. You may wish to look through any introductory book on C++ such as *Guide to Scientific Computing in C++ by J. Pitt-Francis and J. Whiteley; Schaum's Outline of Programming with C++ by J.R. Hubbard; Schaum's Outline of Fundamentals of Computing with C++ by J.R. Hubbard* or *the online tutorial http://www.cplusplus.com/doc/tutorial/*

  If you do not have any prior or very little computational experience, we strongly recommend you to read and do exercises from a simpler book such as e.g. *Jumping into C++ by Alex Allain*, before you start the course.

- **Financial Mathematics**

  We do not require any prior knowledge of financial mathematics or finance but if you would like to undertake some preliminary reading, you can refer to e.g. *Stochastic Calculus for Finance I by S.E. Shreve* published by Springer, 2004 or *Introductory Course on Financial Mathematics* by *M.V. Tretyakov* published by ICP, 2013.

It is important that you are confident in using mathematical techniques and in basic use of computers before you join the MSc programme at the University of Nottingham. The following questions were compiled by convenors of the core modules to demonstrate further what we consider to be the minimum level of knowledge required. You are strongly advised to attempt all the questions and study any areas where your knowledge is lacking using, e.g. the textbooks listed above or the ones you used during your previous study. Please note that the MSc is very intensive and you will not have time to study background material once the course has started.

Please note that we will **not** provide solutions for these questions and we do **not** expect you to send us your solutions for marking/feedback, these questions are purely for your self-study and critical self-assessment.

Probability Questions

1. Let ξ be a discrete random variable with probability distribution $P_\xi(x)$ and distribution function $F_\xi(x)$. Explain why it is possible or not that for some x :

(a) $P_\xi(x) < 0$; (b) $F_\xi(x) > 1$;
(c) $P_\xi(x) = 1$; (d) $P_\xi(x) \leq F_\xi(x)$.

2. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $A, B \in \mathcal{A}$ be two events. Fill in the missing values (note that each line is a different question):

$P(A)$	$P(B)$	$P(A \cup B)$	$P(A \cap B)$	$P(A \setminus B)$	$P(B \setminus A)$	$P(A \triangle B)$
0.6	0.5	0.9				
			0.1	0.4	0.2	
	0.7	1	0			
0.7	0.3		0.3			
0.5	0.5	0.5				
	0.8		0.2	0.1		
0.4		0.8				0.5

Here $A \triangle B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference of A and B.

3. Let $\xi_1, \ldots, \xi_n$ be independent identically distributed Bernoulli random variables taking values 1 and 0 with probabilities p and q, respectively. Let $S_n = \xi_1 + \cdots + \xi_n$.

(a) Evaluate ES_n; (b) Evaluate $Var\, S_n$.

4. Let $\hat{\pi}$ be the empirical distribution for the data set $\mathbb{S}$:

$$13, 2, 5, 1, 27, \quad 5, 34, 7, 2, 4, \quad 9, 7, 1, 19, 6.$$

Calculate

(a) the median and the 20%– and 80%–quantiles of $\hat{\pi}$;

(b) the sample mean and variance for $\mathbb{S}$.

5. Let X and Y be two independent and identically normally $N(0, 1)$ distributed random variables.

(a) Calculate expectation, variance, and covariance of $U := 5\,X + 2Y$ and $V := 2X - 5\,Y$.

(b) What is the distribution of (U, V)? Are U and V independent?

6. Let $X_1, \ldots, X_n$, $(n = 50)$ be independent and identically exponentially distributed random variables with parameter $\lambda = 3$. Use the central limit theorem to get an approximative value of the probability $P(|\overline{X}_n - \mathrm{E}(\overline{X}_n)| > \varepsilon)$, where $\varepsilon = 0.1$ and the sample mean $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

7. Suppose that a random variable ξ has a probability density function

$$f(x) = Ce^{-3x}, \quad 0 < x < \infty,$$

where $C \in \mathbb{R}$ is a constant. Calculate the probability that $1 < \xi < 4$.

8. A bag contains 9 black balls and 5 white balls. Two balls are drawn out at random, one after the other without replacement. Calculate the probabilities that

(a) The second ball is black,

(b) The first ball was white, given that the second ball is black.

9. A company produces lightbulbs at three factories A, B, C. Factory A produces 50 percent of the total number of bulbs, of which 2 percent are defective. Factory B produces 30 percent of the total number of bulbs, of which 4 percent are defective. Factory C produces 20 percent of the total number of bulbs, of which 3 percent are defective. A defective bulb is found among the total output. Find the probability that it came from factory A.

10. A motorist encounters four consecutive traffic lights, each likely to be red or green. Suppose that the traffic lights work independently one from another. Let ξ be the number of green lights passed by the motorist before being stopped by a red light. Find the probability distribution of ξ.

11. Let X be a normally $N(\mu, \sigma^2)$ distributed random variable with mean μ and σ^2. Compute Ee^{aX}, where $a \in \mathbb{R}$.

Linear Algebra Questions

12. Solve the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 &= 8, \\ 3x_1 - 4x_2 &= 4. \end{aligned}$$

13. Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix},$$

where a, b, $c \in \mathbb{R}$.

(a) For which a, b, $c \in \mathbb{R}$, does $\mathbf{A}$ have an inverse?

(b) Find the inverse of $\mathbf{A}$ when it exists.

14. Find the eigenvalues and eigenvectors of the following matrix:

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

Then diagonalise this matrix, i.e., find a matrix S and a diagonal matrix D such that $\mathbf{B} = S^{-1}DS$.

15. Find the dimension and a basis for the following subspace of $\mathbb{R}^4$:

$$U = \operatorname{span}\{(1,1,2,3),(2,4,1,0),(1,5,-4,-9)\}.$$

16. The $n \times n$ matrix M has all of its entries equal to p, except for the entries on the diagonal which are all equal to q. Write down the eigenvalues and eigenvectors of the matrix.

Hint. If you don't know how to do it straightaway, you can first try some small examples and spot the pattern.

17. Determine which of the following matrices are (1) upper triangular, (2) lower triangular, (3) permutation matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Prove that the inverse of a lower/unit lower triangular matrix is also lower/unit lower triangular.

18. A 3×3-matrix

$$A = \begin{bmatrix} 6 & 13 & -17 \\ 13 & 29 & -38 \\ -17 & -38 & 50 \end{bmatrix}$$

has eigenvalues $\lambda_1 \doteq 0.0588$, $\lambda_2 \doteq 0.2007$ and $\lambda_3 \doteq 84.74$. Find the 2-norms $||A||_2$ and $||A^{-1}||_2$, and also the corresponding condition number $\kappa_2(A)$. Which properties of the matrix A did you use in finding the 2-norms? Is this matrix A well-conditioned?

Complex numbers, calculus of a single variable and ODEs

19. Use De Moivre's Theorem to evaluate

$$\left(\frac{1}{2} + \frac{1}{2}\mathrm{i}\right)^8.$$

20. Use Euler's equation to evaluate

(a) $e^{\mathrm{i}\pi/3}$,

(b) $e^{-2+\mathrm{i}\pi/6}$.

21. Given $z = 2 - i$, find z^2 and $1/z$, expressing the answers in the form $x + iy$.

22. Consider the function $f(x) = (1 + 3x)^{1/3}$. Obtain the first three nonzero terms in the Taylor expansion of this function about $x = 0$. Using the expansion evaluate the function at $x = 0.1$.

23. Evaluate the following integrals:

$$\begin{aligned} I_1 &= \int_a^b c^x dx, \\ I_2 &= \int \ln x dx, \\ I_3 &= \int \frac{x+1}{x^2+3x+2} dx, \end{aligned}$$

where $a, b, c \in \mathbb{R}$ are constants.

24. Using the *definition* of the Riemann integral (i.e., not integration rules based on the Fundamental Theorem of Calculus), show that for any constant $c \in \mathbf{R}$ and any $a < b$ with $a, b \in \mathbf{R}$:

$$\int_a^b c dx = c(b-a).$$

25. Determine the following limit if it exists:

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}.$$

26. Does the following integral converge:

$$\int_0^\infty \frac{\sin x}{x^{3/2}} dx?$$

27. Solve the following ODE, given that $y(\pi/2) = 0$:

$$\frac{dy}{dx} = -\frac{1}{x^2} + \sin(x),$$

where $x > 0$.

28. Find the general solution of the nonhomogeneous linear differential equation

$$\frac{d^5y}{dx^5} - 2\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} = e^x.$$

Determine how many solutions of this equation satisfy the conditions:

$$y(0) = -1, \; y'(0) = 0.$$

Explain why.

29. Consider the differential equation

$$\frac{dy}{dx} = \sqrt[3]{y+1} \, .$$

(a) State the regions of the xy-plane in which the conditions of the existence and uniqueness theorem are satisfied (using any appropriate theorem).

(b) Let S be the region of the xy-plane where the conditions of the existence and uniqueness theorem are NOT satisfied. State whether the given equation with the initial condition $y(x_0) = y_0$ with $(x_0, y_0) \in S$ has a solution.

(c) Solve this equation.

(d) Using the result of (c), find whether the given equation with the initial condition $y(x_0) = y_0$ with $(x_0, y_0) \in S$ has a unique solution.

30. Miss X would like to take out a mortgage to buy a house in Nottingham. The bank will charge her interest at a fixed rate of 6.1% per year compounded continuously. Miss X is able to pay money back continuously at a rate of £6000 per year.

(a) Make a continuous model of her economic situation, i.e. write a differential equation together with initial condition for the balance $B(t)$ she owes the bank at time t.

(b) Using the phase portrait, say what the maximum mortgage B_0 Miss X can take that she can repay in finite time.

(c) If the repayment period is 25 years, what is the maximum mortgage B_0 Miss X can take?

31. Consider the boundary value problem:

$$y'' + \mu y = 0, \quad y(0) = y(\pi/2) = 0.$$

(a) For what values of μ does this problem have the trivial solution $y \equiv 0$?

(b) For what values of μ does the problem have nontrivial solutions?

Calculus of many variables and vector calculus

32. Let $f(x, y) = \cos \frac{x}{y}$ for $y \neq 0$. Calculate f_{xx}, f_{yy}, f_{xy} and f_{yx}. Is $f_{xy} = f_{yx}$? Justify your answer.

33. Find and classify the stationary points of the function

$$f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - x^2 - y^2.$$

34. Let a curve be given in the parametrised form by

$$\overrightarrow{r}(t) = (2\cos t, 2\sin t), t \in [0, 2\pi).$$

(a) What is this curve?

(b) Find the equations of the tangents to the curve at each of its points (x_0, y_0).

35. Calculate the gradient of $g(x, y, z) = x^4y^2 + \arctan(yz) + z^2$.

36. Calculate the divergence of $\overrightarrow{f}(x, y, z) = (xye^z, y^2z, x^2ye^z + z^2x)$.

37. Find a normal to the surface S given by $z = x^2y^2 + y + 2$.

38. Evaluate the integral $\iint_D \sin(x^2 + y^2)\mathrm{d}x\mathrm{d}y$ over the domain $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

Partial Differential Equations (PDEs)

39. The one-dimensional heat equation with homogeneous boundary conditions is used to model the temperature profile $u(x,t)$ of a metal bar of length 1 (such that $0 \leq x \leq 1$ and $t \geq 0$). The problem to be solved is given by the following boundary value problem (BVP):

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}\,, \quad u(0,t) = 0, \quad u(1,t) = 0.$$

D is the (constant) thermal diffusivity of the metal rod.

(a) By applying separation of variables, i.e. writing $u(x,t) = X(x)T(t)$, find the general solution of this BVP.

(b) Find the particular solution of this BVP given the initial condition

$$u(x,0) = \sin(2\pi x).$$

40. The one-dimensional wave equation with arbitrary initial conditions is used to model the vertical displacement $u(x,t)$ of a vibrating string of length 1 (such that $0 \leq x \leq 1$ and $t \geq 0$). The problem to be solved is given by the following initial value problem (IVP):

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}\,, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

c is the (constant) propagation speed of the wave, and $f(x)$ and $g(x)$ are arbitrary functions of x.

(a) By applying the change of variables

$$\xi = x - ct, \quad \eta = x + ct,$$

show that the PDE transforms to

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

(b) Hence, show that

$$u(x,t) = F(x - ct) + G(x + ct)\,;$$

i.e., the solution of the IVP is the sum of a right-travelling function F and a left-travelling function G.

(c) By applying the initial conditions given in the IVP, show that the solution of the IVP is given by

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ \mathrm{d}s.$$

Basic Computing Questions

Using any computer language you know, please code, debug and verify the following functions.

41. Code and test a universal function to solve a quadratic equation (think through inputs and outputs for the function).

42. Code a universal function for evaluating a definite integral of a smooth function by (composite) Simpson's rule (think through inputs and outputs for the function). Test the function on integrals for which you know the exact value and make observations about convergence of Simpson's rule.

43. Code a universal function for solving an arbitrary algebraic equation $f(x) = 0$ by the Newton-Raphson method; i.e., using the following iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_0 = a,$$

where a is a user-chosen initial guess.

Your program should stop and return the obtained result to the user once $|f(x)|$ passes below some user-chosen tolerance tol, or abort after some user-chosen maximum allowed number of iterations N. Your code should also abort if $f'(x_n) = 0$ at any iteration.

Test your program with an example you know the answer to.

44. Consider the general first-order ordinary differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y),$$

with domain $0 \le x \le x_{\max}$ and initial condition $y(0) = y_0$.

The forward Euler method is a numerical method for solving this problem and is given by the following iterative scheme:

$$y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0, 1, \ldots, N,$$

where N is the user-chosen number of steps such that $x_N = x_{\max}$, h is the resulting step size where $h = x_{\max}/N$ and $x_n = nh$, and y_n is an approximation for $y(x_n)$.

Code a universal function for solving this problem via the forward Euler method. Test your program with an example you know the answer to.

Experiment with the value of N...what happens once N becomes sufficiently small?